An Introduction to Partially Ordered Structures and Sheaves

Francisco Miraglia

Lógica no Avião

An Introduction to Partially Ordered Structures and Sheaves

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To

Tiago Passos Miraglia

In Memoriam

Editorial Board

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Editorial Preface

I am very glad for the opportunity to write an editorial preface to "An Introduction to Partially Ordered Structures and Sheaves" by Francisco Miraglia. This book with 45 chapters and almost 500 pages of exciting mathematics, featuring partially ordered structures, category theory, spectral spaces, presheaves over topological spaces, change of base and characteristic maps, originated in graduate courses given by the author, as a visiting scholar at the Mathematical Institute of Oxford University, in the academic year 90/91. It was first published by Polimetrica International Scientific Publisher in January 2009, where I was acting as a chief editor. I then had the privilege of writing a more careful, perhaps a bit poetic, Foreword to the book. I will not repeat it here, but instead register some facts. Polimetrica was founded by Giandomenico de Sica in Monza, Italy, with the intention of bringing quality books to light. In my short life as an editor at Polimerica, I had the chance to publish, besides Miraglia's book, also "The Magic Garden of George B And Other Logic Puzzles" by Raymond Smullyan in 2007, which was reprinted by World Scientific in 2015. Polimetrica did publish several books, but unfortunately could not resist the pressure of the editorial market.

I would like to applaud the excellent initiative by the UnB Brasilia group of "Lógica no Avião" (LnA) in the person of Rodrigo Freire (editor), Alexandre Costa-Leite, Edgar Almeida and Gustavo Schmidt in republishing this book. LnA is a non-profit organization dedicated to the promotion and dissemination of high quality work in logic and philosophy, and will make this extraordinary book free for all future generations. I am proud to have contributed to it.

Campinas, July 2020
Walter Carnielli

Foreword

The discussion whether category theory is just a convenient language or has its own intrinsic way to conceptualize mathematical knowledge will be seen as idle after you read this book: you will certainly agree that it is both. Category theory was born under the point of view that many (or all) concepts in Mathematics could be better understood and explained by approaching them in a highly unified way: several definitions, constructions, proofs and concepts are essentially the same, and this is what the tools of category theory reveal.

Interesting enough, the idea of "category" derives from the use it had in philosophy; even if the intention the founders of category theory gave the term is not exactly the one Aristotle, Immanuel Kant, and Charles S. Peirce referred to it, some connections remain.

Perhaps what the ideas of category in the philosophical and mathematical senses share is the concern about existence and predication. In a similar way as on what can be said about a given object or subject, as idealized by the classics, would depend on the attention to certain aspects (such as quantity, substance, relations or states), in the mathematical categories the existence of objects depends on the conceptual environment. The witnesses of their existence are the morphisms (another borrowed term, this one from R. Carnap) that relate the object in question with other objects.

Some authors will say that, in category theory, an object cannot say to exist, but what exists is the concept behind it. This is clearly reflected, for instance, in setting the natural number objects in sheaves: if we suspect that perhaps one of the most fundamental properties of natural numbers is their capacity of defining functions by recursion, an abstract treatment can test this idea and see how far it goes.

It is in the spirit of Mathematics to stretch out the essence of an idea to its ultimate consequences: this way, for example, the category of sets or a topos of sheaves over a topological space generalize at the same time a huge number of concepts.

This book gives an overall, as self-contained as possible, introduction to sheaf theory and its relation to logic. Starting from partially ordered structures, Miraglia shows how to go from lattices to sheaf theory, how this naturally leads to the universal constructions of category theory, and to first-order structures over partially ordered sets.

Sheaves and presheaves over topological spaces - and how this is related to first-order structures - is a central topic of this book.

But the connections to contemporary logic itself go much further: A theory of relations and quantification in some particular categories is also explained; classical existential and universal quantifiers arise from those when certain projections "forget" coordinates, following the seminal ideas of Francis William Lawvere on conceptual mathematics.

There is also a deep methodological point when working in abstract partially ordered structures and sheaves, and this book contributes to making this point clear - as in general such structures are "pointless", the constructions and generalizations from elementary to more sophisticated results have to be more intrinsic, but this, on the other hand, reveals the common relationship among semillatices, distributive lattices, Heyting algebras and frames.

Continuous lattices and their natural topology (the "Scott topology") is a basic topic here; they are known to be connected with computer science, general topology, analysis, algebra and topos theory. This will also be relevant to readers interested in constructive mathematics, and in deeper connections between category theory and Logic, and on how this leads to more abstract views, as for example how the methods of homology, cohomology, algebra and topological K-theory could be seen as a sort of unified theory.

But what is an enjoyable and distinctive characteristic of this book is that all this (and its relationship to category theory) is developed in a smooth way: Miraglia chooses to introduce sheaves directly as mathematical structures, (specifically, as Ω-sets closed under certain gluing properties) without previous requirements on category theory, for the benefit of the non-acquainted reader. This makes this book as peerless introduction not only to concepts and methods, but to the philosophical assumptions, foundations and significance of contemporary logic.

Walter Carnielli
Editor, *Contemporary Logic*
Campinas, September 2006

Preface

These notes originated in graduate courses given by the author, as a visiting scholar, at the Mathematical Institute, Oxford University, in the academic year 90/91. The audience included Angus Macintyre, Alex Wilkie, Richard Kaye, Margerita Otero, Ugo Solitro and Paula D'Aquino, all of which deserve my heartiest thanks for ideas and suggestions. I am also grateful for the hospitality of the Mathematical Institute at Oxford, marvelously represented by Angus Macintyre and Alex Wilkie.

I am happy to acknowledge the contributions of Ugo Solitro, Marcelo Coniglio, Andreas Brunner and Hugo Mariano to the present version of the text, which consists of a considerable revision of the original, distributed every week to the participants of the courses at Oxford. Special thanks are due to Walter Carnielli for his enthusiasm with this project and to Giandomenico de Sica and the editorial staff at Polimetrica for all their help in bringing the book to print.

In January of 1989, Carlos di Prisco organized, at the *Instituto Venezuelano De Investigaciones Científicas (IVIC)* in Caracas, Venezuela, a workshop on Category Theory and Logic. Besides Carlos di Prisco, were in attendance Antonio Mario Sette (University of Campinas, Brazil), Xavier Caicedo (University of Los Andes, Colombia), Ken López-Escobar (University of Maryland, USA) and the author. In the many hours of enjoyable mathematical and cultural discussion, arose the idea of writing a text that could serve as an introduction for the development of the Sheaf Theory and Logic, as well as an introduction to the abstract context of Topoi. The visit to Oxford gave me the opportunity of constructing a proposal in this direction. However, all shortcomings of this attempt are my sole responsibility.

The prerequisites are a knowledge of basic algebra, point set topology and elementary category theory. I expect that a first year graduate student will have enough background to be able to work through the book.

The text is divided into seven parts. In Part I, **Partially Ordered Structures**, we discuss the lattice theoretic basis of sheaf theory. The attempt is to make the text relatively self-contained. On the other hand, to keep size under control, we cut a rather brisk path through partially ordered sets, lattices, distributive lattices, Boolean algebras, Heyting algebras and their complete counterparts, giving indications for further reading.

We have also included, in Chapters 16 and 17 of Part II, a summary of the Category Theory and of limits and colimits of first-order structures over partially ordered sets used in the book.

Part III, **Spectral Spaces**, gives a unified treatment of the spectra of distributive lattices and of commutative rings. It also includes a presentation of the Gleason or projective cover of compact spaces.

Part IV, **Presheaves over Topological Spaces**, is a survey of the main ingredients of sheaves and presheaves over topological spaces. There is, of course, also a presentation of sheaves and presheaves of first order structures, in this setting. Our feeling was that the geometrical model is important in understanding the abstract constructions, which it originated.

Part V, ***L*-sets** generalizes to semilattices, distributive lattices, Heyting algebras and frames, the constructions in Part III. Since in general the algebraic basis we work with do not have points, the treatment has to be more intrinsic. The origin of these ideas are in [**15**], but we build on the development begun in [**50**].

Part VI, **Change of Base**, discusses the process of transporting, along a semilattice morphism, L-sets and presheaves over one base to another. The material includes the fundamental constructs of image, base extension, inverse image, localization, fiber and stalks. These ideas are then applied to the description of regularization functors, that generalize the transport functor associated to double negation in a frame.

With an eye to applications of the material in text to Model Theory in the category of Ω-sets and presheaves, we develop, in Part VII, **Characteristic Maps**, a description of closed subobjects of Ω-sets and presheaves that has proven to be a versatile and useful instrument for the establishment of a theory of relations and quantification in those categories. The final Chapter of this Part introduces the notion of *graded frame*, bringing to the theory of characteristic maps a construct that is inspired by the well-known sequences that occur in Homology, Cohomology, as well as in Algebraic and Topological K-theory.

We have chosen to present the development of Model Theory in the category of L-sets and presheaves over a frame in a distinct volume.

At the end of each chapter we have included exercises, of varying difficulty. Moreover, supplying the proofs of many assertions made in Examples and Remarks are also considered as exercises for the reader.

All results, remarks, examples, exercises and definitions are numbered consecutively within each chapter, beginning anew with every chapter, even if a chapter is divided into sections. We adopt standard conventions concerning cross-references. The symbol $\square$ indicates the end of a proof, of an example or of a remark.

São Paulo, April, 2006
F. Miraglia

Contents

Part 1

Partially Ordered Structures

CHAPTER 1

Fundamentals

1. Sets

We adopt standard notation for unions, intersections and other set theoretic operations. For sets A and B, $A - B$ is their difference, while $A \triangle B$ is their symmetric difference, i.e.,

$$A - B = \{x \in A : x \notin B\} \quad \text{and} \quad A \triangle B = (A \ - \ B) \cup (B \ - \ A).$$

If $A \subseteq B$ is clear from context, write A^c for $B - A$.

$A \subseteq_f B$ stands for A is a *finite* (possibly empty) subset of B. Let

[Fin] $$Fin(B) = \{A \subseteq B : A \subseteq_f B\}$$

be the collection of finite subsets of B.

If A is a set, write card(A) or $\overline{\overline{A}}$ for the cardinal of A.

If $X \xrightarrow{f} Y$ is a map and $S \subseteq X$, $f_{|S} : S \longrightarrow Y$ is the restriction of f to S.

Write $2 = \{0, 1\}$ and identify the set 2^X with the family of subsets of X, via characteristic functions :

$$S \subseteq X \quad \longmapsto \quad \chi_S : X \longrightarrow 2, \text{ where } \chi_S(x) = 1 \text{ iff } x \in S.$$

The identity map as well as the composition of maps will be written as usual. Whenever possible, we omit parentheses from functional notation, writing fx for the value of f at x.

We write $\mathbb{N} = \{0, 1, \ldots\}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ for the natural numbers, integers, rationals, reals and complex numbers respectively, all of which carry their well-known mathematical structure. If $a < b$ are reals, we use standard conventions regarding intervals. Thus, e.g.,

$$[a, b) = \{\ x \in \mathbb{R} : a \leq x < b\}.$$

For $n \in \mathbb{N}$,

$*$ $n = \{0, 1, 2, \ldots, (n-1)\}$, as is standard in Set Theory;

$*$ If $n \geq 1$, write $\underline{n} = \{1, 2, \ldots, n\}$.

Thus, $2 = \{0, 1\}$, while $\underline{2} = \{1, 2\}$. Clearly, $0 = \emptyset$.

1.1. **Partial Maps.**

Write $pF(X, Y)$ for the set of partial maps from X to Y, that is

$$pF(X, Y) = \{f : f \subseteq X \times Y \text{ and } f \text{ is a function}\}.$$

Write dom$f$ and Imf, respectively, for the domain and image of a binary relation f. Note that $\emptyset$ (the function with empty domain) is a member of $pF(X, Y)$.

Write Y^X $(\subseteq pF(X, Y))$ for the set of maps from X to Y. $\square$

One of the basic properties of subsets of $pF(X, Y)$ is described in

LEMMA 1.2. *Assume that $S \subseteq pF(X, Y)$ is* **compatible***, i.e., it satisfies :*

$$\text{For all } f,\, g \in S, \quad f_{|_{dom f \cap dom g}} = g_{|_{dom f \cap dom g}}.$$

Then, there is a **unique** *$h \in pF(X, Y)$ such that*

$$dom\, h = \bigcup_{f \in S} dom f \quad \text{and} \quad h_{|_{dom f}} = f,\ \forall\, f \in S.$$

PROOF. For $x \in U = \bigcup_{f \in S}$, set $hx = fx$, with $x \in dom\, f$, $f \in S$; since the elements of S are compatible, h is a map from U into Y, with the required properties. $\square$

Write $h = \bigvee_{f \in S} f$ or $h = \bigvee S$ for the 'gluing' of the compatible family S given by Lemma 1.2.

1.3. **Equivalence Relations.**

An **equivalence relation** on a set X is a subset $E \subseteq X \times X$, such that for all x, y, $z \in X$

[equ 1] : $x\ E\ y$;

[equ 2] : $x\ E\ y$ implies $y\ E\ x$;

[equ 3] : $x\ E\ y$ and $y\ E\ z$ implies $x\ E\ z$,

where, as usual, $x\ E\ y$ stands for $\langle x, y \rangle \in E$ (called infix notation). If E is an equivalence relation on X and $x \in X$, write

$$x/E = \{y \in X : x\ E\ y\},$$

for the equivalence class of x with respect to E. Write X/E for the set of equivalence classes of elements of X by E. There is a natural surjection $\pi_E : X \longrightarrow X/E$, given by $\pi_E(x) = x/E$. $\square$

1.4. **Products.**

If $\{X_i\}_{i \in I}$ is a family of sets, their **product** is defined to be

$$\prod_{i \in I} X_i = \{I \xrightarrow{s} \bigcup_{i \in I} X_i : \forall\, i \in I,\, s(i) \in X_i\ \},$$

which may be abbreviated by $\prod X_i$. A typical element of $\prod X_i$ is written $\overline{a} = \langle\, a_i \,\rangle$ or $\overline{a} = \langle\, a(i) \,\rangle$. There are natural projection maps $\prod X_i \xrightarrow{\pi_i} X_i$, given by $\overline{a} = \langle\, a_i \,\rangle \mapsto a_i$.

If $J \subseteq I$, there is a natural map,

$$\rho_J : \prod_{i \in I} X_i \longrightarrow \prod_{j \in J} X_j, \quad \rho_J(\overline{a}) = \text{restriction of } \overline{a} \text{ to } J.$$

Hence, ρ_J is the projection that forgets the components outside J. Since we employ the restriction notation when dealing with presheaves, we shall refrain (although appropriate) to write $\rho_J(\overline{a})$ as $\overline{a}_{|_J}$.

Let $X_i \xrightarrow{f_i} Y_i$, $i \in I$, be a family of maps. There is a unique map

$$\prod_{i \in I} f_i : \prod_{i \in I} \longrightarrow \prod_{i \in I} Y_i, \quad \prod f_i(\overline{a}) = \langle\, f_i a_i \,\rangle, \tag{PM}$$

called the **product of the $\boldsymbol{f_i}$**. A map $X \xrightarrow{f} Y$, induces, for any set I, a function

$$f^I : X^I \longrightarrow Y^I, \quad f^I(\overline{a}) = \langle\, f a_i \,\rangle.$$

We shall frequently let $\boldsymbol{f(\overline{a})}$ stand for $\boldsymbol{f^I(\overline{a})}$. $\square$

1.5. **Disjoint Unions.**
Write $\coprod_{i \in I} X_i$ or simply $\coprod X_i$, for the disjoint sum of the family X_i, where
$$\coprod X_i = \bigcup_{i \in I} X_i \times \{i\}.$$
There are canonical maps $X_i \xrightarrow{\alpha_i} \coprod X_i$, given by $x \in X_i \mapsto \langle x, i \rangle$. $\square$

2. Topology

We assume that the reader is familiar with the language of Topology, as for instance in [**12**] or [**77**]. This section is designed to serve as a quick reference, for the convenience of the reader.

Definition 1.6. *A* **topological space** *is a pair $\langle X, \tau \rangle$ where X is a set and τ is a subset of 2^X such that*
[top 1] : $\emptyset$, $X \in \tau$;
[top 2] : τ is closed under finite intersections;
[top 3] : τ is closed under arbitrary unions.
The elements of τ are called **opens** [1] *and τ as a* **topology** *on X.*
A subset of X is **closed** *if its complement is open. The de Morgan laws of elementary set theory guarantee that the closed sets verify the following conditions:*
[clo 1] : $\emptyset$, X are closed;
[clo 2] : The family of closed sets is closed under finite unions;
[clo 3] : The family of closed sets is closed under arbitrary intersections.

Example 1.7. Let $\langle X, \tau \rangle$ be a topological space and A be a subset of X. Define
$$\tau_{|A} = \{C \subseteq A : \exists\, U \in \tau \text{ such that } C = U \cap A\}.$$
$\tau_{|A}$ is a topology, the **induced or subspace topology** on A. $\square$

If τ_1, τ_2 are topologies on X and $\tau_1 \subseteq \tau_2$, we say that τ_2 is **finer** than τ_1.

Note that 2^X is a topology on X, the **discrete topology**, in which all subsets of X are open. It is clearly the finest topology on X.

The family of topologies on X – a subset of 2^{2^X} –, is closed under intersections. Hence, if $S \subseteq 2^X$ is a family of subsets of X, S generates a *unique* topology on X, defined as
$$\tau(S) = \bigcap \{\tau : \tau \text{ is a topology on } X \text{ and } S \subseteq \tau\}.$$
A more "constructive" description of $\tau(S)$ is given by

Lemma 1.8. *For $S \subseteq 2^X$, let* [2]
$$\mathfrak{B}(S) = \{V \subseteq X : \exists\, F \subseteq_f S \textit{ such that } V = \bigcap F\}.$$
Then,
$$\tau(S) = \{U \subseteq X : \exists\, G \subseteq \mathfrak{B}(S) \textit{ such that } U = \bigcup G\}.$$

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[1] This is quite imprecise. "Open" only has meaning after τ is given.
[2] $\subseteq_f$ means "finite subset of", defined in page 15.

Lemma 1.8 may be paraphrased as "the elements of $\tau(S)$ are the union of finite intersections of elements of S".

Proof. Write $T = \{U \subseteq X : \exists\, G \subseteq \mathfrak{B}(S)$ such that $U = \bigcup G\}$; note that $S \subseteq \mathfrak{B}(S) \subseteq T$. It is straightforward that $T \subseteq \tau(S)$. Hence, it suffices to check that T is a topology. It is clear that T is closed under arbitrary unions, as well as that $\emptyset$, $X \in T$ [3]. To see that it is closed under finite intersections, write

$$U = \bigcup G_1 \quad \text{and} \quad V = \bigcup G_2,$$

with $G_i \subseteq \mathfrak{B}(S)$, $i = 1, 2$. Then

$$U \cap V \;=\; \textstyle\bigcup_{A \in G_1, B \in G_2} A \cap B. \tag{*}$$

Since A and B are finite intersections of elements of S, the same is true of $A \cap B$, and (*) entails that $U \cap V \in T$, as needed. $\square$

Definition 1.9. *Let $\langle T, \tau\rangle$ be a topological space and $S \subseteq \tau$.*
a) S is a **basis** *for T iff every open set in T can be written as the union of elements of S* [4].
b) S is a **sub-basis** *for T if every open set in T can be written as the union of elements in $\mathfrak{B}(S)$ (as in 1.8).*

If X is a topological space, write $\Omega(X)$ for the topology (or the set of opens) in X. For $x \in X$,

$$\nu_x = \{U \in \Omega(X) : x \in U\}$$

is the set of **open neighborhoods of x in X**.

Associated to any topological space X there are operations on 2^X, which we now describe. For $A \subseteq X$, define

$$\begin{cases} \text{int } A &= \text{interior of } A \;=\; \bigcup \{U \in \Omega(X) : U \subseteq A\}; \\ \overline{A} &= \text{closure of } A \\ &= \bigcap \{F \subseteq X : A \subseteq F \text{ and } F \text{ is closed in } X\}; \\ \partial A &= \text{frontier of } A \;=\; \overline{A} \cap \overline{X - A} \end{cases}$$

The basic properties of interior and closure are described in the following Lemma, whose proof is left to the reader.

Lemma 1.10. *Let X be a topological space, $x \in X$, A, $B \subseteq X$.*
a) A is open iff $A = \text{int } A$ and A is closed iff $\overline{A} = A$.
b) Interior and closure are increasing *and* idempotent*, that is,*

(1) Increasing *:* $A \subseteq B \;\Rightarrow\; \text{int } A \subseteq \text{int } B$ *and* $\overline{A} \subseteq \overline{B}$*;*

(2) Idempotent *:* $\text{int}(\text{int } A) = \text{int } A$ *and* $\overline{(\overline{A})} = \overline{A}$.

c) $\text{int } A \cup \text{int } B \subseteq \text{int } (A \cup B)$ *and* $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$ [5].
d) $\text{int } (A \cap B) = \text{int } A \cap \text{int } B$ *and* $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
e) $\overline{A} = \{p \in X : \forall\, V \in \nu_p,\;\; V \cap A \neq \emptyset\}$.

[3] Just as above, $X = \bigcap \emptyset$ and $\emptyset = \bigcup \emptyset$ and $\emptyset$ is a finite subset of S and $\mathfrak{B}(S)$.
[4] Some authors require that S be closed under finite intersection.
[5] In general, interior does not preserve unions and closure does not preserve intersection.

f) $\text{int}\, A^c = \left(\overline{A}\right)^c$ *and* $\left(\overline{A^c}\right)^c = \text{int}\, A$.

Definition 1.11. *Let T be a topological space and $A, B \subseteq X$.*
a) A is **clopen** *if it is open and closed in T. Write $B(T)$ for the set of clopen subsets of T.*
b) A is a **regular open set** *if $A = \text{int}\, \overline{A}$. Write $Reg(T)$ for the set of regular opens in T.*
c) A is a **regular closed set** *if $A = \overline{\text{int}\, A}$.*
d) **A is dense in B** *if $B \subseteq \overline{A}$. A is* **dense** *if it is dense in T, that is, $\overline{A} = T$. Write $D(T)$ for the set of dense* **open** *sets in T.*

With notation as in 1.11, it is clear that

[R] $$\{\emptyset, T\} \subseteq B(T) \subseteq Reg(T) \subseteq \Omega(T).$$

Lemma 1.12. *If T is a topological space, then $B(T)$ is closed under complements, as well as finite unions and intersections.*

Proof. It is immediate from the defining properties of a topology (1.6) that $B(T)$ is closed under complements and finite unions and intersections. ☐

Lemma 1.13. *Let T be a topological space.*
a) $A \subseteq T$ is dense iff for all $U \in \Omega(X) - \{\emptyset\}$, $A \cap U \neq \emptyset$.
b) The set $D(T)$ of dense opens in T has the following properties :
(1) *$A, B \in D(T) \;\Rightarrow\; A \cap B \in D(T)$;*
(2) *$A \in D(T)$ and $A \subseteq B \in \Omega(X) \;\Rightarrow\; B \in D(T)$.*

Proof. a) Straightforward from 1.10.(e) and the fact that $\overline{A} = T$.
b) Since closure is increasing (1.10.(b).(1)), (2) is clear. For (1), we use (a). Let $x \in T$ and $V \in \nu_x$; since A is dense in T, we have $V \cap A \neq \emptyset$. But note that $V \cap A$ is open and so must intersect the dense set B. Hence, $V \cap A \cap B \neq \emptyset$, and 1.10.(e) entails $x \in \overline{A \cap B}$, as needed. ☐

Lemma 1.14. *Let $\{U\} \cup \{W_i : i \in I\} \subseteq Reg(T)$, where T is a topological space.*
a) For $A \in \Omega(X)$, A is regular $\Leftrightarrow$ A the interior of a closed set.
b) If A is open in T then
$$\neg A \;=_{def}\; \text{int}\,(T - A)$$
is a regular open, the largest **open** *set in T that is disjoint from A. Moreover, $(A \cup \neg A) \in D(T)$.*
c) The smallest[6] *regular open containing all W_i is*
$$\textstyle\bigvee^*_{i \in I} W_i \;=_{def}\; \text{int}\, \overline{\bigcup_{i \in I} W_i}.$$
In particular, $U \vee^* \neg U = T$.
d) The largest[7] *regular open contained in all W_i is*
$$\textstyle\bigwedge^*_{i \in I} W_i \;=\; \text{int}\, \overline{\bigcap_{i \in I} W_i}.$$

[6]Under inclusion.
[7]Under inclusion.

e) Reg(T) is closed under **finite** *intersections. Hence, if I is finite, then*

$$\bigwedge\nolimits^*_{i \in I} W_i \;=\; \bigcap\nolimits_{i \in I} W_i.$$

f) $U \cap \bigvee^*_{i \in I} W_i \;=\; \bigvee^*_{i \in I} U \cap W_i$.

PROOF. a) $(\Rightarrow)$ is clear; conversely, suppose $A = \text{int}\, F$, with F closed in T. Then, since closure and interior are increasing (1.10.(b)), we get

$$\overline{A} \;=\; \overline{\text{int}\, F} \;\subseteq\; F.$$

Hence, $A \subseteq \text{int}\, \overline{A} \subseteq \text{int}\, F = A$, and $A = \text{int}\, \overline{A}$, as desired.

b) Since A is open, it follows from (a) that $\neg A$ is a regular open. Clearly, $A \cap \neg A = \emptyset$ [8]. If $W \cap A = \emptyset$, then $W \subseteq (T - A)$. Since the interior of a set is the *largest* open set contained in it (see § before 1.10), it follows that $W \subseteq \text{int}\,(T - U) = \neg U$.

To check that $U \cup \neg U$ is dense in T, let $x \in T$ and $V \in \nu_x$. By what has just been proven

$$V \cap U = \emptyset \;\Rightarrow\; V \subseteq \neg U.$$

Hence, $V \cap (U \cup \neg U) \neq \emptyset$, and the conclusion follows from 1.10.(e).

c) By (a), $\bigvee^* W_i \in Reg(T)$; clearly, it contains all W_i. If $V \in Reg(T)$ verifies $W_i \subseteq V$, $i \in I$, then

$$\bigcup\nolimits_{i \in I} W_i \;\subseteq\; V,$$

wherefrom it follows that $\overline{\bigcup_{i \in I} W_i} \;\subseteq\; \overline{V}$. But then,

$$\bigvee\nolimits^*_{i \in I} W_i \;=\; \text{int}\, \overline{\bigcup\nolimits_{i \in I} W_i} \;\subseteq\; \text{int}\, \overline{V} \;=\; V,$$

as needed. Since $U \cup \neg U$ is dense in T, it follows that

$$U \vee^* \neg U \;=\; \text{int}\, \overline{U \cup \neg U} \;=\; \text{int}\, T \;=\; T,$$

as asserted. Item (d) is similar and left to the reader.

e) It is enough to show that if U, $V \in Reg(T)$, then $U \cap V \in Reg(T)$. This amounts to verifying that

$$U \cap V \;=\; \text{int}\, \overline{U \cap V},$$

which reduces to $\text{int}\, \overline{U \cap V} \subseteq U \cap V$. But this follows from items (c) and (d) in 1.10 :

$$\text{int}\, \overline{U \cap V} \;\subseteq\; \text{int}\, (\overline{U} \cap \overline{V}) \;=\; \text{int}\, \overline{U} \cap \text{int}\, \overline{V} \;=\; U \cap V,$$

as desired.

f) From (c) and (d) in 1.10 we get

$$\begin{aligned} \text{int}\, \overline{\bigcup\nolimits_{i \in I} U \cap W_i} \;&=\; \text{int}\, \overline{U \cap \bigcup\nolimits_{i \in I} W_i} \;\subseteq\; \text{int}\, \left(\overline{U} \cap \overline{\bigcup\nolimits_{i \in I} W_i}\right) \\ &=\; \text{int}\, \overline{U} \cap \text{int}\, \overline{\bigcup\nolimits_{i \in I} W_i} \;=\; U \cap \bigvee\nolimits^*_{i \in I} W_i. \end{aligned}$$

By the computation above, the reverse inclusion is equivalent to

$$\text{int}\, \left(\overline{U} \cap \overline{\bigcup\nolimits_{i \in I} W_i}\right) \;\subseteq\; \text{int}\, \overline{U \cap \bigcup\nolimits_{i \in I} W_i}, \tag{*}$$

which we now verify. To do this, it is enough to check that any open set contained in the left hand side of (*) is also contained in its right hand side. Suppose, then

[8] For $\neg A \subseteq T - A$.

that $W \in \Omega(T)$ is such that $W \subseteq \left(\overline{U} \cap \overline{\bigcup_{i \in I} W_i}\right)$. Then, $W \subseteq \overline{U}$, and so $W \subseteq U$, because U is regular. Hence,

$$W \subseteq U \cap \overline{\bigcup_{i \in I} W_i} \qquad (^{**})$$

We now observe

Fact 1.15. *For $U \in \Omega(T)$ and $A \subseteq T$, $U \cap \overline{A} \subseteq \overline{U \cap A}$.*

Proof. Let $p \in U \cap \overline{A}$ and $V \in \nu_p$. Because U is open, $U \cap V \in \nu_p$. Since $p \in \overline{A}$, 1.10.(e) entails $U \cap V \cap A \neq \emptyset$, establishing that $p \in \overline{U \cap A}$.

It is now immediate from Fact 1.15 and (**) that $W \subseteq \overline{U \cap \bigcup_{i \in I} W_i}$, as needed to establish (*), ending the proof. □

There are several ways to measure the "size" of a topological space (besides cardinality). Two of the most common are introduced in

Definition 1.16. *Let T be a topological space.*
a) The **density of T**, *$d(T)$, is the least cardinal κ such that T has a dense subset of cardinality κ. T is* **separable** *if $d(T)$ is at most countable.*
b) The **weight of T**, *$w(T)$, is the least cardinal γ such that T has a basis of cardinal γ. T is* **second countable or Lindelöff** *if its weight is at most countable.*

It is clear that $d(T) \leq w(T)$; but Theorem 1.29 (stated below) implies that there are (many) separable spaces whose weight is uncountable.

Definition 1.17. *Let $f : X \longrightarrow Y$ be a map between topological spaces.*
a) f is **continuous** *if for all $V \in \Omega(Y)$, $f^{-1}(V) \in \Omega(X)$. Write $\mathbb{C}(X, Y)$ for the set of continuous maps from X to Y. When Y is the real line $\mathbb{R}$, write $\mathbb{C}(X)$ for $\mathbb{C}(X, \mathbb{R})$.*
b) f is **closed** *if the image of every closed subset of its domain is closed in its codomain.*
c) f is **open** *if the image of every open subset of its domain is open in its codomain.*

d) A continuous map is a **homeomorphism** *if it is bijective and its inverse is continuous.*

Lemma 1.18. *Let $f : X \longrightarrow Y$ be a continuous map of topological spaces. If f is bijective, the following are equivalent :*
(1) *f is a homeomorphism;* (2) *f is closed;* (3) *f is open.*

Proof. Let $g : Y \longrightarrow X$ be the inverse of f. Recall that inverse image commutes with complement and that for $A \in 2^X$ and $B \in 2^Y$

$$f^{-1}(B) = g(B) \quad \text{and} \quad g^{-1}(A) = f(A).$$

The stated equivalence is immediate from these observations. □

Definition 1.19. *Let X and Y be sets and S, T be disjoint subsets* **of X**. *Let $\mathcal{A} \subseteq 2^X$ and $\mathcal{K} \subseteq Y^X$.*
a) $\mathcal{A}$ **separates** *S and T if there are A, $B \in \mathcal{A}$ such that $S \subseteq A$ and $T \subseteq B$.*

b) $\mathcal{K}$ **separates** S *and* T *iff there is* $f \in \mathcal{K}$ *and* y, y' *in* $Im\, f$ *such that* $S \subseteq f^{-1}(y)$ *and* $T \subseteq f^{-1}(y')$.
c) $\mathcal{A}$ **separates points** *in* X *if distinct points in* X *are separated by* $\mathcal{A}$*. Analogously, for the concept that* $\mathcal{K}$ *separates points in* X.

Definition 1.20. *A topological space* X *is* [9]

* **T0** *iff for all* $x, y \in X$, $\quad x \neq y \;\; \Rightarrow \;\; \nu_x \neq \nu_y$.
* **T1** *iff* $\forall\, x, y \in X$, $\; x \neq y \;\; \Rightarrow \;\; \nu_x \,-\, \nu_y \neq \emptyset$.
* **Hausdorff or T2** *iff distinct points in* X *are separated by* $\Omega(X)$.
* **regular or T3** *iff it is* $T1$ *and for all* $x \in X$ *and closed* F *in* X,
$$x \notin F \;\; \Rightarrow \;\; \{x\} \text{ and } F \text{ are separated by } \Omega(X).$$
* **completely regular** [10] *iff it is* $T1$ *and for all* $x \in X$ *and all closed* F *in* X,
$$x \notin F \;\; \Rightarrow \;\; \{x\} \text{ and } F \text{ are separated by } C(X, [0,1]).$$
* **normal or T4** *iff it is* $T1$ *and disjoint closed sets in* X *are separated by* $\Omega(X)$.

Lemma 1.21. *a) A space is* $T1$ *iff all points in* X *are closed.*
b) A $T1$ *space is regular iff for all* $x \in X$ *and all* $V \in \nu_x$*, there is* $U \in \nu_x$ *such that* $\overline{U} \subseteq V$.

If $i \leq j$ then $Tj \Rightarrow Ti$; T4 $\Rightarrow$ completely regular, comes from the famous Urysohn separation theorem for normal spaces :

Theorem 1.22. (Urysohn) *X is a normal space iff any pair of disjoint closed sets is separated by* $C(X, [0,1])$.

An important property that will appear quite frequently is *compactness*, a topological version of finiteness.

Definition 1.23. *Let* T *be a topological space and* $A \subseteq B \subseteq T$.
a) A *is* **compact** [11] *if every open covering of* A *has a finite subcovering.*
b) ***A* is relatively compact in *B*** *if* $(\overline{A} \cap B)$ *is compact in* B *(in the topology induced by* T*)* [12].
c) A *is* **relatively compact** *if* $\overline{A}$ *is compact in* T.

Lemma 1.24. *Let* T *be a topological space.*
a) Compactness is preserved by finite unions [13].
b) If $U \in \Omega(T)$ *and* $A \subseteq U$*, then* A *is compact in* U *iff it is compact in* T.
b) The intersection of a closed set and a compact set is compact. In particular, a closed subset of a compact set is compact.
c) In a Hausdorff space, all compact subsets are closed. Hence, in a Hausdorff space, the intersection of compacts is compact.

[9] Terminology as in 1.19.
[10] Sometimes T3$\frac{1}{2}$ space; $[0,1]$ is the closed unit real interval.
[11] Some authors use *quasi-compact* when T is not Hausdorff.
[12] $(\overline{A} \cap B)$ is the closure of A in B, in the induced topology from T as 1.7.
[13] However, in general, not by intersections.

d) A compact Hausdorff space is normal (1.20).
e) Compactness is preserved by continuous image.
f) Let $f : X \longrightarrow Y$ be a continuous map. If X is compact and Y is Hausdorff, then f is a homeomorphism $\Leftrightarrow$ f is bijective.

Proof. We prove b), (d) and (f) leaving the other items to the reader.
b) Let $C = F \cap K$, with $F = \overline{F}$ and K compact in T. Let $W = F^c$; then W is open in T. If V_i, $i \in I$, is an open covering of C, the family $\{V_i : i \in I\} \cup \{W\}$ is an open covering of K; hence it has a finite subcovering. Removing W from this finite subcovering, one obtains a finite subcovering C by the V_i, as needed.

d) Let C, E be disjoint closed sets in T. By (b), C and E are compact. Fix $p \in C$; for each $q \in E$, select disjoint open neighborhoods U_q, V_q of p and q, respectively. This is possible because T is Hausdorff. Since E is compact, there is $\beta \subseteq_f E$ such that

$$E \subseteq \bigcup_{q\in\beta} V_q.$$

Because β is finite, $U_p = \bigcap_{q\in\beta} U_q$ is an open set containing p. Now note that

$$U_p \cap \bigcup_{q\in\beta} V_q = \emptyset,$$

providing disjoint opens, one containing p and the other E. This reasoning shows that T is *regular*. Now repeat the argument, using regularity. For each $p \in C$, there are disjoint opens U_p, V_p, with $p \in U_p$ and $E \subseteq V_p$. Compactness yields $\alpha \subseteq_f C$ such that $C \subseteq \bigcup_{p\in\alpha} U_p$. But then

$$\bigcap_{p\in\alpha} V_p \quad \text{and} \quad \bigcup_{p\in\alpha} U_p$$

are disjoint opens, the first containing E and the second C, establishing normality.
f) It is enough to prove ($\Leftarrow$). We show that f is closed and conclude by 1.18. Let F be a closed set in X; then, F is compact (by (b)) and so $f(F)$ is compact in Y (by (e)). Now, (c) entails that $f(F)$ is closed, ending the proof. $\square$

1.25. **The Product Topology.** Let X_i, $i \in I$, be a family of topological spaces. The product $X = \prod X_i$ (1.4) carries a natural topology, **the product topology**, that we now describe.

Recall that $Fin(I)$ is the family of finite subsets of I (see [Fin] in page 15). For $\alpha \in Fin(I)$, define

$$\mathcal{O}(\alpha) = \prod_{i\in\alpha} \Omega(X_i).$$

Write $U = \langle U_i \rangle_{i\in\alpha}$ for a typical element of $\mathcal{O}(\alpha)$. Now define

$$\mathfrak{p}(U) = \{s \in \prod X_i : s(i) \in U_i, \text{ for all } i \in \alpha\}. \tag{1}$$

Note that $\mathfrak{p}(U)$ is the product of the family

$$\{X_j : j \in I - \alpha\} \cup \{U_i : i \in \alpha\}.$$

Consequently,

$$\begin{cases} \mathfrak{p}(U) = X & \text{if } \alpha = \emptyset, \\ \quad \text{and} & \\ \mathfrak{p}(U) = \emptyset & \text{if } U_i = \emptyset, \text{ for some } i \in \alpha. \end{cases} \tag{2}$$

For α, $\beta \in Fin(I)$, $U \in \mathcal{O}(\alpha)$ and $V \in \mathcal{O}(\beta)$, define

$$U \wedge V \in \mathcal{O}(\alpha \cup \beta) \quad \text{and} \quad U \vee V \in \mathcal{O}(\alpha \cap \beta),$$

by the following prescriptions :

$$(U \wedge V)_i \;=\; \begin{cases} U_i & \text{if } i \in \alpha - \beta; \\ V_i & \text{if } i \in \beta - \alpha; \\ V_i \cap U_i & \text{if } i \in \alpha \cap \beta. \end{cases}$$

$$(U \vee V)_i \;=\; U_i \cup V_i,\; i \in \alpha \cap \beta.$$

We have

FACT 1.26. *For* $\alpha, \beta \in Fin(I)$, $U \in \mathcal{O}(\alpha)$ *and* $V \in \mathcal{O}(\beta)$
(1) $\mathfrak{p}(U) \cap \mathfrak{p}(V) \;=\; \mathfrak{p}(U \wedge V)$;
(2) $\mathfrak{p}(U) \cup \mathfrak{p}(V) \;=\; \mathfrak{p}(U \vee V)$.

Proof. (1) For $s \in X$

$$\begin{array}{lll} s \in \mathfrak{p}(U) \cap \mathfrak{p}(V) & \text{iff} & \forall\, i \in \alpha,\, s(i) \in U_i \text{ and } \forall\, j \in \beta,\, s(j) \in V_j \\ & \text{iff} & \forall\, k \in \alpha \cup \beta,\; \begin{cases} s(k) \in U_k & \text{if } k \in \alpha - \beta \\ s(k) \in V_k & \text{if } k \in \beta - \alpha \\ s(k) \in U_k \cap V_k & \text{if } k \in \alpha \cap \beta \end{cases} \\ & \text{iff} & s \in \mathfrak{p}(U \wedge V). \end{array}$$

The proof of (2) is similar and left to the reader.

The product topology on $X = \prod X_i$ is the topology generated (1.8) by the family

$$\mathfrak{p} = \{\mathfrak{p}(U) : U \in \mathcal{O}(\alpha) \text{ and } \alpha \in Fin(I)\}.$$

By Fact 1.26, $\mathfrak{p}$ is closed under finite intersections and unions. Hence, a subset $C \subseteq X$ is open in the product topology iff it can be written as a union of elements of $\mathfrak{p}$. Some of the fundamental properties of the product topology are described in

FACT 1.27. *a) The canonical projections,* $\pi_i : \prod X_i \longrightarrow X_i$*, are continuous.*
b) Let Y *be a topological space and* $f : Y \longrightarrow \prod X_i$ *be a map. The following are equivalent :*

(1) f *is continuous;* (2) *For all* $i \in I$, $f \circ \pi_i$ *is continuous.*

Proof. a) If A is open in X_i, then

$$\pi_i^{-1}(A) = \mathfrak{p}(U),$$

where $\alpha = \{i\}$ and $U = \langle A \rangle$.
b) Since composition preserves continuity, (a) entails that (1) $\Rightarrow$ (2). For the converse, note that if $\alpha \in Fin(I)$ and $U \in \mathcal{O}(\alpha)$, then

$$f^{-1}(\mathfrak{p}(U)) \;=\; \textstyle\bigcap_{i \in \alpha} (f \circ \pi_i)^{-1}(U_i),$$

which is open because α is finite. Since all opens in $\prod X_i$ are unions of elements in $\mathfrak{p}$ and inverse image preserves arbitrary unions, f is continuous. $\square$

We now mention two important structural properties of product spaces: preservation of compactness and the characterization of their density (1.16.(a)) in terms of that of its components. For a proof of these results, the reader may consult [**12**].

THEOREM 1.28. (Tychonoff) *If* X_i, $i \in I$, *is a family of compact spaces, then* $\prod X_i$*, with the product topology, is compact.*

THEOREM 1.29. (Hewitt, Marczewski, Pondiczery) *Let m be an infinite cardinal and let X_i, $i \in I$, be spaces such that $d(X_i) \leq m$. If $card(I) \leq 2^m$, then $d(\prod X_i) \leq m$.* □

Exercises

1.30. If T is a topological space and $U \in \Omega(T)$, then
a) $\neg\neg U = \text{int}\, \overline{U}$.
b) $U \in D(T)$ iff $\neg\neg U = T$.
c) $U \in Reg(T)$ iff $\neg\neg U = U$. □

1.31. a) If $f : X \longrightarrow Y$ is a continuous map, then

$$U \in B(Y) \;\Rightarrow\; f^{-1}(U) \in B(X).$$

b) Give an example to show that the above property is false for regular opens. □

1.32. Let X_i, $i \in I$, be a family of topological spaces and let B_i be a basis [14] for the topology on X_i, $i \in I$. In analogy with 1.25, for $\alpha \in Fin(I)$, define

$$\mathcal{B}(\alpha) \;=\; \prod_{i \in \alpha} B_i$$

and write $U = \langle U_i \rangle_{i \in \alpha}$ for a typical element of $\mathcal{B}(\alpha)$. As in 1.25, set

$$\mathfrak{p}(U) \;=\; \{s \in \prod_{i \in I} X_i : s_i \in U_i, \text{ for all } i \in \alpha\},$$

and let

$$\mathfrak{b} \;=\; \{\mathfrak{p}(U) : U \in \mathcal{B}(\alpha) \text{ and } \alpha \in Fin(I)\}.$$

a) $\mathfrak{b}$ is a basis for the product topology on $\prod_{i \in I} X_i$.
b) If the B_i, $i \in I$, are closed under finite intersections, the same is true of $\mathfrak{b}$ [15].
c) If the B_i, $i \in I$, are closed under finite unions, the same is true of $\mathfrak{b}$.
d) If B_i, $i \in I$, consists of clopen sets, then all elements of $\mathfrak{b}$ are clopen in the product topology on $\prod_{i \in I} X_i$.
e) If B_i, $i \in I$, consists of compact sets and all X_i are compact, then all elements of $\mathfrak{b}$ are compact in the product topology on $\prod_{i \in I} X_i$. □

1.33. Let X, Y be sets, S, T be disjoint subsets of X and $\mathcal{A} \subseteq 2^X$, $\mathcal{K} \subseteq Y^X$.
a) Prove that 2^X separates any disjoint pair of subsets of X.
b) $\mathcal{K}$ separates S and T iff $\{f^{-1}(y) : f \in \mathcal{K} \text{ and } y \in Y\}$ separates S and T.
c) With notation as in 1.3, for $x, y \in X$, define

$$x \; E \; y \quad \text{iff} \quad \text{For all } f \in \mathcal{K}, \quad fx = fy.$$

(i) Show that E is an equivalence relation on X.
(ii) Show that if $f \in \mathcal{K}$, then the rule $\widehat{f}(x/E) = fx$ yields a well-defined map $\widehat{f} : X/E \longrightarrow Y$, such that $\widehat{f} \circ \pi_E = f$.
(iii) Show that $\widehat{\mathcal{K}} = \{\widehat{f} : f \in \mathcal{K}\}$ separates points in X/E. □

[14] That is, every open in X_i is an unions of elements of B_i.
[15] Fact 1.26 may be useful.

CHAPTER 2

Partial Orders

Partial Orders occur very frequently in Mathematics and are at the foundation of all that will henceforth be discussed. We shall cut a rather brisk path through the basic facts we will need. General references for the material presented here are [**3**], [**5**], [**21**] and [**60**].

There is a perhaps inevitable cluster of definitions and nomenclature which has to presented and acquired. We hope the examples will prove helpful in obtaining an understanding of these. Almost all that is described in this chapter will appear repeatedly in future work.

1. Pre-Orders and Partial Orders

DEFINITION 2.1. *Let L be a set. A binary relation, $\leq$, on L is a* **pre-order** *on L iff for all a, b, $c \in L$ we have*
[po1] : $a \leq a$;
[po2] : $a \leq b$ and $b \leq c \;\Rightarrow\; a \leq c$.

A pre-order $\leq$ on L that satisfies, for all $a, b \in L$
[po3] : $a \leq b$ and $b \leq a \;\Rightarrow\; a = b$.
is called a **partial order (po)** *on L. We often say that $\langle L, \leq \rangle$ is a* **poset** [1]. *As usual, $a < b$ stands for $a \leq b$ and $a \neq b$.*

If $\langle L, \leq \rangle$, $\langle R, \preceq \rangle$ are pre-ordered sets, a map $f : L \longrightarrow R$ is a **morphism** [2], *if for all x, $y \in L$*

[I] $$x \leq y \;\Rightarrow\; fx \preceq fy.$$

A morphism $f : L \to R$ is an **embedding** *if for all x, $y \in L$*

[E] $$x \leq y \;\Leftrightarrow\; fx \preceq fy.$$

2.2. **Notation.** For $a \leq b$ in a pre-ordered L,

$$\begin{array}{rclrcl} [a,b] &=& \{x \in L : a \leq x \leq b\}; & (a,b) &=& \{x \in L : a < x < b\}; \\ (a,b] &=& \{x \in L : a < x \leq \mathrm{b}\}; & [a,b) &=& \{x \in L : a \leq x < b\}; \\ a^{\rightarrow} &=& \{x \in L : a \leq x\}; & a^{\leftarrow} &=& \{x \in L : x \leq a\}. \ \square \end{array}$$

The proof of the next result is straightforward.

---

[1]Partially ordered set.
[2]Or increasing.

Lemma 2.3. *Let $\langle L, \leq \rangle$ be a pre-ordered set. Define a binary relation E on L by*

$$x \; E \; y \quad \text{iff} \quad x \leq y \text{ and } y \leq x$$

Then, E is an equivalence relation on L. For $x, y \in L$, set

$$x/E \leq y/E \quad \text{iff} \quad x \leq y.$$

Then, $\langle L/E, \leq \rangle$ is a poset and the natural surjection, π_E, is a morphism of pre-ordered sets. Moreover, we have the following universal property : If $\langle P, \preceq \rangle$ is a poset and $f : L \longrightarrow P$ is a morphism, then there is a **unique** *morphism, $f_E : L/E \longrightarrow P$, such that $f_E \circ \pi_E = f$.*

$$\begin{tikzcd}
L \arrow[r, "\pi_E"] \arrow[d, "f"'] & L/E \arrow[dl, "f_E"] \\
P &
\end{tikzcd}$$

□

The poset in 2.3 is called the **poset associated** to the pre-ordered set $\langle L, \leq \rangle$.

Pre-orders on a set are in duality with a special type of topology on its carrier, as follows :

Proposition 2.4. *There is a natural bijective correspondence between the pre-orders on a set X and subsets of 2^X which are closed under arbitrary unions and intersections.*

Proof. Let $\Vdash$ be a pre-order on X. With notation as in 2.2, define

$$\tau_{\Vdash} = \{U \subseteq X : \forall \, x \; (x \in U \;\; \Rightarrow \;\; x^{\rightarrow} \subseteq U\}. \tag{I}$$

It is readily verified that $\tau_{\Vdash}$ is closed under arbitrary unions and intersections. In fact, $\tau_{\Vdash}$ is a **topology** on X, of a very special kind : each $x \in X$ has a minimal neighborhood, namely $x^{\rightarrow}$.

Conversely, given a family $\mathcal{P} \subseteq 2^X$, which is closed under unions and intersections, define

$$x \Vdash_{\mathcal{P}} y \quad \text{iff} \quad \forall \, U \in \mathcal{P}, \, x \in U \;\; \Rightarrow \;\; y \in U. \tag{II}$$

Again, it is straightforward that $\Vdash_{\mathcal{P}}$ is a reflexive and transitive relation on X. We may now ask for the relation between $\mathcal{P}$ and $\tau = \tau_{\Vdash_{\mathcal{P}}}$.

If $U \in \mathcal{P}$, $x \in U$ and $x \Vdash_{\mathcal{P}} y$, then by (II) above, $y \in U$. Thus, by (I), $U \in \tau$ and so $\mathcal{P} \subseteq \tau$. For the converse, first note that for $x \in X$

$$x^{\rightarrow} = \{y \in X : x \Vdash_{\mathcal{P}} y\} = \bigcap \, \{U \in \mathcal{P} \colon x \in U\},$$ [3]

[3] In the pre-order $\Vdash_{\mathcal{P}}$.

which is an element of $\mathcal{P}$ since it is closed under intersections. Thus, given $V \in \tau$, we may write $V = \bigcup \{x^{\rightarrow} : x \in V\}$, a union of elements in $\mathcal{P}$, showing that $V \in \mathcal{P}$ and so $\tau = \mathcal{P}$. A similar reasoning shows that $\Vdash_{\mathcal{T}_{\Vdash}} = \Vdash$, ending the proof. $\square$

We have used the forcing symbol $\Vdash$ to indicate a pre-order in the proof of 2.4 because of the connection between pre-orders and Kripke models.

It is clear that if $S \subseteq L$ and $\leq$ is a po on L, then S is a poset with the order induced by L; when no confusion can arise, the induced order is still written $\leq$.

Example 2.5. We change notation a little for the sake of clarity. If R is a po on L, we may define a new po on L, R^{op}, given by

$$\langle x, y \rangle \in R \;\;\Leftrightarrow\;\; \langle y, x \rangle \in R^{op}.$$

R^{op} is the **inverse or opposite order** of R. Clearly, $(R^{op})^{op} = R$. Concepts defined for R, correspond, by duality, to concepts defined for R^{op} : lower bound to upper bound, inf to sup, bounded to bounded, etc. We shall have a chance to see other pairs of dual concepts below. It is useful to keep this duality in mind, since a result proven for R will also yield its dual for R^{op}. $\square$

Definition 2.6. *Let $\langle L, \leq \rangle$ be a poset, let S be a subset of L and let a, b be elements of L.*

$*$ *a is the* **maximum (minimum)** *of S, if $a \in S$ and $\forall\, s \in S$, $s \leq a$ (resp., $a \leq s$). Notation : $a = max\ S$ (resp., $a = min\ S$).*

The symbols $\top$ (top) and $\perp$ (bottom) will always denote max L and min L, respectively. Write L^{-} for $L - \{\perp, \top\}$.

$*$ *a is* **an upper (lower) bound** *for S, if $\forall\ s \in S$, $s \leq a$ (resp., $a \leq s$). The (possibly empty) set of upper (resp., lower) bounds for S is denoted by $S^{\rightarrow}$ (resp., $S^{\leftarrow}$) :*

$$\begin{cases} S^{\rightarrow} & = \;\; \{x \in L : \forall\ s \in S,\ (s \leq x)\} \\ S^{\leftarrow} & = \;\; \{x \in L : \forall\ s \in S,\ (x \leq s)\}. \end{cases}$$

When $S = \{b\}$, following the notation in 2.2, write $b^{\rightarrow}$ and $b^{\leftarrow}$ in place of $S^{\rightarrow}$ and $S^{\leftarrow}$, respectively. Notice that

$$S^{\rightarrow} \;\;=\;\; \textstyle\bigcap_{s \in S}\ s^{\rightarrow},$$

and similarly for $S^{\leftarrow}$.

S is **bounded** *if $S^{\rightarrow} \neq \emptyset$ and $S^{\leftarrow} \neq \emptyset$. The reader can surely imagine the definitions of bounded above or below.*

$*$ *a is* **the least upper bound, supremum (sup) or join** *of S if $a = min\ S^{\rightarrow}$;*

$$sup\ S, \;\; \bigvee S \;\;\; or \;\;\; \textstyle\bigvee_{s \in S}\ s,$$

stand for the join of S in L (whenever it exists). Duality yields the concept of **greatest lower bound, infimum (inf) or meet** *of S, written*

$$inf\ S, \;\; \bigwedge S \;\;\; or \;\;\; \textstyle\bigwedge_{s \in S}\ s.$$

It is easily verified that

$$a = min\ S^{\rightarrow} \;\;\; iff \;\;\; a = inf\ S^{\rightarrow} \;\;\; iff \;\;\; a = sup\ S.$$

Similar relations hold for the meet.

$*$ *a is* **maximal (minimal)** *in S, if* $a \in S$ *and*

$$\forall\, s \in S,\, s \geq a \;\;\Rightarrow\;\; s = a \;(resp.,\, s \leq a \;\;\Rightarrow\;\; s = a).$$

$*$ *If L has bottom* $\perp$, *a is an* **atom** *in L iff a is a minimal element distinct from* $\perp$.

$*$ *S is an* **upper (lower) set** *iff* $S = \bigcup_{s \in S} s^{\rightarrow}$ *(resp.,* $\bigcup_{s \in S} s^{\leftarrow}$*).*

If $\langle L, \leq \rangle$ is a poset and $S \subseteq L$, it is clear that

$$\bigvee S \text{ exists in } L \quad \text{iff} \quad \bigvee \bigcup_{s \in S} s^{\leftarrow} \text{ exists in } L,$$

in which case they are equal. A similar comment holds for meets.

Remark 2.7. Every subset $S \subseteq L$ generates an upper and a lower set, given respectively by

$$S\!\uparrow\, = \bigcup_{s \in S} s^{\rightarrow} \quad \text{and} \quad S\!\downarrow\, = \bigcup_{s \in S} s^{\leftarrow}.$$

With this notation we have, for all $S \subseteq L$:
(i) S is an upper (lower) set iff $S = S\!\uparrow$ (resp., $S\!\downarrow$);
(ii) $(S\!\uparrow)\!\uparrow\, = S\!\uparrow$ and $(S\!\downarrow)\!\downarrow\, = S\!\downarrow$.
(iii) $\bigvee S = \bigvee S\!\downarrow$ and $\bigwedge S = \bigwedge S\!\uparrow$,
where the equations mean that one side is defined iff the other is and they are equal. □

Remark 2.8. When dealing with sups and infs we must be careful of the posets in which they are computed (see 2.11, below). A completely unambiguous notation would be very cumbersome. Common sense and care are the proposed alternatives. □

Example 2.9. One of the most basic examples of poset is the power set of a set X, 2^X, with the inclusion relation, $\subseteq$. If one wishes to deal directly with characteristic functions, this order is given by

$$f \leq g \quad \text{iff} \quad \forall\, x \in X,\, fx \leq gx.$$

If $S \in 2^X$ then $\bigcup S$ and $\bigcap S$ are, respectively, sup S and inf S in this po. Further, in this po, $\emptyset = \perp$ and $X = \top$.

Let $S = \{\text{singletons in } X\} = \{\{x\} : x \in X\}$; if X has more than one element, every element in S is maximal and minimal, but S has no top and no bottom.

Inside 2^X there are many interesting subposets. As examples, let λ be a cardinal, $\lambda \leq \mathrm{card}(X)$ (the cardinal of X); define

$$\begin{cases} 2^X_\lambda &=& \{A \in 2^X : \mathrm{card}(A) < \lambda\} \cup \{X\}; \\ B_\lambda(X) &=& \{A \in 2^X : \mathrm{card}(A) < \lambda \text{ or } \mathrm{card}(A^c) < \lambda\}, \end{cases}$$

both with the po induced from 2^X. Note that $2^X_\gamma = 2^X$, when γ is a cardinal not less than the successor of card(X). Moreover, if $\lambda = \omega$ (the cardinal of the natural numbers), then

$$2^X_\omega = Fin(X) \cup \{X\},$$

where $Fin(X)$ is the set of finite subsets of X ([Fin], page 15). Further, $B_\omega(X)$ is the poset of subsets of X which are finite or cofinite (the complement of a finite set). $\square$

EXAMPLE 2.10. The natural orders on $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ are, of course, all pos. Definition 2.6 generalizes familiar concepts in these examples. If A denotes any of these number sets, let $A^* = A \cup \{-\infty, \infty\}$, with the canonical order. [4] $\square$

EXAMPLE 2.11. Let T be a topological space.
a) For x, $y \in T$, define $x \leq y$ iff $x \in \overline{\{y\}}$; $\leq$ is a po iff T is a T0 space, called the **specialization po**.
b) Let $\Omega(T)$ be the set of opens in T. Clearly, $\Omega(T) \subseteq 2^T$ and the inclusion po in $\Omega(T)$ is that induced by 2^X. As above, top and bottom in $\Omega(T)$ are T and $\emptyset$, respectively. For $S \subseteq \Omega(T)$,

$$\bigwedge S = \mathrm{int}(\bigcap S) \quad \text{and} \quad \bigvee S = \bigcup S,$$

are the sup and inf of S in $\Omega(T)$, while int $(*)$ is the *interior* operation, as in section 1.2 and 1.10.

It is easy to find topological spaces and families of opens S where $\mathrm{int}(\bigcap S) \neq \bigcap S$. This exemplifies the caution mentioned in 2.8. $\square$

EXAMPLE 2.12. Let $pF(X, Y)$ be the set of partial maps from X to Y (1.1). Define [5]

$$f \leq g \quad \text{iff} \quad dom\, f \subseteq dom\, g \text{ and } g_{|dom\, f} = f.$$

This is clearly a partial order, called the **extension po**. For a non-empty $S \subseteq pF(X, Y)$ we have :
a) $S^{\rightarrow} \neq \emptyset$ iff for all f, $g \in S$, $f_{|dom f \cap dom g} = g_{|dom f \cap dom g}$.
In that case, $\bigvee S$ exists (by 1.2) as the gluing of the compatible family of maps S. Thus, this poset satisfies a familiar property : every non-empty subset with an upper bound has a least upper bound.
b) $f \in S^{\leftarrow}$ iff all elements of S are extensions of f.
Hence, $S^{\leftarrow}$ is compatible and all non empty subsets of $pF(X, Y)$ have an infimum, namely $\bigvee S^{\leftarrow}$.
c) Any element of Y^X is maximal, although $pF(X, Y)$ has no top, whenever X has more than one element.

In set-theoretical forcing, it is customary to use the opposite of this order : $f \leq g$ iff f is an extension of g. The results are then dual to the ones described above. $\square$

EXAMPLE 2.13. If X, Y are sets and κ is a cardinal, define

$$pF_\kappa(X, Y) = \{f \in pF(X, Y) : \mathrm{card}(dom\, f) < \kappa\}.$$

$pF_\kappa(X, Y)$ inherits the extension po of $pF(X, Y)$, with which it becomes a poset in its own right. In particular, $pF_\omega(X, Y)$ consists of all partial maps from X to Y with finite domain, with the extension po. $\square$

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[4] For all $x \in A$, $-\infty < x < \infty$.
[5] $f_{|*}$ is the *restriction* of f, as in page 15.

EXAMPLE 2.14. Let M be a module over the commutative ring R. Let $\mathrm{Sub}(M)$ be the set of submodules of M, partially ordered by inclusion. If S is a set of submodules of M, there always exist $\bigwedge S$ (the intersection of the submodules in S) and $\bigvee S$ (the submodule generated by $\bigcup S$). Similarly, one can consider the poset of ideals of a ring, the poset of subspaces of a vector space and the poset of closed subspaces of a Hilbert space. $\square$

EXAMPLE 2.15. Let $\langle M, \Sigma, \mu \rangle$ be a measure space, that is M is a set, $\Sigma \subseteq 2^M$ is a σ - algebra and μ is a countably additive map, $\mu : \Sigma \longrightarrow \mathbb{R}_+$ (positive reals). Actually, for our purposes, it would be sufficient that μ be a finitely additive vector measure. We assume that $\mu M < \infty$.

A **partition** of M in Σ is a **finite** set $P \subseteq \Sigma - \{\emptyset\}$, such that for all distinct $p, q \in P$ we have $p \cap q = \emptyset$ and $\bigcup P = \bigcup_{p \in P} p = M$. Let $\mathcal{P}$ be the set of partitions of M in Σ. For $P, Q \in \mathcal{P}$, define

$$P \preceq Q \quad \text{iff} \quad \exists\, b : Q \longrightarrow P \text{ such that } \forall\, q \in Q, \quad q \subseteq bq.$$

The relation $\preceq$ is a po on $\mathcal{P}$, called **refinement**. Every pair (and thus every finite subset) of partitions has a least upper bound, namely,

$$P \vee Q = \{p \cap q : p \in P,\, q \in Q \text{ and } p \cap q \neq \emptyset\}.$$

Some authors allow partitions which are countable subsets of Σ, with refinement as above; others, require only that $\Sigma_{p \in P}\ \mu p = \mu M$. In general, we still have only finite sups. We remark that for finite measure spaces, only finite partitions are needed for integration theory. $\square$

EXAMPLE 2.16. **Products.** If X_i, $i \in I$, are posets, $\prod X_i$ has a natural po given by

$$\langle a_i \rangle \leq \langle b_i \rangle \quad \text{iff} \quad \forall\, i \in I,\, a_i \leq b_i.$$

In particular, if M is a poset, any power M^I is also a poset in a natural way. Important examples occur as sub posets of $\mathbb{R}^I$ or ${\mathbb{R}^*}^I$, as for instance, continuous real functions on a topological space and lower or upper semicontinuous on a topological space.

Recall that a map $f : T \longrightarrow \mathbb{R}^*$ from a topological space to the extended reals is **lower semicontinuous** ($f \in \mathrm{LSC}(T)$) iff

[LSC] For all $r \in \mathbb{R}^*$, the set $\{x \in T : fx > r\}$ is open in T.

For upper semicontinuity ($\mathrm{USC}(T)$) we require that $\{x : fx < r\}$ be open in T.

Under the order induced from the product ${\mathbb{R}^*}^I$, both $\mathrm{LSC}(T)$ and $\mathrm{USC}(T)$ have sups for all subsets. On the other hand, the same will be true for continuous functions on compact spaces iff T is extremely disconnected, a property we will study later on. We have relative versions of the above for real valued functions, that is, every non-empty subset with an upper bound has a sup.

Other classical Banach function spaces are associated to "quotients" of posets of the sort $\mathbb{R}^I$: if f, g are measurable real valued functions we may define

$$f \leq g \ \text{ iff } \ \mu\{x : fx > gx\} = 0.$$

As it stands, this is a pre-order (2.1). It becomes a po if we declare equal all functions which are equal almost everywhere.

The reader will find a wealth of information on the functional analytic properties of these structures in [**16**], [**42**], [**78**], [**63**] and [**64**]; [**17**] also has considerable information on the relation between these structures and continuous lattices. $\square$

EXAMPLE 2.17. Let $\{X_i : i \in I\}$ be a family of posets and assume that I is partially ordered by $\preceq$. The disjoint union $\coprod X_i$ (see 1.5) carries a po, defined as follows : [6]

$$\langle x, i \rangle \leq \langle y, j \rangle \quad \text{iff} \quad i \prec j \quad \textit{or} \quad i = j \text{ and } x \leq y \text{ (in } X_i). \qquad \square$$

EXAMPLE 2.18. Let $\langle L, \leq \rangle$ be a poset and $n \geq 1$ an integer. The product L^n carries, besides the natural (coordinate-wise) partial order presented in 2.16, the following partial order, called **lexicographic po**, defined as follows :

For $\overline{x} = \langle x_1, \ldots, x_n \rangle$ and $\overline{y} = \langle y_1, \ldots, y_n \rangle$ in L^n, set [7]

$$\begin{cases} \alpha(\overline{x}, \overline{y}) = \{k \in \underline{n} : x_k \neq y_k\}; \\ \textit{and} \\ \gamma(\overline{x}, \overline{y}) = \begin{cases} \min \alpha(\overline{x}, \overline{y}) & \text{if } \alpha(\overline{x}, \overline{y}) \neq \emptyset \\ 0 & \text{if } \gamma(\overline{x}, \overline{y}) = \emptyset. \end{cases} \end{cases}$$

Note that $\alpha(\overline{x}, \overline{y}) = 0$ iff $\overline{x} = \overline{y}$. Now define, with $\gamma = \gamma(\overline{x}, \overline{y})$,

$$\overline{x} \leq \overline{y} \quad \text{iff} \quad \gamma(\overline{x}, \overline{y}) = 0 \quad \text{or} \quad x_\gamma \leq y_\gamma.$$

It is straightforward to check that this defines a partial order on L^n. The same construction maybe obtained substituting $n \geq 1$ for an arbitrary *well-ordered* (see 2.19) set of indices. $\square$

2. Chains and Well-Founded Posets

Many interesting mathematical structures arise by considering partial orders in which certain subsets have maximum or minimum. We shall discuss two examples of this sort : *chains and well-orderings.* While at it, we also present the notion of *well-foundedness.*

DEFINITION 2.19. *Let $\langle L, \leq \rangle$ be a non-empty poset.*

a) L is a **chain** *if for all a, $b \in S$, $a \leq b$ or $b \leq a$. Chains are also called total or linear orders.*

A subset $S \subseteq L$ is a **chain in L** *if S, with the induced order, is a chain.*

b) L is **well-founded** *if for all non-empty subsets S of L, there is $x \in S$ such that $x^{\leftarrow} \cap S = \{x\}$.*

c) L is **well-ordered** *if all non-empty subsets of L have a minimum.*

d) L is a **tree** *if for all $x \in L$, $x^{\leftarrow}$ is well-ordered* [8].

REMARK 2.20. A famous statement involving chains in posets is

Zorn's Lemma : *If $\langle V, \leq \rangle$ is a non-empty poset in which every chain has an upper bound, then V has a maximal element.*

[6] As usual, $\prec$ is the *strict* order derived from $\preceq$.

[7] $\underline{n} = \{1, 2, \ldots, n\}$ is defined in page 15.

[8] With the ordering induced by L.

It is easily seen that the above statement is equivalent to : If $\langle V, \leq \rangle$ is a poset in which every chain has a lower bound, then V has a minimal element. As is well-known, Zorn's Lemma is equivalent to the Axiom of Choice ([**51**]). Another statement of this sort is

Well-Ordering Axiom (WOA) : *Every non-empty set can be well-ordered.*

WOA means that if X is a non-empty set, then there is a partial order on X which is a well-ordering. As is the case with Zorn's Lemma, WOA is another example of an equivalent to the Axiom of Choice ([**51**]). In [**62**] the reader will find a plethora of statements that are equivalent to the Axiom of Choice. $\Box$

LEMMA 2.21. *a) Any well-ordered poset is a chain.*
b) All well-orderings are well-founded and have $\perp$ [9].
c) A chain is well-ordered iff it is well-founded.

PROOF. a) Since $\{a, b\}$ has a minimum, either $a \leq b$ or $b \leq a$.
b) Let L be a well-ordered poset. It follows immediately from the definition that L has a least element. If $S \neq \emptyset$ in L and $x = \min S$, it is clear that $x^{\leftarrow} \cap S = \{x\}$.
c) It is enough to verify that a well-founded chain L is well-ordered. If $S \neq \emptyset$ in L, select $x \in S$ such that $x^{\leftarrow} \cap S = \{x\}$. Hence, for $y \in S$, we cannot have $y < x$. Since L is a chain, we conclude that $x \leq y$, and so $x = \min S$. $\Box$

A well-known example of a well-ordered set is $\mathbb{N}$, with its natural order. There are many well-founded posets that are not chains (and thus, by 2.21.(c), not well-ordered). Here is a family of examples :

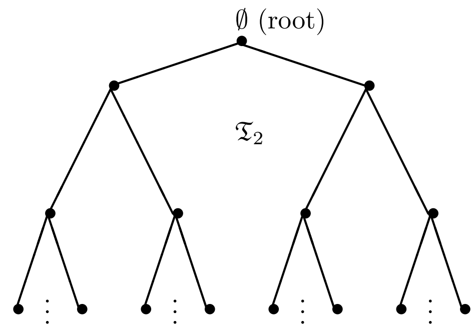
EXAMPLE 2.22. Let Y be a set. With notation as in 1.1, let
$$\mathfrak{T}_Y = \{f \in pF(\mathbb{N}, Y) : \exists\, n \in \mathbb{N} \text{ such that } dom\, f = n\}.$$
We consider $\mathfrak{T}_Y$ partially ordered by the extension po induced from $pF(\mathbb{N}, Y)$. Since Y will remain fixed, write $\mathfrak{T}$ for $\mathfrak{T}_Y$. For $f \in \mathfrak{T}$, define a map $l : \mathfrak{T} \longrightarrow \mathbb{N}$, given by $l(f) = dom\, f$, called the **length** of f.
Fact 1. *For all* $f, g \in \mathfrak{T}$, $f < g \;\Rightarrow\; l(f) < l(g)$.
Proof. If $f < g$ in the extension po of 2.12, then $n = dom\, f$ is *properly* contained in $m = dom\, g$. Hence, in $\mathbb{N}$, $l(f) = n < m = dom\, g$.
Fact 2. $\mathfrak{T}$ *is a tree* (the Y-branching tree over $\mathbb{N}$).
Proof. Fix $f \in \mathfrak{T}$; note that for $g \in \mathfrak{T}$
$$g \leq f \quad \text{iff} \quad g = f_{|k},$$
where $k \leq n$ [10]. Hence, $f^{\leftarrow} = \{f_{|k} : k \leq n\}$, which is well-ordered in the extension po, completing the proof of Fact 2.

For a non-empty $S \subseteq \mathfrak{T}$, consider
$$\alpha(S) = \{l(f) : f \in \mathfrak{T}\}.$$

[9] As in 2.6, $\perp$ is the least element of a poset.
[10] But not in $pF(\mathbb{N}, Y)$; n has many subsets which are not of the form $k \leq n$.

Since $\mathbb{N}$ is well-ordered and $\alpha(S)$ is a non-empty subset of $\mathbb{N}$, there is $s \in S$ such that $l(s) = \min\ \alpha(S)$. It follows immediately from Fact 1 that $s^{\leftarrow} \cap S = \{s\}$, verifying that $\mathfrak{T}$ is well-founded.

When $Y = 2 = \{0, 1\}$, $\mathfrak{T}_2$ is the **binary tree**, whose schematic diagram follows.



The above can be generalized to Y-branching trees over well-ordered sets. $\square$

Remark 2.23. If L is a poset with $\bot$, then the condition that it be well-founded can be written, in the interval notation of 2.2, as

For all $\emptyset \neq S \subseteq L$, there is $x \in S$ such that $[\bot, x) \cap S = \emptyset$.

Since we did not want to use the hypothesis $\bot \in L$ in the definition of well-founded (2.19.(b)), the equivalent formulation therein was employed. In truth, this question is irrelevant : any poset L can be isomorphically embedded in a poset with $\bot$, by adding a new point and declaring it to be less than all the elements of L. Hence, whenever convenient, it shall be assumed that posets have $\bot$ [11]. $\square$

One important method at our disposal in well-founded posets is the possibility of *proof by induction*.

Theorem 2.24. *For a subset S of a well-founded poset L with $\bot$, the following are equivalent :*

(1) *$S = L$;*

(2) *For all $x \in L$, $\quad [\bot, x) \subseteq S \quad \Rightarrow \quad x \in S$.*

Proof. Clearly, (1) $\Rightarrow$ (2). For the converse, let $A = (L - S)$; if $A \neq \emptyset$, there is $a \in A$, such that $a^{\leftarrow} \cap A = \{a\}$. Thus, $[\bot, a) \subseteq S$, with a in the complement of S, violating (2). $\square$

3. Directed Sets. Filters and Ideals

Our next theme is the study of *directed* subsets of partially ordered sets, leading to a general definition of *filter* and *ideal*.

[11] The same comment applies, of course, to $\top$.

DEFINITION 2.25. *Let S be a subset of a poset L.*
a) S is **up directed** *if*
[ud] *For all a, $b \in S$, there is $c \in S$ such that $a \leq c$ and $b \leq c$.*
The concept of **down directed** *is dual, that is,*
[dd] *For all a, $b \in S$, there is $c \in S$ such that $c \leq a$ and $c \leq b$.*
We abbreviate up directed by ud and down directed by dd. Some authors use right (left) directed for ud (resp., dd) sets.
b) S is **up cofinal (ucof)** *in L if*
[$ucof$] *For all $a \in L$, there is $c \in S$ such that $a \leq c$.*
c) S is **down cofinal (dcof)** *if*
[$dcof$] *For all $a \in L$, there is $c \in S$ such that $c \leq a$.*
d) S is **cofinal** *if it is up and down cofinal.*

The notion of directed set is instrumental to construct the concepts of *filter* and *ideal* in a poset.

DEFINITION 2.26. *Let S be a subset of a poset L.*
a) S is a **filter** *in L if $S \neq \emptyset$ and the following hold :*
[$Fi1$] *: S is dd in L;*
[$Fi2$] *: $\forall\ a, b \in L$, ($a \in S$ and $a \leq b \Rightarrow b \in S$).*
For $a \in L$, $a^{\rightarrow}$ is the **principal filter** *generated by a in L.*
b) S is an **ideal** *in L if $S \neq \emptyset$ and the following hold :*
[$Id1$] *: S is ud in L;*
[$Id2$] *: $\forall\ a, b \in L$ ($a \in S$ and $b \leq a \Rightarrow b \in S$).*
If $a \in L$, $a^{\leftarrow}$ is the **principal ideal** *generated by a in L.*
A filter or ideal is said to be **proper** *if it is distinct from L.*

Clearly, Definitions 2.25 and 2.26 presents a set of pairs of dual concepts. Notice that a filter is down directed upper set, while an ideal is a ud lower set. In fact, some authors call a dd set a **filter base** and a ud set an **ideal base**. The reason is the following result, whose proof is left to the reader.

LEMMA 2.27. *With notation as in* 2.7*, the following conditions are equivalent, for a subset S of a poset L :*
(1) *S is dd (ud);*
(2) *$S\!\uparrow$ is dd (resp., $S\!\downarrow$ is ud);*
(3) *$S\!\uparrow$ is a filter (resp., $S\!\downarrow$ is an ideal).*

EXAMPLE 2.28. Let X be a set and $(2^X, \subseteq)$ be the power set of X, partially ordered by inclusion. For $S \subseteq 2^X$,
a) S is ucof (dcof) iff $X \in S$ (resp., $\emptyset \in S$). This will always happen in a po with $\top$ (resp., $\bot$).
b) The notions of filter and ideal in this po coincide with the usual ones in set theory [12] : S is a filter (ideal) in 2^X iff

[12] We shall discuss filters and ideals in lattices in Chapter 3.

(*) $a, b \in S \;\Rightarrow\; a \cap b \in S$ (resp., $a \cup b \in S$);
(**) $a \in S$ and $a \subseteq b$ (resp., $b \subseteq a$) $\;\Rightarrow\; b \in S$. $\Box$

EXAMPLE 2.29. If X is a set, $S \subseteq 2^X$ and $x \in X$, define

$$S_x = \{U \in S : x \in U\}.$$

Let $L = 2^X - \{\emptyset\}$, partially ordered by $\subseteq$. For $S \subseteq L$ we have :

a) S is dcof or left cofinal in L iff $\forall\, x \in X$, $\{x\} \in S_x$.

b) S is a filter in L iff S is a filter in 2^X, such that $\emptyset \notin S$ (i.e., S is a proper filter).

c) S is an ideal in L iff $S \cup \{\emptyset\}$ is an ideal in 2^X. $\Box$

EXAMPLE 2.30. Let T be a topological space and $L = \Omega(T) - \{\emptyset\}$, the non empty open sets of T, partially ordered by $\subseteq$. With notation as in Example 2.29, and for $S \subseteq L$, we have :

a) S is dcof or left cofinal in L iff $\forall\, t \in T$, S_t is a neighborhood basis of t in T. In particular, $\bigcup S$ is a dense open in T. Conversely, if V is a dense open in T, then $S = \Omega(V) - \{\emptyset\}$ is dcof in L.

b) S is a filter (ideal) in $\Omega(T)$ iff it satisfies conditions (*) and (**) of 2.28 above, relative to $\Omega(T)$. The concepts of filter/ideal in L correspond to that of **proper** filter/ideal in $\Omega(T)$. $\Box$

EXAMPLE 2.31. Let $L = pF(X, Y)$, partially ordered by the extension po, as in 2.12.

a) Left cofinal subsets of L are not very interesting. On the other hand, ucof or right cofinal subsets furnish 'samples' of the possible values of elements in L. The set

$$D(x) = \{f \in L : x \in dom\, f\}$$

is clearly up cofinal in L. As another example, fix $g \in L$ and consider

$$D(g) = \{f \in L : f \geq g \quad\text{or}\quad \{x \in X : fx \neq gx\} \neq \emptyset\}.$$

Simple calculations will show that $D(g)$ is up cofinal in L.

b) A subset $S \subseteq L$ is a proper filter iff

i) $\emptyset \in S$;

ii) S contains all extensions of its members;

iii) $\forall\, f, g \in S$, $\{x \in X : fx = gx\} \neq \emptyset$.

c) S is an ideal in L iff S consists of a family of pairwise compatible maps, together with the 'gluing' of every finite subset of S and all restrictions of elements of S to subsets of their domains. Thus, S is an ideal iff :

$*$ $\forall\, f, g \in S$, $f_{|dom f \cap dom g} = f_{|dom f \cap\, dom g}$ and $f \vee g \in S$;

$*$ $\forall\, f \in S$, $\forall\, A \subseteq dom\, f$, $f_{|A} \in S$.

For a non-empty $S \subseteq L$, the following are equivalent :

(a) $\bigvee S$ exists in L; (b) S is bounded in L;

(c) S is contained in some ud set T; (d) S is contained in some ideal I.

If the above equivalent conditions hold, then the ud subset T and the ideal I can be chosen so that $\bigvee S = \bigvee T = \bigvee I$. Hence, in $pF(X, Y)$, **every non-empty ud subset has a supremum**.

By Lemma 1.2, an ideal S in L furnishes a function $\bigvee S \in L$; the domain of $\bigvee S$ might be very small ($\{\emptyset\}$ is an ideal). One way to guarantee that $dom \bigvee S = X$ is to require that for all ucof $H \subseteq L$, we have $S \cap H \neq \emptyset$. Such ideals are usually called **generic**.

In expositions of forcing in set theory, it is the reverse order of the one above that is used (see, for instance, [**40**]). We are then led to consider filters, dcof (or dense) subsets and generic filters. $\square$

Example 2.32. Let $\langle L, \leq \rangle$ be a poset and $\mathcal{I}(L)$ ($\mathcal{F}(L)$) be the set of ideals (resp., filters) in L, partially ordered by inclusion. Up directed subsets of $\mathcal{I}(L)$ or $\mathcal{F}(L)$ have a least upper bound, their set theoretic union, which, for ud subsets, preserves the property of being an ideal or a filter. The intersection of ideals or filters will not, in general, be an ideal or filter. To see this, let

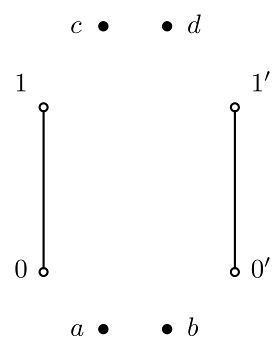
$$L = \{a,\, b\} \coprod (0,\, 1) \coprod (0,\, 1) \coprod \{c,\, d\},$$

with the po determined by the following clauses :

i) The order in each copy of (0, 1) is its natural order;

ii) a and b are smaller, while c and d are larger than all elements in $(0,\, 1) \coprod (0,\, 1)$;

iii) Elements in the distinct copies of (0, 1), as well as a, b, c and d, are unrelated.



Since $\{c,\, d\} \subseteq a^{\rightarrow} \cap b^{\rightarrow}$, this intersection is not dd, because both copies of (0, 1) have no last element. Similarly, $c^{\leftarrow} \cap d^{\leftarrow}$ is not ud.

Observe that for all $I \in \mathcal{I}(L)$ and all $a \in L$, $a \in I \;\Leftrightarrow\; a^{\leftarrow} \subseteq I$. Hence,

$$I = \bigvee \{a^{\leftarrow} : a \in I\}.$$

For all $a, b \in I$, there is $c \in I$ such that $a \leq c$ and $b \leq c$. Therefore, for all $I \in \mathcal{I}(L)$, $\{a^{\leftarrow} : a \in I\}$ is ud ($a^{\leftarrow} \subseteq c^{\leftarrow}$ and $b^{\leftarrow} \subseteq c^{\leftarrow}$). Moreover, for all $a \in L$ and ud subsets $S \subseteq \mathcal{I}(L)$,

$$a^{\leftarrow} \leq \bigvee S \quad \text{iff} \quad a \in \bigcup_{I \in S} I.$$

Analogous properties hold for $\mathcal{F}(L)$, with principal filters in place of principal ideals. Properties similar to the ones described here hold in the posets of Example 2.14, with the appropriate modifications. We shall return to this theme after Definition 2.43. $\square$

4. The Countable Chain Condition

Definition 2.33. *Let* $\langle L, \leq \rangle$ *be a poset and* $a, b \in L$.
$*$ *a and b are* **up-compatible (compatible)** *if* $a^{\rightarrow} \cap b^{\rightarrow} \cap L^{-} \neq \emptyset$ (*resp.,* $a^{\leftarrow} \cap b^{\leftarrow} \cap L^{-} \neq \emptyset$) [13].
$*$ *a and b are* **up-incompatible (resp., incompatible)**, *written* $a \perp^{*} b$ (*resp.,* $a \perp b$) *if they are not up-compatible* (*resp., compatible*).

L is **up-ccc (ccc)** (*countable chain condition*) *iff every set of pairwise up-incompatible* (*resp., incompatible*) *elements is at most countable. Hence,* $\leq$ *is up ccc iff* $\leq^{op}$ *is ccc and vice-versa.*

Remark 2.34. Terminology here is not quite standard. There is always the question of duality and of what is the “natural” way in which “increasing” or “decreasing” are understood. The examples below will help, hopefully, to make matters clearer. ☐

Example 2.35. If T is a topological space and $L = \Omega(T)$ with the inclusion po, then :
i) $U \perp^{*} V$ iff $U \cup V = T$.
ii) $U \perp V$ iff $U \cap V = \emptyset$.
Hence, our notion of ccc corresponds to what is known in the literature as a ccc topological space : any set of pairwise disjoint opens is at most countable. ☐

Example 2.36. If $L = pF(X, Y)$, with the extension partial order of 2.12, then :
a) $f \perp^{*} g$ iff they do not have a common extension
iff $\{x \in X : fx \neq gx\} \neq \emptyset$;
b) $f \perp g$ iff $\{x \in X : fx = gx\} = \emptyset$.
In the literature in Set Theory, sub posets of $pF(X, Y)$ are said to be ccc if they are up-ccc in our terminology. ☐

Remark 2.37. If α is a cardinal, it should be clear how to define the concept of α-ccc (the α chain condition). As above we would have a pair of dual up/down (or left/right) conditions. ☐

Theorem 2.39 below, of independent interest, will guarantee that certain posets are ccc. From here on, we assume that the well-ordering axiom (WOA) in 2.23 is part of the axioms of Set Theory. We first recall the concept of *regular cardinal.*

Definition 2.38. *Let K be a set.* $card(K)$ *is* **regular** *if it is infinite and for all* $\{A_i : i \in I\} \subseteq 2^{K}$,

$$\begin{array}{c} card(I) < card(K) \\ \text{and} \\ card(A_i) < card(K),\ \forall\, i \in I \end{array} \quad \Rightarrow \quad card\Big(\bigcup_{i \in I} A_i\Big) < card(K).$$

[13] Recall from 2.1 that $L^{-} = L - \{\perp, \top\}$.

THEOREM 2.39. (Erdös-Rado) *Let A be a set, $n \geq 1$ a positive integer and K an infinite collection of subsets of A of cardinality n. Assume that card(K) is regular. Then there is a $B \subseteq_f A$ and $K' \subseteq K$ such that*

(1) *card(K) = card(K');* (2) *For all α, $\beta \in K'$, $\alpha \cap \beta = B$.*

PROOF. By induction on $n \geq 1$. If $n = 1$, then either

$*$ There is $a \in K$ that is common to card(K) elements of K; in this case, set $B = \{a\}$;

$*$ No such element exists and set $B = \emptyset$.

Suppose the result true for $(n-1) \geq 1$ and that the elements of K have cardinal n. Assume A is well ordered and for $\alpha \in K$, let a_α be the least element of α in this order. Consider the set

$$C = \{\alpha - \{a_\alpha\} : \alpha \in K\}.$$

There are two possibilities :

1. card(C) < card(K) : For $u \in C$, define

$$K_u = \{\alpha \in K : \alpha - \{a_\alpha\} = u\}.$$

Then $K = \bigcup \{K_u : u \in C\}$; since card$(K)$ is regular (2.38) and card(C) < card(K), there is $B \in C$ such that card(K_B) = card(K). Now observe that for all α, $\beta \in K_B$, $\alpha \cap \beta = B$.

2. card(C) = card(K) : Since the elements of C have cardinal $(n-1)$, by induction there is $D \subseteq_f A$ and $C' \subseteq C$, with card(C') = card(C) = card(K) and $U \cap V = D$, $\forall\, U$, $V \in C'$. Consider the cardinality of

$$T = \{a_\alpha : \alpha - \{a_\alpha\} \in C'\}.$$

Again there are two possibilities :

card(T) < card(K) : For $a \in T$, let $K_a = \{\beta : (\beta - \{a\}) \in C'\}$; clearly, $\bigcup \{K_a : a \in T\}$ has cardinality equal to card(K) (it is larger than or equal to card(C')). Since card(T) is strictly less the regular card(K), there is $a \in T$ such that card(K_a) = card(K). It is straightforward to check that for all β, γ in K_a, $\beta \cap \gamma = D \cup \{a\}$.

card(T) = card(K) : For $a \in T$, select $\alpha \in K$ with $(\alpha - \{a\}) \in C'$. It is clear that this selection produces a subset $K' \subseteq K$ such that for all α, $\beta \in K'$, we have $\alpha \cap \beta = D$, ending the proof □

COROLLARY 2.40. *Let A be a set and K an uncountable collection of finite subsets of A. Then there is $B \subseteq_f A$ and an uncountable $K' \subseteq K$ such that $\forall\, \alpha$, $\beta \in K'$, $\alpha \cap \beta = B$. If card(K) is regular, then there is $K' \subseteq K$ with the above property, with card(K') = card(K).*

PROOF. For each $n \geq 1$, let $K_n = \{\alpha \in K : \text{card}(\alpha) = n\}$; then $K = \bigcup K_n$ and so, being uncountable, at least one of the K_n must be uncountable. If card(K) is regular, then card(K_n) = card(K), for some $n \geq 1$. If card(K) is not regular, one can always choose a regular uncountable cardinal below card(K) [14]. The full statement of the Corollary now follows from 2.39. □

[14]The first uncountable cardinal, ω_1, is regular.

REMARK 2.41. Although Theorem 2.39 is true if card(K) is countable (a regular cardinal), this is not the case of Corollary 2.40. It is not very hard to find examples of this. $\square$

As an application of these results, we show that the posets $pF_\omega(X,Y)$ (2.13) are up-ccc, if Y is finite.

COROLLARY 2.42. *If X, Y are sets, then $pF_\omega(X,Y)$ is up-ccc in the extension po, whenever Y is finite.*

PROOF. Write $Z = pF_\omega(X,Y)$ and suppose T is an uncountable set of up-incompatible elements in Z. Since for finite D, Y^D is finite, $K = \{dom\, t : t \in T\}$ is uncountable. By Corollary 2.40, there is $B \subseteq_f X$ and an uncountable $K' \subseteq K$ such that $u \cap v = B$, for all $u, v \in K'$. Let $T' = \{t \in T : dom\, t \in K'\}$; if $s, t \in T'$, then 2.36.(a) implies that $s_{|B} \neq t_{|B}$, because $s \perp^* t$. But this is impossible, since T' is uncountable and Y^B is finite. $\square$

We shall apply 2.42 in 18.6 to show that all dyadic spaces are ccc. More information on combinatorics and chain conditions can be found in [**40**] and [**9**].

5. Continuous and Algebraic Posets. Compactness

Analogy with topology produces the following

DEFINITION 2.43. *Let $\langle L, \leq \rangle$ be a poset, $a, b \in L$ and $S \subseteq L$.*
a) a is **compact** *(also algebraic or finite) in L iff for all* **ud** *$S \subseteq L$,*

$$S^{\rightarrow} \subseteq a^{\rightarrow} \quad \Rightarrow \quad \textit{There is } s \in S \textit{ such that } a \leq s.$$

For $b \in L$, set $cp(b) = \{a \in L : a$ is compact and $a \leq b\}$.
b) a is **way below** *b, in symbols $a \ll b$, iff for all* **ud** *$S \subseteq L$,*

$$S^{\rightarrow} \subseteq b^{\rightarrow} \quad \Rightarrow \quad \textit{There is } s \in S \textit{ such that } a \leq s.$$

For b in L, set $b^{\twoheadleftarrow} = \{a \in L : a \ll b\}$.

Note that a is compact iff $a \ll a$.

c) $\leq$ is a **continuous po** *on L (or L is a continuous poset) iff*

[*cpo*1] *: All non-empty, up directed subsets of L, have a sup in L;*

[*cpo*2] *: For all a in L, $a^{\twoheadleftarrow}$ is up directed and $a = \bigvee a^{\twoheadleftarrow}$.*

d) L is an **algebraic poset** *iff it is a continuous poset such that for all $a \in L$, $cp(a)$ is up directed and $a = \bigvee cp(a)$.*
e) A poset with $\top$ is **compact** *if it is algebraic and $\top$ is compact.*

One can find much information on this topic in [**17**], where its connections with lattice theory, computer science, general topology, analysis, C^*-algebras and Topos theory are presented and explored. The material in [**34**] is also interesting, with a different point of view. Here we will be content with discussing the above properties relative to the examples already given and presenting another two, important for future work.

REMARK 2.44. Our definition of compact and "way below" is not that in [**17**] since we wanted to state it without the assumption of "completeness". The following observations are, therefore, in order :
a) Since the sets S in the definition of the way below relation are ud, we have [15]

$$a \ll b \;\Leftrightarrow\; \forall \text{ ud } S \subseteq L,\;\; S^{\rightarrow} \subseteq a^{\rightarrow} \;\;\Rightarrow\;\; \exists\, F \subseteq_f S, \text{ with } F^{\rightarrow} \subseteq a^{\rightarrow}.$$

Similarly for the notion of compactness. This is closer to the usual topological definition of compactness.
b) In the presence of $[cpo1]$ we may rewrite the definition of $a \ll b$ as :

$$\forall \text{ ud } S \subseteq L,\;\; b \leq \bigvee S \quad\Rightarrow\quad \exists\, s \in S\ (a \leq s).$$

Similarly for the concept of compactness.
c) If L has sups for all finite subsets (or equivalently, for each pair of its elements), then

$$a \ll b \quad \text{iff} \quad \forall\, \boldsymbol{S \subseteq L},\;\; S^{\rightarrow} \subseteq a^{\rightarrow} \;\;\Rightarrow\;\; \exists\, F \subseteq_f S \text{ with } a \leq \bigvee F.$$

This holds because, in this case, given $S \subseteq L$, we can form

$$S' = \{\sup F : F \subseteq S \text{ and } F \text{ is finite}\},$$

which is ud and satisfies $S^{\rightarrow} = (S')^{\rightarrow}$. Similarly, of course, for compactness.
d) If L has sups for all its subsets then

$$a \ll b \quad \text{iff} \quad \forall\, S \subseteq L,\;\; b \leq \bigvee S \;\;\Rightarrow\;\; \exists\, F \subseteq_f S, \text{ with } a \leq \bigvee F. \qquad \square$$

PROPOSITION 2.45. *In a poset* $\langle L, \leq \rangle$ *the relation* $\ll$ *has the following properties : for* $x, y, u, z \in L$
a) $x \ll y \;\;\Rightarrow\;\; x \leq y$.
b) $u \leq x \ll y \leq z \;\;\Rightarrow\;\; u \ll z$. *If* x *is compact and* $z \leq x \leq y$, *then* $z \ll x \ll y$.
c) If L *has finite sups then :*
  (i) $x \ll z$, $y \ll z \;\;\Rightarrow\;\; sup\{x, y\} \ll z$. *In particular,* $z^{\twoheadleftarrow}$ *is ud.*
  (ii) *If* x *and* y *are compact, so is* $x \vee y$. *In particular,* $cp(z)$ *is ud.*

PROOF. For item (a), just consider the ud set $\{a\}$. The remaining assertions follow straightforwardly from the definitions. $\square$

We now relate the notion of algebraic compactness with its topological version, presented in Definition 1.23.

PROPOSITION 2.46. *Let* T *be a topological space and* V, $U \in \Omega(T)$.
a) In $\langle \Omega(T), \subseteq \rangle$, *consider the following conditions :*
  (1) $V \ll U$; (2) V *is relatively compact* (1.23) *in* U.
Then, (2) $\Rightarrow$ (1). *If* T *is regular* (1.20), *these conditions are equivalent.*
b) U *is compact in the poset* $\langle \Omega(T), \subseteq \rangle$ *iff it is (topologically) compact.*
c) Consider the following conditions :
  (1) $\langle \Omega(T), \subseteq \rangle$ *is a continuous poset;*
  (2) *Every open set in* T *is the union of relatively compact opens in* U.
  (3) T *is* **locally compact**, *i.e., for* $U \in \Omega(T)$ *and* $x \in U$,

[15] $\subseteq_f$ means "finite subset of", defined in page 15.

[lc] *There is* $V \in \nu_x$ [16] *such that* $\overline{V} \subseteq U$ *and* $\overline{V}$ *is compact.*

Then (3) $\Rightarrow$ (2) $\Rightarrow$ (1). *If* T *is regular, these conditions are equivalent.*

d) The following are equivalent :

(1) $\langle\, \Omega(T), \subseteq \,\rangle$ *is an algebraic poset;*

(2) T *has a basis of compact opens, that is, every open in* T *is the union of compact opens.*

Proof. Recall that $\Omega(T)$ has arbitrary sups and so we may use the displayed statement in 2.44.(d) as the definition of $\ll$ in $\Omega(T)$. We discuss (a), leaving the other items to the reader.

a) For $V \subseteq U$, assume that $\overline{V} \cap U$ is compact in U (or, equivalently, in T; see 1.24.(b)). If $S \subseteq \Omega(U)$ is a covering of U, then there is $C \subseteq_f S$ such that $V \subseteq \overline{V} \cap U \subseteq \bigcup C$, and $V \ll U$, establishing that (2) $\Rightarrow$ (1).

To show that (1) $\Rightarrow$ (2), suppose that T is regular and let $S \subseteq \Omega(U)$ be an open covering of $F = \overline{V} \cap U$. The covering S is refined in two ways :

$*$ By Lemma 1.21.(b), each $W \in S$ may be written as a union of opens whose closure is contained in W; let S' be the covering of F obtained through the refinement of S by this method;

$*$ Let $S'' = S' \cup \{\overline{V}^{\,c} \cap U\}$.

Since $\overline{V}^{\,c} \cap U$ is the complement of F in U – it is clearly open –, S'' is an open covering of U. It follows from (1) that there is $B \subseteq_f S''$, such that $V \subseteq \bigcup B$. Since $\overline{V}^{\,c} \cap U$ has empty intersection with V, we may as well assume that it is not in B. Since closure commutes with finite unions (1.10.(d)), we get

$$F \subseteq \textstyle\bigcup_{W \in B} \overline{W}. \tag{*}$$

Now, for each $W \in B$, select $A \in S$ such that $\overline{W} \subseteq A$. Because B is finite, one obtains a finite subset of S that, by (*), is a covering of F, ending the proof. $\square$

Example 2.47. In 2^X, a set is compact iff it is finite. Again, we have a continuous poset, which is algebraic.

In $pF(X, Y)$, an element is compact iff its domain is finite; it is easily verified that $pF(X, Y)$ is an algebraic poset. $\square$

Example 2.48. An important example of a continuous poset which is **not** algebraic is the real unit interval [0, 1], with its usual order. We have $a \ll b$ iff $a < b$ or $a = 0$. On the other hand, the only compact element of [0, 1] is 0. Another example of this sort (but harder to verify), is furnished by LSC(T), the lower semicontinuous functions on a compact space T. $\square$

Example 2.49. Let A be an algebra in the sense of universal algebra, i. e., A is a set, together with a set of n-ary operations on A, for each $n \geq 0$.

If A is an algebra, a **congruence** θ on A is an equivalence relation such that for all $n \geq 1$, all n-ary operations α on A and all $\langle\, a_1, \ldots, a_n \,\rangle$, $\langle\, b_1, \ldots, b_n \,\rangle$ in A^n,

$$a_i \;\theta\; b_i,\ 1 \leq i \leq n, \quad \Rightarrow \quad \alpha(a_1, \ldots, a_n) \;\theta\; \alpha(b_1, \ldots, b_n).$$

[16] The filter of opens neighborhoods of x in T; see section 1.2.

Let Con(A) be the set of congruences on A. With the containment po inherited from A^2, Con(A) is a poset. Clearly, Con(A) is closed under arbitrary intersections. Thus, every subset $S \subseteq A \times A$ generates a congruence on A, given by

$$\theta_S = \bigcap \{\theta \in \mathrm{Con}(A) : S \subseteq \theta\},$$

the smallest congruence relation containing S. Note that the equality relation (the diagonal of $A \times A$) is the $\perp$ of Con(A), while $A \times A$ is its $\top$. Hence, in Con(A) all subsets have sup and inf.

For each $\langle x, y \rangle$ in $A \times A$, let θ_{xy} be the congruence generated by $\{\langle x, y \rangle\}$. Clearly, $\theta_{xy} = \theta_{yx}$. Note that θ_{xy} is compact in Con(A) and for all θ in Con(A)

$$\theta = \bigvee \{\theta_{xy} : \langle x, y \rangle \in \theta\}.$$

Thus, Con(A) is an algebraic lattice. This example has many interesting instances: groups, rings, etc. □

Exercises

2.50. a) If L is a poset, show that $\mathcal{I}(L)$ and $\mathcal{F}(L)$ (2.32) are continuous algebraic posets.
b) If M is a module over a commutative ring, show that the lattice of submodules of M (2.14) is an algebraic poset. □

2.51. Let I be a set and $[0,1]$ be the real unit interval. In the power $[0,1]^I$, with the coordinate-wise order (2.16), show that [17]

$$(x_i) \ll (y_i) \quad \text{iff} \quad \begin{cases} \forall\, i \in I,\, x_i \ll y_i \\ \text{and} \\ \exists\, F \subseteq_f I \text{ with } x_i = 0 \text{ if } i \in (I - F). \end{cases}$$ □

2.52. One of the basic examples of poset is the family of subsets of a set, 2^X, partially ordered by inclusion. We can produce further posets by considering subsets $S \subseteq 2^X$, with the induced order. Show that this, in fact, produces all posets, except for isomorphism. □

[17] Recall that $\subseteq_f$ denotes "finite subset of".

CHAPTER 3

Lattices

Recall that $A \subseteq_f B$ means that A is a finite subset of B (page 15).

Definition 3.1. *Let $(L, \leq)$ be a poset.*
a) L is a **semilattice** *iff for all $F \subseteq_f L$, $\bigwedge F$ exists in L. When $F = \{x, y\}$, write $x \wedge y$ for inf $\{x, y\}$* [1]*. Thus, all semilattices have $\top = \bigwedge \emptyset$.*
b) L is a **join semilattice** *iff for all $F \subseteq_f L$, $\bigvee F$ exists in L; for $x, y \in L$, $x \vee y$ indicates sup $\{x, y\}$* [2]*. Note that all join semilattices have $\perp = \bigvee \emptyset$.*
c) L is a **lattice** *iff for all $F \subseteq_f L$, both $\bigwedge F$ and $\bigvee F$ exist in L. Hence, all lattices have $\perp$ and $\top$.*

If L and P are lattices, **a morphism**, *$f : L \longrightarrow P$, is a map such that for all $F \subseteq_f L$*

$$f(\bigvee F) = \bigvee f(F) \quad \text{and} \quad f(\bigwedge F) = \bigwedge f(F),$$

where, as usual, $f(F) = \{fx : x \in F\}$.
d) A **sublattice** *of L is a subset P of L, such that for all $F \subseteq_f P$, $\bigvee F$ and $\bigwedge F$,* **computed in *L***, *are in P. Thus, $\bigvee F$ and $\bigwedge F$* **in *P*** *exist, and are identical to the sup and inf taken in L.*

Note that the canonical injection $P \longrightarrow L$ is a morphism iff P is a sublattice of L.

Lattices and their morphisms are a category, denoted by $\boldsymbol{\mathcal{L}}$. We write $L \in Ob(\boldsymbol{\mathcal{L}})$ and $f \in \mathcal{M}(\boldsymbol{\mathcal{L}})$ to indicate that L is a lattice and that f is a lattice morphism.

Standard references on lattices are [**21**], [**5**] and [**3**]; [**60**] is also very interesting. In general, the definitions will allow for the absence of $\perp$ and/or $\top$. It is simple to modify the above to cover that : just require that **non-empty** finite sets have inf and/or sup. We will be mainly interested in lattices with $\perp$ and $\top$. The following is straightforward :

Lemma 3.2. *Let $L \xrightarrow{f} P$ be a map between lattices.*
a) f is a morphism iff it verifies
$*$ $f(\perp) = \perp$ *and* $f(\top) = \top$;
$*$ $\forall\, x, y \in L, \quad f(x \vee y) = fx \vee fy \quad \text{and} \quad f(x \wedge y) = fx \wedge fy.$
b) If f is a morphism, then f is increasing (i.e., a poset morphism) and the image of f is a sublattice of P.

[1] Also called the *meet of x and y*.
[2] Also called the *join of x and y*.

c) If f is increasing, then it is a lattice morphism iff
$*$ $f(\bot) = \bot$ *and* $f(\top) = \top$;
$*$ $\forall\, x, y \in L$, $f(x \vee y) \leq fx \vee fy$ *and* $fx \wedge fy \leq f(x \wedge y)$.

Many of the examples in Chapter 2 are lattices : 2^X_λ and $B_\lambda(X)$ (2.9), linear orders, the opens of a topological space (2.11), congruences on an algebra (2.49) and the posets presented in 2.14. We also have examples of semilattices and join semilattices.

EXAMPLE 3.3. If L_i, $i \in I$, is a family of lattices, the set-theoretical product, $\prod L_i$, has a natural lattice structure, where the operations of $\wedge$ and $\vee$ are defined coordinatewise :

$$\forall\, i \in I, \quad [s \wedge t](i) = s(i) \wedge t(i) \quad \text{and} \quad [s \vee t](i) = s(i) \vee t(i).$$

Note that $\top$ and $\bot$ in $\prod L_i$ are the constant sequences $\top$ and $\bot$, respectively. The canonical projections $\prod L_i \xrightarrow{\pi_i} L_i$ are lattice morphisms.

If $f : D \longrightarrow L$ is a lattice morphism, define

$$D \times_f D = \{\langle a, b \rangle \in D \times D : fa = fb\}.$$

With the operations induced by $D \times D$, $D \times_f D$ is a lattice, called **the fibered product** of D over f. The restriction of the canonical projections π_1, π_2 from $D \times D$ to D are lattice morphisms, still indicated by π_i, $i = 1, 2$. Note that $f \circ \pi_1 = f \circ \pi_2$. This construction yields a proof that monomorphisms in the category $\boldsymbol{\mathcal{L}}$ are precisely the injective lattice morphisms (Exercise 3.15.(d)). $\square$

EXAMPLE 3.4. We shall now construct the **coproduct** of a family of lattices, $\{L_i : i \in I\}$, written $\coprod_{i \in I} L_i$. The question here is to obtain a lattice with $\bot$ and $\top$. The formal definition of coproduct of a family of object in a category appears in Example 16.30.(b), while that of initial object in a category in Definition 16.9.

If $\boldsymbol{I = \emptyset}$, set

$$\coprod_{i \in I} L_i \ =_{def}\ 2 \ = \ \{\bot, \top\},$$

where 2 is the two-element lattice with $\bot < \top$. Note that for all lattices L, there is a *unique* lattice morphism from 2 to L, taking $\bot$ to $\bot$ and $\top$ to $\top$. Hence, 2 is the **initial object** in the category of lattices [3].

From here on **assume that $\boldsymbol{I \neq \emptyset}$** and let

$$T \ = \ \{s \in \textstyle\prod L_i : \{i \in I : s_i \neq \top\} \text{ is finite}\},$$

with the partial order induced by $\prod L_i$. Define

$$B \ = \ \{s \in T : \{i \in I : s_i = \bot\} \neq \emptyset\}.$$

Now note that :

(1) B is closed under meets;

(2) $T \ = \ B \cup \bigcup_{s \in T - B} \{s\}$, and this union is disjoint.

Hence, there is a *unique* equivalence relation θ on T, whose equivalence classes are precisely B and, for $s \in T - B$, the singleton $\{s\}$. Whence, for $s, t \in T$, $s\ \theta\ t$ iff $s, t \in B$ or $s, t \in T - B$ and $s = t$.

Set

[3] It is a general fact that an empty coproduct, if it exists, is an initial object.

$$\coprod_{i\in I} L_i \ =_{def}\ T/\theta \ =_{def}\ \{s/\theta : s \in T\}.$$

We now define the operations of meet and join in $\coprod_{i\in I} L_i$ as follows : for $s, t \in T$

(3) $s/\theta \wedge t/\theta \ = \ (s \wedge t)/\theta$;

(4) $s/\theta \vee t/\theta \ = \ \begin{cases} s/\theta & \text{if } t \in B; \\ t/\theta & \text{if } s \in B; \\ (s \vee t)/\theta & \text{if } s, t \in T - B. \end{cases}$

It is straightforward that the operations defined above verify the axioms in Exercise 3.15.(a) and hence, by its item (b), $\langle \coprod_{i\in I} L_i, \wedge, \vee \rangle$ is a lattice, wherein $\bot/\theta$ is class of any sequence in T that has $\bot$ as one of its coordinates (i.e., $\bot/\theta = B$) and $\top/\theta$ is the class of the element of T whose all coordinates are equal to $\top$. The reader can also check that the corresponding partial order in $\coprod_{i\in I} L_i$ is given by

$$s/\theta \leq t/\theta \quad \text{iff} \quad s \in B \quad \text{or} \quad s \in T - B \text{ and } s \leq t \text{ in } T.$$

For each $i \in I$ and $a \in L_i$, let $\check{a}$ be the element of T that has a in the i^{th} coordinate and $\top$ in all others. Now define

$$\alpha_i : L_i \longrightarrow \coprod_{i\in I} L_i, \text{ given by } \alpha_i(a) \ = \ \check{a}/\theta.$$

It is easily verified that α_i is a lattice morphism. This construction has the following universal property :

Lemma 3.5. *If $f_i : L_i \longrightarrow L$, $i \in I$, are lattice morphisms, then there is a* unique *lattice morphism, $f : \coprod_{i\in I} L_i \longrightarrow L$, such that for all $i \in I$ the following diagram is commutative :*

$$\begin{tikzcd} L_i \arrow[rr, "\alpha_i"] \arrow[dr, "f_i"'] & & \coprod_{i\in I} L_i \arrow[dl, "f"] \\ & L & \end{tikzcd}$$

This unique f is written $\coprod_{i\in I} f_i$ and called the **coproduct of the f_i**.

Proof. We give only a sketch, leaving details to the reader. For $s \in T$, define

$$f(s/\theta) \ = \ \bigwedge_{i\in I} f_i(s_i). \tag{*}$$

Note that if $s \in T$, then $\{i \in I : s_i \neq \top\}$ is *finite*, and so the (apparently) infinitary meet in the right side of (*) is in fact finite. Since for all $s \in B$, $f(s/\theta) = \bot$, the definition in (*) is independent of representatives. It is straightforward that f preserves $\wedge$ and $\vee$, being, therefore, a lattice morphism. Moreover, for all $i \in I$ and $a \in L_i$

$$f(\alpha_i(a)) \ = \ f(\check{a}) \ = \ f_i(a),$$

and the diagram in the statement is indeed commutative. To verify the uniqueness of f, let $g : \coprod_{i\in I} L_i \longrightarrow L$ be a lattice morphism making the diagram in the statement commutative. For $s \in T$, let $J \subseteq_f I$ be such that for all $i \notin J$ we have $s_i = \top$. Then, $s \ = \ \bigwedge_{j\in J} \check{s_j}$ and $s/\theta \ = \ \bigwedge_{j\in J} \check{s_j}/\theta$. Hence,

$$g(s/\theta) = \bigwedge_{j\in J} g(\check{s_j}) = \bigwedge_{j\in J} g(\alpha_j(s_j)) = \bigwedge_{j\in J} f_j(s_j)$$
$$= \bigwedge_{i\in I} f_i(s_i) = f(s/\theta),$$

ending the proof. □

It is the universal property described by Lemma 3.5 that guarantees that the system $\langle \coprod_{i\in I} L_i,\ \{\alpha_i : i \in I\}\rangle$ is the **coproduct of the L_i in the category of lattices**, as stipulated by Definition 16.29 and Example 16.30.(b). □

EXAMPLE 3.6. Let T be a topological space. Recall from 1.11 that $B(T)$ and $Reg(T)$ are the set of clopens and regular opens in T. We have (see [R] in page 19)

$$\{\emptyset, T\} \subseteq B(T) \subseteq Reg(T) \subseteq \Omega(T)$$

as *partially ordered sets*. By 1.12,

$B(T)$ is a **sublattice** of $\Omega(T)$.

It follows from 1.14 that $Reg(T)$ is a lattice with $\bot = \emptyset$, $\top = T$ and

$$U \wedge V = U \cap V \quad \text{and} \quad U \vee V = \text{int}\,(\overline{U \cup V}).$$

However, $Reg(T)$ *is not* a sublattice of $\Omega(T)$. □

Recall from 2.49, that a **congruence** θ on a lattice L is an equivalence relation on L, such that for all $x, y, a, b \in L$,

$$x\ \theta\ a \text{ and } y\ \theta\ b \ \Rightarrow\ (x \wedge y)\ \theta\ (a \wedge b) \quad \text{and} \quad (x \vee y)\ \theta\ (a \vee b).$$

Write $Con(L)$ for the lattice of congruences on L.

EXAMPLE 3.7. Let $L \xrightarrow{f} P$ be a lattice morphism. For $x, y \in L$, define

$$x\ \theta_f\ y \quad \text{iff} \quad fx = fy.$$

θ_f is a congruence on L, the **congruence generated by f**. □

A congruence relation θ on L gives rise to a quotient L/θ, the set of equivalence classes of the elements of L by θ. For $x \in L$, write x/θ for the equivalence class of x in L/θ. The following are straightforward :

a) $\forall\, x, y \in L$, $\ x\ \theta\ (x \wedge y)$ iff $\ y\ \theta\ (x \vee y)$. This allows us to define

$$x/\theta \le y/\theta \quad \text{iff} \quad x\ \theta\ (x \wedge y),$$

yielding a partial order $\le$ on L/θ.

b) The assignments,

$$x/\theta \vee y/\theta = (x \vee y)/\theta \quad \text{and} \quad x/\theta \wedge y/\theta = (x \wedge y)/\theta,$$

define operations on L/θ that, together with $\bot = \bot/\theta$ and $\top = \top/\theta$, make it the lattice associated to $(L/\theta, \le)$. Further, the map

$$\pi_\theta : L \longrightarrow L/\theta, \text{ given by } x \mapsto x/\theta$$

is a lattice morphism. The lattice L/θ is called the **quotient** of L by θ and π_θ is the quotient morphism.

PROPOSITION 3.8. (Fundamental theorem for morphisms) *Let L be a lattice, θ be a congruence on L and $f : L \longrightarrow P$ be a lattice morphism. The following conditions are equivalent :*

(1) *There is a* unique $g : L/\theta \longrightarrow P$ *such that* $g \circ \pi_\theta = f$;

(2) $\theta \subseteq \theta_f$ (3.7).

$$\begin{tikzcd} L \arrow[rr, "\pi_\theta"] \arrow[dr, "f"'] & & L/\theta \arrow[dl, "g"] \\ & P & \end{tikzcd}$$

Proof. Left to the reader. □

As an application of Proposition 3.8, we establish that the category of lattices has coequalizers [4], namely :

Corollary 3.9. *Let f, $g : L \longrightarrow R$ be lattice morphisms. Then, there is a lattice C and a lattice morphism , $\gamma : R \longrightarrow C$, satisfying the following* (universal) *property :*

(1) $\gamma \circ f \;=\; \gamma \circ g$;

(2) *If $R \xrightarrow{\alpha} T$ is a lattice morphism such that $\alpha \circ f \;=\; \alpha \circ g$, there is a* unique *lattice morphism, $\widehat{\alpha} : C \longrightarrow T$ such that the following diagram is commutative :*

[coeq]

$$\begin{tikzcd} L \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & R \arrow[rr, "\gamma"] \arrow[dr, "\alpha"'] & & C \arrow[dl, "\widehat{\alpha}"] \\ & & T & \end{tikzcd}$$

Proof. Set $K \;=\; \{\langle\, fx, gx \,\rangle \in R^2 : x \in L\}$ and let θ be the *lattice congruence* generated by K on R (2.49). Set $C \;=\; R/\theta$ and let $\gamma = \pi_\theta : R \longrightarrow C$ be the canonical quotient map. Since $K \subseteq \theta$, it is clear that γ verifies condition (1) in [coeq]. Let $\alpha : R \longrightarrow T$ be a lattice morphism such that $\alpha \circ f = \alpha \circ g$. Then,

$$\Sigma \;=\; \{\langle\, r, s \,\rangle \in R^2 : \alpha s = \alpha r\}$$

is, by Example 3.7, a lattice congruence on R; moreover, the hypothesis on α implies that $K \subseteq \Sigma$. Since θ is congruence generated by K, it follows that $\theta \subseteq \Sigma$. But then Proposition 3.8 guarantees that there is a unique $\widehat{\alpha}$ making the diagram in the statement commutative, ending the proof. □

[4] The formal definition of this notion appears in 16.30.(a).

The diagram $\gamma : R \longrightarrow C$ constructed in Corollary 3.9 is the **coequalizer of the pair $\langle\, \boldsymbol{f}, \boldsymbol{g}\, \rangle$**.

We now turn to the consideration of filters and ideals in a lattice. The definitions of filter and ideal in a poset are in 2.26.

LEMMA 3.10. *Let L be a lattice and $\emptyset \neq A \subseteq L$.*
a) A is a filter in L iff for all $x, y \in L$
$[lf\ 1]$ *:* $x, y \in A \;\Rightarrow\; x \wedge y \in A$;
$[lf\ 2]$ *:* $x \in A \;\Rightarrow\; x^{\rightarrow} \subseteq A$;
b) A is an ideal in L iff for all $x, y \in A$
$[li\ 1]$ *:* $x, y \in A \;\Rightarrow\; x \vee y \in A$;
$[li\ 1]$ *:* $x \in A \;\Rightarrow\; x^{\leftarrow} \subseteq A$.
c) A filter A (ideal) is proper iff $\perp \notin A$ (resp., $\top \notin A$).

PROOF. We prove only (a). Suppose A is a (poset) filter. It is clear that $[Fil\ 2]$ in 2.26 is identical with $[lf\ 2]$. If $x, y \in A$, down-directedness yields $c \in A$ such that $c \leq a, b$. But then $c \leq (a \wedge b)$ and $[Fil\ 2]$ implies that $a \wedge b \in A$. For the converse, just observe that any set satisfying $[lf\ 1]$ is dd. □

EXAMPLE 3.11. If T is a topological space, 1.13 implies that $D(T)$, the set of dense opens in T, is a proper filter in $\Omega(T)$. □

Recall that $a^{\rightarrow}$ (resp., $a^{\leftarrow}$) is the **principal filter (ideal)** generated by a.

With the help of 3.17.(c), we can define the filter or ideal generated by a subset of a lattice.

DEFINITION 3.12. *Let L be a lattice and let S be a subset of L. The* **filter generated** *by S, $\mathfrak{f}(S)$, is defined as*
$$\mathfrak{f}(S) = \bigcap \{F \subseteq L : F \text{ is a filter and } S \subseteq F\}.$$
Similarly, the **ideal generated** *by S, $\mathfrak{i}(S)$, is given by*
$$\mathfrak{i}(S) = \bigcap \{I \subseteq L : I \text{ is an ideal and } S \subseteq I\}.$$

Note that $\mathfrak{f}(\emptyset) = \{\top\}$ and $\mathfrak{i}(\emptyset) = \{\perp\}$. The following Lemma describes some of the fundamental properties of these concepts, as well as the basic method for constructing filters and ideals.

LEMMA 3.13. *Let L be a lattice and S be a subset of L. With notation as above,*
a) $\mathfrak{f}(S) = \{x \in L :$ There is a finite $D \subseteq S$, such that $x \geq \bigwedge D\}$.
b) $\mathfrak{i}(S) = \{x \in L :$ There is a finite $D \subseteq S$, such that $x \leq \bigvee D\}$.
c) The following are equivalent :
(1) *$\mathfrak{f}(S)$ is a proper filter.*
(2) *(Finite intersection property, fip) If $D \subseteq_f S$, then $\bigwedge D \neq \perp$.*
d) The following are equivalent :
(1) *$\mathfrak{i}(S)$ is a proper ideal.*
(2) *(Finite union property, fup) If $D \subseteq_f S$, then $\bigvee D \neq \top$.*

PROOF. The reasonings for filters and ideals are entirely similar and we treat only the case of filters. Let A be the right hand side of the equality in (a). If F is a filter in L containing S, then $A \subseteq F$, since F is closed under finite meets. Thus, it suffices to show that A is a filter. It is clear that condition $[lf\ 2]$ in 3.10 is satisfied. For $[lf\ 1]$, if $x, z \in A$, let D and K be finite subsets of S, such that

$$x \geq \bigwedge D \quad \text{and} \quad z \geq \bigwedge K.$$

By 3.15.(c), $(x \wedge y) \geq \bigwedge (D \cup K)$ and so $(x \wedge y)$ is in A, as needed. Item (c) follows from (a), because a filter F is proper iff $\perp \notin F$. $\square$

LEMMA 3.14. *Let $L \xrightarrow{f} P$ be a lattice morphism. If F is a filter in K, then $f^{-1}(F)$ is a filter in L, which is proper if F is proper in K.*

PROOF. It is straightforward that $f^{-1}(F)$ is a filter in L. Now note that $\perp \in f^{-1}(F)$ implies $f(\perp) = \perp \in F$, verifying the properness claim. $\square$

Exercises

3.15. a) For all x, y, z in a lattice L

(i) $(x \vee y) \vee z = x \vee (y \vee z)$; $(x \wedge y) \wedge z = x \wedge (y \wedge z)$.

(ii) $x \vee y = y \vee x$; $x \wedge y = y \wedge x$.

(iii) [5] $x \vee (x \wedge y) = x = x \wedge (x \vee y)$.

(iv) $x \vee \perp = x$ and $x \wedge \perp = \perp$; $x \vee \top = \top$ and $x \wedge \top = x$.

b) Show, conversely, that a set L with two operations, $\wedge$, $\vee$, two distinguished elements $\perp$, $\top$, and with properties (i) to (iv) in (a) is a lattice. Thus, lattices can be **equationally defined**.

c) Let D, $K \subseteq_f L$ and x, y, z, t be elements of L. Show that :

$$(i) \left\{ \begin{array}{lcl} \bigvee (D \cup K) & = & (\bigvee D) \vee (\bigvee K) \\ \bigwedge (D \cup K) & = & (\bigwedge D) \wedge (\bigwedge K). \end{array} \right.$$

Generalize the above for a finite family of finite subsets of L.

(ii) $D \subseteq K \;\Rightarrow\; \bigvee D \leq \bigvee K$ and $\bigwedge K \leq \bigwedge D$.

(iii) $x \leq z$ and $y \leq t \;\Rightarrow\; x \wedge y \leq z \wedge t$ and $x \vee y \leq z \vee t$.

d) Show that a morphism of lattices is a monomorphism iff it is injective. Discuss the nature of an epimorphism.

e) Give an example of a subposet $P \subseteq L$ such that both L and P are lattices but P **is not** a sublattice of L. $\square$

3.16. Let L_i, $i \in I$ be a family of lattices. Suppose that I is a chain and consider

$$\coprod L_i = \bigcup_{i \in I} \{i\} \times L_i.$$

In $\coprod L_i$, define

$$\langle i, u \rangle \leq \langle j, w \rangle \quad \text{iff} \quad i < j \text{ or } i = j \text{ and } u \leq w.$$

---

[5]The *absorption laws.*

The relation $\leq$ is a po. Is $\coprod L_i$ a lattice ? Is there an analogue of the morphisms α_i described in Example 3.4 for the direct sum ? $\square$

3.17. (Compare with 2.32) Let L be a poset.

a) If L is a semilattice, then the intersection of a family of filters in L is a filter.

b) If L is a join semilattice, the intersection of a family of ideals in L is an ideal.

c) If L is a lattice, the intersection of a family of filters (ideals) in L is a filter (resp., ideal).

d) If A_i, $i \in I$, is an upward directed (by inclusion) family of filters or ideals in L, then $\bigcup A_i$ is a filter or an ideal in L, respectively.

e) If L is a lattice, show that $\mathcal{F}(L)$ and $\mathcal{I}(L)$, the posets of filters and ideals in L, respectively, are lattices. $\square$

CHAPTER 4

Distributive Lattices

We now come to the type of lattice that is central in all that follows.

DEFINITION 4.1. *A lattice L is* **distributive** *if it satisfies the following equivalent conditions :*
$[\wedge, \vee]$: *For all $x, y, z \in L$,* $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.
$[\vee, \wedge]$: *For all $x, y, z \in L$,* $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

The notions of morphism and sublattice are the same as in the category of lattices (3.1). *In particular,* $L \xrightarrow{f} P$ *is an* **isomorphism** *iff there is* $P \xrightarrow{g} L$ *such that* $f \circ g = Id_P$ *and* $g \circ f = Id_L$. *If L and P are isomorphic we write* $L \approx P$ *or* $L \approx_f P$, *if there is need to display the isomorphism f explicitly.*

Definition 4.1 and Exercise 3.15 guarantee that distributive lattices can be equationally defined.

EXAMPLE 4.2. Except for the class of lattices in Example 2.10, all other examples in chapter 3 are distributive lattices. □

EXAMPLE 4.3. The lattices in 2.10 are important mathematical objects. Some satisfy an alternative to distributivity, known as **the modular law** : $\forall\, x, y, z \in L$

$$x \leq z \quad \text{implies} \quad x \vee (y \wedge z) = (x \vee y) \wedge z.$$

For closed subspaces of a Hilbert space, there is the operation of taking a subspace to its orthogonal complement, $x \mapsto x^{\perp}$, satisfying :
(*i*) $x \wedge x^{\perp} = \perp$; $x \vee x^{\perp} = \top$; $(x^{\perp})^{\perp} = x$.
(*ii*) $(x \wedge y)^{\perp} = x^{\perp} \vee y^{\perp}$.
(*iii*) $(x \vee y)^{\perp} = x^{\perp} \wedge y^{\perp}$.
(*iv*) (*The orthomodular law*) $x \leq y \;\Rightarrow\; x \vee (x^{\perp} \wedge y) = y$.
The interested reader may consult [**5**], [**21**] and [**35**]. □

Distributive lattices and their morphisms form a category, denoted by $\boldsymbol{\mathcal{D}}$. The expressions $L \in \boldsymbol{\mathcal{D}}$ and $f \in \mathcal{M}(\boldsymbol{\mathcal{D}})$ indicate that L is a distributive lattice and that f is a morphism in $\boldsymbol{\mathcal{D}}$.

REMARK 4.4. There is a neat characterization of distributivity, due to Birkhoff, in terms of having or not certain finite lattices as sublattices. See, for instance, Theorem II.5.9, pg. 51 of [**3**]. □

EXAMPLE 4.5. The product (3.3) and the coproduct (3.4) of distributive lattices is distributive. Hence, $2 = \{\perp, \top\}$ is an initial object in the category $\boldsymbol{\mathcal{D}}$.

If $f : L \longrightarrow P$ is a lattice morphism, with L distributive, then Im f is a distributive sublattice of P. $\square$

EXAMPLE 4.6. If X is a topological space, the lattice of opens in X, $\Omega(X)$, is a *distributive* lattice, because it is a sublattice of 2^X.

Let Y be a topological space and $f : X \longrightarrow Y$ be a continuous map. Define $f^* : \Omega(Y) \longrightarrow \Omega(X)$ by $U \mapsto f^{-1}(U)$. It is well known that f^* preserves intersections and unions, being, therefore, a lattice morphism from $\Omega(Y)$ to $\Omega(X)$. Of course, if f is a homeomorphism, f^* will be an isomorphism.

Recall from 3.6, that $B(X)$, $Reg(X)$ are the lattice of clopens and regular opens, respectively, in X. Since $B(X)$ is a sublattice of $\Omega(X)$ it is also distributive. On the other hand, by 1.14.(f), $Reg(X)$ is a distributive lattice, although not a sublattice of $\Omega(X)$ or of 2^X.

If we restrict f^* to $B(Y)$, we get a lattice morphism from $B(Y)$ to $B(X)$ (1.31.(a)). However, the inverse image of a regular open is not, in general, regular (1.31.(b)). For instance, for the sine function, sin : $[0,\, \pi] \longrightarrow [0,\, 1]$, the inverse image of $(0,\, 1)$ is $(0,\, \pi/2) \cup (\pi/2,\, \pi)$, which is not regular.

There is a map $f_* : \Omega(X) \longrightarrow \Omega(Y)$, closely associated to f^*. Define

$$f_*(V) = \bigcup \{W \in \Omega(Y) : f^*(W) \subseteq V\}.$$

f_* is **not** direct image, as f might not be an open map. In fact,

$$f_*(V) = \mathrm{int}(Y - f(X - V)).$$

The connection between f_* and f^* is described by the adjunction :

(*) For all $\langle V, W \rangle \in \Omega(X) \times \Omega(Y)$, $\quad f^*W \subseteq V \quad$ iff $\quad W \subseteq f_*V$.

A general way of obtaining such adjoint pairs is described in Theorem 7.8. $\square$

LEMMA 4.7. *Let L be a distributive lattice, $a \in L$, and S, $T_1, \ldots, T_n$ be finite subsets of L. Then,*

a) $a \vee (\bigwedge S) = \bigwedge \{a \vee s : s \in S\}$.

b) $a \wedge (\bigvee S) = \bigvee \{a \wedge s : s \in S\}$.

c) $\bigvee_{i=1}^n (\bigwedge T_i) = \bigwedge \{\bigvee_{i=1}^n f(i) : f \in \prod T_i\}$

d) $\bigwedge_{i=1}^n (\bigwedge T_i) = \bigvee \{\bigwedge_{i=1}^n f(i) : f \in \prod T_i\}$

PROOF. Parts (a) and (b) come from the distributivity, by induction on the number of elements of S. Items (c) and (d) can be proven, for $n = 2$, by induction on the number of elements of T_1, the basis step being provided by (a) and (b). Finally, induction on n will finish the proof. $\square$

REMARK 4.8. If L is a distributive lattice and θ is a *lattice congruence* on L, then the quotient lattice, L/θ, defined right before Proposition 3.8, **is distributive** and called the **quotient of L be θ**. As before, $\pi_\theta : L \longrightarrow L/\theta$ is a lattice morphism and the statement of 3.8 is valid, *ipsis litteris*, for distributive lattices. In particular, Corollary 3.9 holds $-$ with the same proof $-$, for distributive lattices, that is, in this category every pair of morphisms has a coequalizer. $\square$

We now explore the relationship between congruences and filters or ideals in a distributive lattice.

Definition 4.9. *Let L be a distributive lattice and $f : L \longrightarrow P$ be a lattice morphism.*
a) If $F \subseteq L$ is a filter in L, define, for $x, y \in L$,

$$x \sim_F y \quad \textit{iff} \quad \textit{there is } z \in F \textit{ such that } x \wedge z = y \wedge z.$$

b) If $I \subseteq L$ is an ideal in L, define, for $x, y \in L$,

$$x \sim_I y \quad \textit{iff} \quad \textit{there is } z \in I \textit{ such that } x \vee z = y \vee z.$$

c) If θ is a congruence on L, set

$$F_\theta = \{x \in L : x\ \theta\ \top\} \quad \textit{and} \quad I_\theta = \{x \in L : x\ \theta\ \bot\}.$$

d) In the case θ is the congruence generated in L by f (3.7), the sets F_θ and I_θ are called the **cokernel** *and* **kernel** *of f, respectively :*

$$\textit{coker } f = \{x \in L : fx = \top\} \quad \textit{and} \quad \textit{ker } f = \{x \in L : fx = \bot\}.$$

Proposition 4.10. *Let L be a distributive lattice.*
a) For θ, $\theta' \in Con(L)$, $\quad \theta \subseteq \theta' \ \Rightarrow\ F_\theta \subseteq F_{\theta'}$ and $I_\theta \subseteq I_{\theta'}$.
b) If A is a filter or ideal in L, then $\sim_A$ is a congruence in L. Moreover,

$$\begin{cases} F_{\sim_A} = A & \textit{if } A \textit{ is a filter;} \\ I_{\sim_A} = A & \textit{if } A \textit{ is an ideal.} \end{cases}$$

c) For filters or ideals F, G in L, $\quad F \subseteq G$ iff $\sim_F \subseteq \sim_G$.

Proof. Item (a) is straightforward. In the remaining items, we treat only the case of filters; the proofs for ideals are analogous.
b) $\underline{\sim_A \text{ is an equivalence relation :}}$ Reflexivity and symmetry are immediate. For transitivity, let $x, y, t \in L$ with $x \sim_A y$ and $y \sim_A t$. Choose u and z in A such that $x \wedge u = y \wedge u$ and $y \wedge z = t \wedge z$. Since A is a filter, we have $u \wedge z \in A$, and so

$$\begin{aligned} x \wedge (u \wedge z) &= (x \wedge u) \wedge z = (y \wedge u) \wedge z = (y \wedge z) \wedge u = (t \wedge z) \wedge u \\ &= t \wedge (u \wedge z), \end{aligned}$$

showing that $x \sim_A t$.
$\underline{\sim_A \text{ is a congruence :}}$ Suppose $x \sim_A a$ and $y \sim_A b$; let u, z be elements of A such that $x \wedge u = a \wedge u$ and $y \wedge z = b \wedge z$. Then, $u \wedge z \in A$ and we have

$$\begin{aligned} (u \wedge z) \wedge (x \vee y) &= (x \wedge u \wedge z) \vee (y \wedge u \wedge z) \\ &= (a \wedge u \wedge z) \vee (b \wedge u \wedge z) \\ &= (u \wedge z) \wedge (a \vee b). \end{aligned}$$

showing that $(x \vee y) \sim_A (a \vee b)$. Similar calculations, in fact not involving distributivity, will verify that $\sim_A$ is a congruence with respect to $\wedge$. For the displayed equation, note that

$$\begin{aligned} x \in F_{\sim_A} \quad &\text{iff} \quad x \sim_A \top \quad \text{iff} \quad \text{for some } z \in A,\ x \wedge z = \top \wedge z = z \\ &\text{iff} \quad \text{for some } z \in A,\ x \geq z. \end{aligned}$$

Since A is a filter, we conclude that $A = F_{\sim_A}$, as desired. Item (c) is a straightforward consequence of (a) and (b). ☐

4.11. **Notation.** If A is a filter or an ideal in a distributive lattice L, we adopt the following notation, where $x \in L$:
$*$ x/A is the equivalence class of x by the congruence $\sim_A$;

$*$ L/A for the quotient lattice of L by the congruence $\sim_A$;
$*$ $\pi_A : L \longrightarrow L/A$ for the canonical quotient morphism. $\square$

Corollary 4.12. *Let L be a distributive lattice and $\theta \in Con(L)$. With notation as in* 4.9 *and* 4.11*,*
a) If A is a filter in L, then for all x, $y \in L$

$$x/A \leq y/A \quad \textit{iff} \quad \exists\, z \in A \textit{ such that } x \wedge z \leq y.$$

b) If A is an ideal in L, then for all x, $y \in L$

$$x/A \leq y/A \quad \textit{iff} \quad \exists\, z \in A \textit{ such that } x \leq y \vee z.$$

c) F_θ is a filter and I_θ is an ideal in L. Furthermore, $\sim_{F_\theta} \cup \sim_{I_\theta} \subseteq \theta$.
d) The cokernel of a morphism is a filter and its kernel an ideal in L.
e) If A is a filter in L and $\theta \in Con(L)$, then $\sim_A \subseteq \theta$ iff $A \subseteq F_\theta$.
A corresponding statement holds for ideals.

Proof. We treat only the case of filters. For (a), note that if x, $y \in L$,

$$\begin{aligned} x/A \leq y/A \quad &\text{iff} \quad x/A \wedge y/A = (x \wedge y)/A = x/A \quad \text{iff} \quad (x \wedge y) \sim_A x \\ &\text{iff} \quad \exists\, z \in A \text{ such that } x \wedge y \wedge z = x \wedge z. \end{aligned}$$

To conclude, note that the last equation is equivalent to $(x \wedge z) \leq y$.
c) It is straightforward that F_θ and I_θ are, respectively, a filter and an ideal. We show that $\sim_{F_\theta} \subseteq \theta$, leaving the analogous calculations for $\sim_{I_\theta}$ to the reader. For $x, y \in L$,

$$x \sim_{F_\theta} y \quad \text{iff} \quad \exists\, z \text{ such that } z\ \theta \top \quad \text{and } x \wedge z = y \wedge z.$$

From $x\ \theta\ x$, $y\ \theta\ y$ and $z\ \theta\ \top$, we get $(x \wedge z)\ \theta\ x$ and $(y \wedge z)\ \theta\ y$. Since $x \wedge z = y \wedge z$, transitivity yields $x\ \theta\ y$, as required. Item (d) follows immediately from (c), while (e) is a consequence of 4.10 and (c). $\square$

For quotients by filters and ideals, Proposition 3.8 takes the following form, when the source of a morphism is distributive :

Corollary 4.13. *Let L be a distributive lattice, F be a filter on L and $L \xrightarrow{f} P$ be a lattice morphism. The following are equivalent:*

(1) *There is a* unique $g : L/F \longrightarrow P$ *such that $g \circ \pi_F = f$;*

(2) *$F \subseteq coker\ f$.*

$$\begin{tikzcd} L \arrow[rr, "\pi_F"] \arrow[dr, "g"'] & & L/F \arrow[dl, "f"] \\ & P & \end{tikzcd}$$

A corresponding statement holds for ideals.

Proof. By Proposition 3.8, the existence of g is equivalent to $\sim_F \subseteq \theta_f$, while 4.12.(e) implies that this condition is equivalent to $F \subseteq F_{\theta_f} =$ coker f. $\square$

Complementing Proposition 4.10, we prove

LEMMA 4.14. *Let L be a lattice. With notation as in* 4.9*, the following are equivalent :*
(1) *L is a distributive lattice.*
(2) *For all filters A in L, the relation $\sim_A$ is a congruence in L.*
(3) *For all ideals A in L, the relation $\sim_A$ is a congruence in L.*
(4) *For all* principal *ideals A in L, $\sim_A$ is a congruence in L.*
(5) *For all* principal *filters A in L, $\sim_A$ is a congruence in L.*

PROOF. By 4.10, it is enough to verify that (5) $\Rightarrow$ (1), i. e., if x, y, $z \in L$, then

$$x \wedge (y \vee z) \leq (x \wedge y) \vee (x \wedge y). \tag{I}$$

Let $A = x^{\rightarrow}$; since $x \wedge y = x \wedge (x \wedge y)$, we have $y \sim_A x \wedge y$; analogously, $z \sim_A x \wedge z$. Since $\sim_A$ is a lattice congruence, we conclude that

$$(y \vee z) \sim_A [(x \wedge y) \vee (x \wedge z)].$$

Hence, there is $t \in x^{\rightarrow}$, such that

$$t \wedge (y \vee z) = t \wedge [(x \wedge y) \vee (x \wedge z)] \leq (x \wedge y) \vee (x \wedge z). \tag{II}$$

But (II) and $x \leq t$ immediately imply (I), as desired. □

EXAMPLE 4.15. Let L_i, $i \in I$, be a family of distributive lattices and $F \subseteq 2^I$ be a filter in 2^I (usually one says a filter on I). For $s, t \in \prod L_i$, define,

$$s \sim_F t \quad \text{iff} \quad \{i \in I : s(i) = t(i)\} \in F.$$

Then, $\sim_F$ is a congruence on $\prod L_i$; the quotient lattice, written $\prod L_i/F$, is called **the reduced product** of the L_i by F. The elements of this quotient will be written s/F. For $s, t \in \prod L_i$, we have

$$s/F \leq t/F \quad \text{iff} \quad \{i \in I : s(i) \leq t(i)\} \in F. \qquad \square$$

EXAMPLE 4.16. If T is a topological space and $S \subseteq T$, let

$$\nu_S = \{U \in \Omega(T) : S \subseteq U\}.$$

This is a filter in $\Omega(T)$, **the neighborhood filter** of S in T. If $S = \{x\}$, write ν_x for ν_S (as in section 1.2).

With the topology induced by T, S is a topological space and there is a canonical surjective lattice morphism,

$$\pi_S : \Omega(T) \longrightarrow \Omega(S), \text{ given by } \pi_S(U) = U \cap S.$$

Clearly, coker $\pi_S = \nu_S$. It is left as an exercise for the reader to verify that $\Omega(S)$ isomorphic to the quotient of $\Omega(T)$ by the congruence generated by π_S.

By Proposition 4.10, the congruence generated by ν_S is contained in that arising from π_S. If S *is open* in T, then they are the same, and so $\Omega(T)/\nu_S$ is isomorphic to $\Omega(S)$. In general, these congruences may be different. To see this, let $T = \mathbb{R}$ and $S = [0, 1)$, the unit interval, closed at 0 and open at 1. Let

$$U = (1/2, 2) \quad \text{and} \quad V = (-1, 0) \cup (1/2, 1).$$

Clearly, $U \cap S = V \cap S$; however, any open set W containing S will contain an open interval around 0 and, hence, will have non-empty intersection with $(-1, 0)$. But the latter interval is disjoint from U. This shows that U and V are not equivalent with respect to the congruence generated by ν_S. Proposition 4.10 guarantees that, in fact, the congruence associated to π_S is distinct from all that arise from filters on $\Omega(T)$. $\Box$

Example 4.17. As a simpler example of the fact that not all congruences on a distributive lattice come from filters or ideals, consider the real unit interval $[0, 1]$ and $P = \{0, 1/2, 1\}$, with the usual ordering. Define $f : [0, 1] \longrightarrow P$ by

$$fx = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x = 1 \\ 1/2 & \text{otherwise} \end{cases}$$

If θ is the congruence generated by f, then $F_\theta = \{1\}$ and $I_\theta = \{0\}$, which are, respectively, the cokernel and kernel of f. $\Box$

The poset of congruences in a lattice yield important examples of distributive lattices. Before proving this result, we show

Lemma 4.18. *Let A be an algebra (2.49) and let $\{\theta_i\}$, $i \in I$, be a family of congruences in A. Then, the congruence generated by $\bigcup \theta_i$, $\bigvee \theta_i$, is given by*

$$\bigvee \theta_i = \left\{ \langle x, y \rangle \in A^2 : \begin{array}{c} \exists\, s_0, \dots, s_n \in A,\ i_1, \dots, i_n \in I \text{ such that} \\ s_0 = x,\ s_n = y \text{ and } \langle s_{k-1}, s_k \rangle \in \theta_{i_k}, \\ \forall\ 1 \le k \le n. \end{array} \right\}$$

Proof. Let S be the right hand side of the stated equality. First observe that if θ is a congruence containing $\bigcup \theta_i$ then it must contain S, because every pair of successive terms $\langle s_k, s_{k+1} \rangle$ appearing in the definition of S will be in θ. Quite clearly, $\theta_i \subseteq S$, for all $i \in I$. Thus, it suffices to show that S is a congruence on A.

S is an equivalence relation : It is immediate that S is reflexive. If $\langle x, y \rangle \in S$, there are $s_0, s_1, \dots, s_n$ in A and $i_1, \dots, i_n$ in I with the properties described above. To verify that $\langle y, x \rangle \in S$, just consider $t_0, \dots, t_n$ given by $t_k = s_{n-k}$, $0 \le k \le n$.

For transitivity, suppose $\langle x, y \rangle, \langle y, z \rangle \in S$. Let $s_0, \dots, s_n$ in A and $i_1, \dots, i_n$ in I "connect" x and y as in the statement; suppose further that $t_0, \dots, t_m$ in A and $k_1, \dots, k_m$ in I do the same for y and z. Recall that $s_n = y = t_0$; now :

a) Define elements $u_j \in A$, $0 \le j \le m + n$, by

$$u_j = \begin{cases} s_j & 0 \le j \le n \\ t_{j-n} & n + 1 \le j \le m + n \end{cases}$$

b) Define elements $l_j \in I$, $1 \le j \le m + n$, by

$$l_j = \begin{cases} i_j & 1 \le j \le n \\ k_{j-n} & n + 1 \le j \le n + m \end{cases}$$

It is easily verified that the finite sequences in (a) and (b) "connect" x and z, proving the transitivity of S.

S is a congruence : Let α be a $(m+1)$-ary operation in A and let $\langle x, \vec{a} \rangle \in A^{m+1}$, where $\vec{a} = \langle a_1, \dots, a_m \rangle$. Suppose $\langle x, y \rangle \in S$, with $s_0, \dots, s_n$ in A and $i_1, \dots, i_n$ in I, "connecting" x and y, as above. Then, the sequences $\alpha(s_k, \vec{a})$ $(0 \le k \le n)$ in A and $i_1, \dots, i_n$ in I "connect" $\alpha(x, \vec{a})$ and $\alpha(y, \vec{a})$, and so $\langle \alpha(x, \vec{a}), \alpha(y, \vec{a}) \rangle$ is in S.

The above argument can be carried out for any coordinate, holding the others fixed. Changing one coordinate at a time, from 1 to $m+1$, we can show that for all $\vec{x}$, $\vec{z} \in A^{m+1}$, if $\langle x_k, z_k \rangle \in S$ for all $k \leq m+1$, then $\langle \alpha(\vec{x}), \alpha(\vec{z}) \rangle \in S$, completing the proof that S is a congruence on A. $\square$

Recall that $Con(A)$ is the poset of congruences on an algebra A. In Example 2.49, it is shown that $Con(A)$ has sups and infs for all of its subsets, being therefore a lattice. We now prove

PROPOSITION 4.19. *If L is a (not necessarily distributive) lattice, then $Con(L)$ is a distributive lattice.*

PROOF. First observe that in any lattice,

$$\text{For all } x,\, y,\, z, \quad (x \wedge y) \vee (x \wedge z) \leq x \wedge (y \vee z).$$

Thus, if θ, β and γ are congruences on a lattice L, it suffices to show that

$$\theta \cap (\beta \vee \gamma) \quad \subseteq \quad (\theta \cap \beta) \vee (\theta \cap \gamma).$$

Let $\langle x, y \rangle \in \theta$ and $\langle x, y \rangle \in (\beta \vee \gamma)$; this last assertion means, by 4.18, that there are $s_0, s_1, \ldots, s_n$ in L such that $s_0 = x$, $s_n = y$ and for all $k \leq n$, $\langle s_k, s_{k+1} \rangle$ is in β or γ. We wish to produce, again by 4.18, a sequence "connecting" x and y, such that each pair of successive terms is either in $(\theta \cap \beta)$ or in $(\theta \cap \gamma)$.

For $1 \leq k \leq n$, define a sequence in L, as follows :

$$t_k \quad = \quad x \vee (s_1 \wedge y) \vee \ldots \vee (s_k \wedge y).$$

Fact. *For all $1 \leq k \leq n$,*

$*\ \langle s_{k-1}, s_k \rangle \in \beta \quad \Rightarrow \quad \langle t_{k-1}, t_k \rangle \in \theta \cap \beta;$

$*\ \langle s_{k-1}, s_k \rangle \in \gamma \quad \Rightarrow \quad \langle t_{k-1}, t_k \rangle \in \theta \cap \gamma.$

Proof. We treat only the case in which $\langle s_{k-1}, s_k \rangle \in \beta$, the other being similar. Observe that the sequence t_k is increasing, as well as that

$$t_k = t_{k-1} \vee (s_k \wedge y), \quad 1 \leq k \leq n. \tag{I}$$

Suppose $s_{k-1}\ \beta\ s_k$; taking the meet with y, we get

$$(y \wedge s_{k-1})\ \beta\ (y \wedge s_k),$$

and so the join with t_{k-1} on both sides yields

$$[t_{k-1} \vee (y \wedge s_{k-1})]\ \beta\ [t_{k-1} \vee (y \wedge s_k)].$$

Hence, the absorption laws (3.15.(iii)) and (I) imply $\langle t_{k-1}, t_k \rangle \in \beta$. It remains to show that $\langle t_{k-1}, t_k \rangle \in \theta$. From $x\ \theta\ y$ we get

$$(x \wedge s_k)\ \theta\ (y \wedge s_k),$$

and thus,

$$[t_{k-1} \vee (x \wedge s_k)]\ \theta\ [t_{k-1} \vee (y \wedge s_k)]. \tag{II}$$

Since $t_{k-1} \geq x$, (I) and (II) imply the desired conclusion.

To complete the proof, note that $t_n = x \vee y$ and $t_0 = x$. It follows from the Fact that the sequences t_k in L and $i_1, \ldots, i_n$ in I, "connect" x and $(x \vee y)$, by steps which are either in $(\theta \cap \beta)$ or in $(\theta \cap \gamma)$. We conclude that for all $x, y \in L$,

$$\langle x, y \rangle \in \theta \cap (\beta \vee \gamma) \quad \Rightarrow \quad \langle x,\ x \vee y \rangle \in (\theta \cap \beta) \vee (\theta \cap \gamma).$$

But then, it is also true that $\langle y,\ x \vee y \rangle \in (\theta \cap \beta) \vee (\theta \cap \gamma)$. Now, symmetry and transitivity will finish the proof. $\square$

The filters and ideals essential in establishing the important connections between Lattice theory, Logic and Topology are defined in

Definition 4.20. *Let L be a distributive lattice.*
a) A **proper** *filter A is* **prime** *iff for all x, y in L,*

$$x \vee y \in A \quad \Rightarrow \quad x \in A \text{ or } y \in A.$$

b) A **proper** *ideal is* **prime** *if for all x, $y \in L$,*

$$x \wedge y \in A \quad \Rightarrow \quad x \in A \text{ or } y \in A.$$

c) A proper filter (ideal) is **maximal** *iff for all proper filters (resp., ideals) G, $A \subseteq G \;\; \Rightarrow \;\; A = G$.*

Remark 4.21. We shall consider **ultrafilter** as synonymous with **maximal filter**. Reduced products by ultrafilters on a set are called **ultraproducts**. □

Lemma 4.22. *Let L be a distributive lattice.*
a) If $L = A \cup B$, with $A \cap B = \emptyset$, then A is a prime filter iff B is a prime ideal.
b) Every maximal filter or ideal in L is prime.
c) If $L \xrightarrow{f} K$ is a lattice morphism, then the inverse image of a prime filter is a prime filter.

Proof. Item (a) is left to the reader. In (b), we discuss only the case of filters. Let $F \subseteq L$ be a maximal filter, with $(x \vee y) \in F$. Set $S = F \cup \{x\}$ and $T = F \cup \{y\}$. Notice that if either S or T have the fip (3.13), then they generate a proper filter containing F. Since F is maximal, we would get $x \in F$ or $y \in F$, the desired conclusion. Suppose then that *both* S and T *do not* have the fip. Select $s_1, \ldots, s_n$ and $t_1, \ldots, t_m$ in F such that

$$x \wedge \bigwedge_{i=1}^{n} s_i = \perp = y \wedge \bigwedge_{k=1}^{m} t_k.$$

Since F is a filter, finite intersections of elements of F are in F, and so the above equations are equivalent to the existence of z, $w \in F$ such that

$$x \wedge z = \perp = y \wedge w.$$

But now observe that since $(z \wedge w)$ and $(x \vee y)$ are in F, we get

$$(x \vee y) \wedge (z \wedge w) = \perp,$$

a contradiction, because F is a proper filter.
c) By 3.14, the inverse image of a proper filter is a proper filter. For F prime in K, write G for $f^{-1}(F)$. If x, $y \in L$ are such that $x \vee y \in G$, then $fx \vee fy \in F$, and so either $fx \in F$ or $fy \in F$. Hence, $x \in G$ or $y \in G$, and G is prime in L. □

Remark 4.23. Suppose $A \subseteq L$ is a filter in a lattice L and $x \in L$. To show that $A \cup \{x\}$ has the fip, it is enough to check that for $z \in A$, $\;x \wedge z \neq \perp$. A dual statement holds for ideals and the fup property. □

One of the main results of this Chapter is the following

Theorem 4.24. (M. Stone, G. Birkhoff) *Let L be a distributive lattice with $\perp$ and $\top$. Let F be a filter in L and S a non-empty ud subset of L, such that $F \cap S = \emptyset$. Then, there is a prime filter P, such that $F \subseteq P$ and $P \cap S = \emptyset$.*

PROOF. Let
$$V = \{G \subseteq L : G \text{ is a filter, } F \subseteq G \text{ and } G \cap S = \emptyset\}$$
ordered by inclusion. Since $F \in V$, V is not empty. By 3.17 (or 2.32), V satisfies the hypothesis of Zorn's Lemma (2.20). Thus, there is a filter P, which is maximal in V. Clearly, P is proper filter ($S \neq \emptyset$), $F \subseteq P$ and $P \cap S = \emptyset$. It remains to verify that P is prime.

Suppose x, y in L are such that $(x \vee y) \in P$. If neither x nor y is in P, then the filters generated by $P \cup \{x\}$ and $P \cup \{y\}$ contain P properly and so both have non-empty intersection with S. Hence, there are z, $t \in P$, u, $v \in S$ such that
$$(x \wedge t) \leq u \quad \text{and} \quad (y \wedge z) \leq v.$$
Since S is ud, we can select $w \in S$ such that u, $v \leq w$. Since $t \wedge z$ is in P, we get
$$(x \vee y) \wedge (z \wedge t) \;\leq\; (x \wedge t) \vee (y \wedge z) \;\leq\; u \vee v \;\leq\; w.$$
Hence, $w \in P \cap S$, an impossibility which ends the proof. □

There is, of course, a statement dual to the above, involving a proper ideal in L and a non empty dd subset of L, which are disjoint. This result is actually a consequence of Theorem 4.24. With an similar argument one proves

THEOREM 4.25. *Every proper filter (ideal) in a distributive lattice is contained in a maximal filter (resp., ideal).*

COROLLARY 4.26. *Let L be a distributive lattice.*
a) If $a \leq b$ is false in L, then there is a prime filter containing a, to which b does not belong.
b) Every filter (ideal) in L is the intersection of the prime filters (resp., ideals) that contain it. In particular, the intersection of all prime filters in L is $\{\top\}$ and the intersection of all prime ideals is $\{\bot\}$.

PROOF. a) Just notice that the filter $a^{\rightarrow}$ does not intersect the ideal $b^{\leftarrow}$ and apply Theorem 4.24.
b) Let F be a filter and $V = \{P \subseteq L : P \text{ is a prime filter and } F \subseteq P\}$; clearly, $\bigcap V$ contains F. Now, if $a \notin F$, the ideal $a^{\leftarrow}$ is disjoint from F and so Theorem 4.24 yields a prime filter P containing F and disjoint from $a^{\leftarrow}$. Thus, $\bigcap V = F$. □

It is clear that the intersection of a prime filter/ideal in L with a sublattice of L is a prime filter/ideal in the sublattice. The lifting of primes in sublattices of L to primes in L is described in

THEOREM 4.27. *Let L be a distributive lattice and let D be a sublattice of L. If P is a prime filter (ideal) in D, then there is a prime filter (resp., ideal) Q in L such that $Q \cap D = P$.*

PROOF. Assume that P is a prime filter in D and let $I = D - P$; I is a prime ideal in D (4.22.(a)) and so a non-empty upward directed set in L. Since $P \neq D$, P has the fip in L and so generates a proper filter F in L. Since P is closed under finite meets, $F = \mathfrak{f}(P)$ (3.13.(a)) is given by
$$F = \{x \in L : \exists\, y \in P \text{ such that } y \leq x\}.$$

It follows that $F \cap I = \emptyset$. By Theorem 4.24, there is a prime filter Q in L such that $F \subseteq Q$ and $Q \cap I = \emptyset$. Quite clearly, $Q \cap D = P$. $\square$

Exercises

4.28. Let $f : X \longrightarrow Y$ be a continuous map of topological spaces. With notation as in 4.6 and for $\langle V, W \rangle \in \Omega(X) \times \Omega(Y)$,
a) Show that $f^*W \subseteq V$ iff $W \cap f(X - V) = \emptyset$.
b) Show that $f_*V = int(Y - f(X - V))$.
c) Prove formula $(*)$ in 4.6. $\square$

4.29. Let X be a topological space and $x \in X$.
a) Let $\widehat{0} : X \longrightarrow \{0\}$ be the constant map with value 0. Determine $\hat{0}_*$ and $\hat{0}^*$.
b) Let $\check{x} : \{0\} \longrightarrow X$ be the continuous map which sends 0 to x. Determine $\check{x}^*$ and $\check{x}_*$. $\square$

4.30. Let L be a lattice. Show that the following are equivalent :
(1) L is a distributive lattice;
(2) The lattice of ideals in L, $\mathcal{I}(L)$, is a distributive lattice;
(3) The lattice of filters in L, $\mathcal{F}(L)$, is a distributive lattice. $\square$

4.31. Let L be a lattice. A filter F in L is **irreducible** if it is proper and for all filters G_1, G_2 in L

$$F = G_1 \cap G_2 \;\Rightarrow\; F = G_1 \text{ or } F = G_2.$$

a) If L is a distributive lattice, a filter is irreducible iff it is prime.
b) Give examples of lattices with non-prime irreducible filters. $\square$

CHAPTER 5

Boolean Algebras

It is customary to introduce algebraic constructions by requiring that certain systems of equations have a solution. If L is a distributive lattice with $\perp$ and $\top$ and b is an element of L, consider the system of equations in one unknown,

$$\begin{cases} x \wedge b = \perp \\ x \vee b = \top \end{cases} \tag{*}$$

The above system may be generalized, for $a \leq b \leq c$ in L, by

$$\begin{cases} x \wedge b = a \\ x \vee b = c \end{cases} \tag{**}$$

We have

LEMMA 5.1. *Let L be a distributive lattice.*
a) If the systems $(*)$ and $(**)$ have a solution in L, then it is unique.
b) The following are equivalent :

(1) *For all $b \in L$, system $(*)$ has a solution in L.*

(2) *For all $a \leq b \leq c$ in L, system $(**)$ has a solution in L.*

PROOF. a) It is enough to verify uniqueness of solutions for systems of the type (**). Fix $a \leq b \leq c$ in L. Note that if z is a solution of (**), then $a \leq z \leq b$. If x and t satisfy (**), then $a \leq (x \wedge t)$ and so

$$x = x \wedge c = x \wedge (t \vee b) = (x \wedge t) \vee (x \wedge b) = (x \wedge t) \vee a = x \wedge t,$$

showing that $x \leq t$. Since the argument is symmetrical, we conclude that $x = t$, as claimed.
b) Clearly, (2) $\Rightarrow$ (1). For the converse, let $\neg\, b$ be the *unique* solution of (*) with parameter b. To get a solution for the system (**), with parameters $a \leq b \leq c$, set $t = a \vee (\neg\, b \wedge c)$; then

$$t \wedge b = [a \vee (\neg\, b \wedge c)] \wedge b = (a \wedge b) \vee (b \wedge \neg\, b \wedge c) = a \vee \perp = a,$$

while

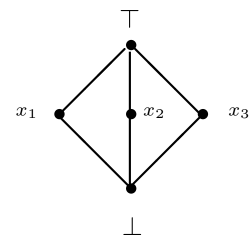
$$\begin{aligned} t \vee b &= a \vee (\neg\, b \wedge c) \vee b = (\neg\, b \wedge c) \vee b = (\neg\, b \vee b) \wedge (c \vee b) \\ &= \top \wedge c = c, \end{aligned}$$

completing the proof. □

REMARK 5.2. A lattice may be distributive and systems of the type (*) might not have solutions for $b \neq \perp, \top$. We shall see a whole category of these, when we treat Heyting algebras in the next chapter. However, just as an example, consider

$L = [0, 1]$, the real unit interval. If $b \neq 0, 1$, then the system (*) with parameter b has no solution in L.

If L is not distributive, it may happen that all systems of type (*) have a solution in L, but *uniqueness is lost*. As an example, consider the lattice $L = \{\perp, x_1, x_2, x_3, \top\}$, where the x_is are unrelated.



If $i = 1, 2, 3$ and j, k are the elements in $\{1, 2, 3\}$ distinct from i, then the system

$$\begin{cases} x \wedge x_i = \perp \\ x \vee x_i = \top \end{cases}$$

has both x_j and x_k as solutions. Another example of this sort is the lattice of closed subspaces of a Hilbert space of dimension greater than 1 (4.3). *We do not know if a lattice in which every system of type* $(*)$ (*or* (**)) *has a* unique *solution must be distributive.* $\square$

Definition 5.3. *Let L be a distributive lattice with $\perp$ and $\top$. An element $b \in L$ is* **complemented or clopen** *iff the system*

$$x \wedge b = \perp \quad \textit{and} \quad x \vee b = \top \tag{$*$}$$

has a solution in L. The unique *solution of this system* (5.1.(*a*)) *is written $\neg b$ and called the* **complement** *of b in L. Write $B(L)$ for the set of clopen (or complemented) elements in L.*

A **Boolean algebra (BA)** *is a distributive lattice with $\perp$ and $\top$ in which every element is complemented. A subset S of a BA B is a a* **sub-Boolean algebra (sub-BA)** *of B iff $\perp, \top \in S$ and S is closed under meets, joins and complementation.*
A sublattice of L is a **sub-BA of L** *iff it is a sub-BA of $B(L)$.*

A **morphism of BAs** *is a morphism in the category of distributive lattices.*

Standard references on Boolean algebras are [**3**], [**54**], [**60**] and [**69**].

It follows from Lemma 5.1.(b) that Boolean algebras are the distributive lattices with $\perp$, $\top$, such that all systems of type (**), have a unique solution.

The class of BAs may be defined equationally by adding a new unary operation, $\neg$, to the lattice operations, and adding the following axioms to those of a distributive lattice with $\perp$ and $\top$:

$*$ $\forall x$ ($x \vee \neg x = \top$ and $x \wedge \neg x = \perp$).
$*$ $\neg \perp = \top$ and $\neg \top = \perp$.

Example 5.4. If X is a set, then $2^X$and $B_\lambda(X)$ are BAs (2.9). In particular, $2 = \{\perp, \top\}$ (subsets of a singleton) is a BA.

If T is a topological space, the lattice of clopens in T (3.6, 4.6), $B(T)$, is precisely $B(\Omega(T))$, and hence a BA. It will be shown latter that all Boolean algebras are isomorphic to $B(X)$, for some topological space X.

The lattice of regular opens in T, $Reg(T)$, is also a BA (1.14) although **not** a sub-BA of $\Omega(T)$. $\square$

Definition 5.5. *If B is a Boolean algebra and $a, b \in B$, define the operation of* **implication** *by the rule*

$$a \to b = \neg\, a \vee b.$$

We also define the operations of **equivalence** *and* **symmetric difference** *by*

$$a \leftrightarrow b = (a \to b) \wedge (b \to a) \qquad \text{and} \qquad a \bigtriangleup b = (a \wedge \neg\, b) \vee (b \wedge \neg\, a),$$

respectively.

Lemma 5.6. *Let $L \xrightarrow{f} K$ be a morphism of distributive lattices and let $x, y \in L$.*

a) If $x \in B(L)$, then $fx \in B(K)$ and $f(\neg\, x) = \neg\, f(x)$.

b) If L is a Boolean algebra, then,

$$\text{i) } f(\neg\, x) = \neg\, f(x); \qquad \text{ii) } f(x \to y) = fx \to fy;$$

$$\text{iii) } f(x \leftrightarrow y) = fx \leftrightarrow fy; \quad \text{iv) } f(x \bigtriangleup y) = fx \bigtriangleup fy.$$

Moreover, f is a BA-morphism from L into $B(K)$.

c) If L is a BA, $f(L)$ is a sub-BA of K.

Proof. a) Since f preserves $\wedge$ and $\vee$, as well as $\perp$ and $\top$, we have

$$\begin{cases} x \wedge \neg\, x = \perp \\ x \vee \neg\, x = \top \end{cases} \quad \text{implies} \quad \begin{cases} fx \wedge f(\neg\, x) = \perp \\ fx \vee f(\neg\, x) = \top \end{cases}$$

which, in view of the distributivity of K and 5.1, shows that $f(\neg\, x)$ is $\neg\, f(x)$. Items (b) and (c) follow from (a) and the definitions of $\to$, $\leftrightarrow$ and $\bigtriangleup$. $\square$

Lemma 5.7. *Let B be a BA and let x, y, z be elements in B. Then,*

a) (The law of double negation) $\neg\neg\, x = x$.

b) $(x \wedge y) \leq z$ *iff* $x \leq (\neg\, y \vee z)$.

c) $x \wedge y = \perp$ *iff* $x \leq \neg\, y$.

d) (The contrapositive law) $x \leq y$ *iff* $\neg\, y \leq \neg\, x$.

e) $\neg\,(x \wedge y) = \neg\, x \vee \neg\, y$; $\neg\,(x \vee y) = \neg\, x \wedge \neg\, y$.

f) For all $S \subseteq B$,

(1) *If $\bigvee S$ exists in B, then both $\bigwedge_{s \in S} \neg\, s$ and $\bigvee_{s\in S} (x \wedge s)$ exist in B and*

$$x \wedge \bigvee S = \bigvee_{s\in S} (x \wedge s) \quad \text{and} \quad \neg \bigvee S = \bigwedge_{s\in S} \neg\, s.$$

(2) *If $\bigwedge S$ exists in B, then both $\bigvee_{s\in S} \neg\, s$ and $\bigwedge_{s\in S} (x \vee s)$ exist in B and*

$$x \vee \bigwedge S = \bigwedge_{s\in S} (x \vee s) \quad \text{and} \quad \neg \bigwedge S = \bigvee_{s\in S} \neg\, s.$$

Proof. a) Follows immediately from the uniqueness of complements.

b) If $x \wedge y \leq z$, then

$$x = x \wedge \top = x \wedge (y \vee \neg\, y) = (x \wedge y) \vee (x \wedge \neg\, y) \leq z \vee (x \wedge \neg\, y)$$
$$= (z \vee x) \wedge (z \vee \neg\, y) \leq z \vee \neg\, y.$$

Conversely, if $x \leq z \vee \neg y$, then

$$x \wedge y \leq (z \vee \neg y) \wedge y = (z \wedge y) \vee (\neg y \wedge y) = (z \wedge y) \vee \perp = z \wedge y \leq z,$$

as desired. Item (c) is an immediate consequence of (b), with $z = \perp$.

d) If $x \leq y$, then

$$x \wedge \neg y \leq y \wedge \neg y = \perp,$$

and hence $x \wedge \neg y = \perp$. But then, (c) implies that $\neg y \leq \neg x$. The converse is similar, using (a).

e) With distributivity, it is straightforward to check that the sated formulas are solutions of the system of equations defining complements.

f) For the first equality in (1), let $b \in B$ satisfy $(x \wedge s) \leq b$, for all $s \in S$. By (b), we get $s \leq \neg x \vee b$. Thus, $\bigvee S \leq \neg x \vee b$; another application of (a) yields $x \wedge \bigvee S \leq b$. Hence, $\bigvee_{s \in S} (x \wedge s)$ exists in B and is equal to $x \wedge \bigvee S$.

For the version of de Morgan's law in (1), let $b \in B$ be such that $b \leq \neg s$, for all $s \in S$. Then, $\forall\, s \in S$, $b \wedge s = \perp$ and so $\bigvee_{s \in S} (b \wedge s) = b \wedge \bigvee S = \perp$. Hence, $b \leq \neg\, (\bigvee S)$. It remains to show that $\neg\, (\bigvee S) \leq \bigwedge_{s \in S} \neg s$. The method is the same as above and will be omitted. The proof of (2) is similar, and in fact, a consequence of (1), by complementation. $\square$

The basic properties of the operation $\triangle$ are described in

LEMMA 5.8. *Let B be a Boolean algebra. For x, y, $z \in B$,*

a) $\langle B, \triangle, \perp \rangle$ is an Abelian group of exponent 2 (i.e., $x \triangle x = \perp$), whose zero is $\perp$.

b) $x \triangle y = \perp$ iff $x = y$; $x \triangle y = \top$ iff $x = \neg y$.

c) $x \wedge (y \triangle z) = (x \wedge y) \triangle (x \wedge z)$.

d) $\langle B, \triangle, \wedge, \perp, \top \rangle$ is a commutative ring with identity $\top$, with $\wedge$ as product and $\triangle$ as addition.

e) $(x \wedge y) \wedge (x \triangle y) = \perp$; $x \triangle y = x \vee y$ iff $x \wedge y = \perp$.

f) $x \vee y = x \triangle y \triangle (x \wedge y) = (x \triangle y) \vee (x \wedge y)$.

g) $(y \triangle z) \leq x$ iff $x \vee y = x \vee z$.

PROOF. Items (a) through (d) are left to the reader.

e) For the first equation, we compute as follows, recalling (b) and (c) :

$$(x \wedge y) \wedge (x \triangle y) = (x \wedge y \wedge x) \triangle (x \wedge y \wedge y) = (x \wedge y) \triangle (x \wedge y) = \perp.$$

It $x \wedge y = \perp$, then, by 5.7.(c), $x \leq \neg y$ and $y \leq \neg x$. Hence,

$$x \triangle y = (x \wedge \neg y) \vee (y \wedge \neg x) = x \vee y,$$

as claimed. Conversely, if $x \vee y = x \triangle y$, then $x \leq x \triangle y$. Thus, using (c), we get

$$x = x \wedge (x \triangle y) = x \triangle (x \wedge y),$$

and so the cancellation law in the group $\langle B, \triangle, \perp \rangle$ yields $x \wedge y = \perp$, as desired.

f) For a, $b \in B$, we have

$$a = a \wedge \top = a \wedge (b \vee \neg b) = (a \wedge b) \vee (a \wedge \neg b).$$

Hence,

$$\begin{aligned} x \vee y &= [(x \wedge y) \vee (x \wedge \neg y)] \vee [(y \wedge x) \vee (y \wedge \neg x)] \\ &= (x \wedge y) \vee [(x \wedge \neg y) \vee (y \wedge \neg x)] = (x \wedge y) \vee (x \bigtriangleup y) \\ &= (x \wedge y) \bigtriangleup (x \bigtriangleup y), \end{aligned}$$

where the last equality comes from item (e).

g) First assume that $(y \bigtriangleup z) \leq x$. Then,

$$y \bigtriangleup z = x \wedge (y \bigtriangleup z) = (x \wedge y) \bigtriangleup (x \wedge z), \tag{I}$$

and so, since B is an Abelian group under $\bigtriangleup$, (I) is equivalent to

$$y \bigtriangleup (x \wedge y) = z \bigtriangleup (x \wedge z). \tag{II}$$

Hence, item (f) and (II) yield

$$x \vee y = x \bigtriangleup y \bigtriangleup (x \wedge y) = x \bigtriangleup z \bigtriangleup (x \wedge z) = x \vee z.$$

For the converse, assume that $x \vee y = x \vee z$. This equality can be written as

$$x \bigtriangleup y \bigtriangleup (x \wedge y) = x \bigtriangleup z \bigtriangleup (x \wedge z),$$

which upon cancellation of x on both sides, leads us back to (II). But we know that (II) is equivalent to (I), ending the proof. $\square$

The fundamental properties of implication and equivalence are in

LEMMA 5.9. *Let B be a Boolean algebra. For x, y and $z \in B$, the following hold :*

a) $x \leq (y \to z)$ *iff* $x \wedge y \leq z$;

b) $x \to y = max\, \{a \in B : x \wedge a \leq y\}$.

c) $x \leq y$ *iff* $x \to y = \top$; $\neg x = (x \to \bot)$.

d) (modus ponens) $x \wedge (x \to y) \leq y$.

e) (contrapositive) $(x \to y) = (\neg y \to \neg x)$.

f) $x \leq (y \leftrightarrow z)$ *iff* $x \wedge y = x \wedge z$.

PROOF. (a) follows immediately from the definition of $(y \to z)$, while (b), (c) and (d) are simple applications of (a).

e) To show that $(x \to y) \leq (\neg y \to \neg x)$, it is enough to verify, by (a), that

$$\neg y \wedge (x \to y) \leq \neg x,$$

or equivalently, by Lemma 5.7.(c), that

$$x \wedge \neg y \wedge (x \to y) = \bot.$$

But this follows immediately from the law of *modus ponens* (item (d)). Similarly, the reverse inequality is equivalent to

$$x \wedge (\neg y \to \neg x) \leq y. \tag{I}$$

Since $y = \neg\neg y$ (5.7.(a)), (5.7.(c)) implies that (I) is equivalent to

$$\neg y \wedge x \wedge (\neg y \to \neg x) = \bot,$$

which is, once again, a consequence of the law of *modus ponens*.

f) If $x \leq (y \to z) = (y \to z) \wedge (z \to y)$, then (a) yields

$$x \wedge y \leq z \quad \text{and} \quad x \wedge z \leq y,$$

and so $x \wedge y = x \wedge z$. The converse is similar and left to the reader. $\square$

Our next result characterizes filters, ideals and the congruences they generate in terms of the operations of implication, equivalence and symmetric difference.

LEMMA 5.10. *Let B be a Boolean algebra and x, $y \in B$.*
a) A downward directed subset S of B is a filter iff

$$(F) \qquad \begin{cases} \top \in S \quad \textit{and} \\ \forall\, x, y \in B, \quad x,\, (x \to y) \in S \;\Rightarrow\; y \in S. \end{cases}$$

b) An upward directed subset S of B is an ideal iff

$$(I) \qquad \begin{cases} \bot \in S \quad \textit{and} \\ \forall\, x, y \in B, \quad x,\, (x \triangle y) \in S \;\Rightarrow\; y \in S. \end{cases}$$

c) With notation as in 4.9
(1) *If F is a filter in B, then $x \sim_F y$ iff $(x \leftrightarrow y) \in F$.*
(2) *If I is an ideal in B, then $x \sim_I y$ iff $(x \triangle y) \in I$.*

PROOF. a) Suppose that S satisfies (F); if $x \in S$ and $x \leq y$, then, by 5.9.(c), $(x \to y) = \top \in S$, and so, $y \in S$. Moreover, if x, $y \in S$, since it is dd, there is $t \in S$, such that $t \leq (x \wedge y)$. But then, we conclude from what has just been proven, that $(x \wedge y) \in S$, and S is a filter (by 3.10.(a)).

Conversely, if S is a filter, it suffices to show that $x \in S$ and $(x \to y) \in S$, entails $y \in S$. But this follows immediately from the law of *modus ponens* (5.9.(d)). The proof of item (b) is analogous and we omit it.
c) We treat the case of ideals, leaving that of filters to the reader. For $x, y \in B$, we have suppose that $x \sim_I y$. Then, there is $t \in I$, such that

$$x \vee t = t \vee y.$$

By Lemma 5.8.(g), $(x \triangle y) \leq t$, and so $(x \triangle y) \in I$. Conversely, another application of 5.9.(g) yields

$$x \vee (x \triangle y) = y \vee (x \triangle y),$$

showing that $(x \triangle y) \in I$ implies $x \sim_I y$. □

A characterization of BAs among distributive lattices is given by

PROPOSITION 5.11. *Let L be a distributive lattice with $\bot$ and $\top$. Then, L is a BA iff for all $\theta \in Con(L)$, the congruence generated by F_θ is equal to θ.*

PROOF. By Exercise 5.18.(b), any congruence with respect to the lattice operations in a BA is a congruence with respect to $\neg$.

Suppose L is a BA and $\theta \in Con(L)$. By Corollary 4.12, it is enough to prove that $\theta \subseteq \sim_{F_\theta}$. To simplify notation, set

$$F = F_\theta = \{x : x\ \theta\ \top\} \quad \text{and} \quad \alpha = \sim_{F_\theta}.$$

It must be shown that $x\ \theta\ y$ implies $x \sim_F y$. Taking joins with $\neg\, x$ on both sides of $(x\ \theta\ y)$, yields $(\neg\, x \vee x)\ \theta\ (\neg\, x \vee y)$. Thus, $(\neg\, x \vee y)$ is in F. Similarly, we have $(\neg\, y \vee x) \in F$. F being a filter, we conclude

$$(x \leftrightarrow y) = (\neg\, x \vee y) \wedge (\neg\, y \vee x) \in F.$$

It follows from Lemma 5.10.(c) that $x \sim_F y$.

For the converse, we have to find a complement for each $x \in L$. Consider the ideal $I = x^{\leftarrow}$ and the congruence $\sim_I$ on B (4.10). By 5.18.(a), there is a filter $F \subseteq B$ such that

$$\forall\, a, b \in B, \quad a \sim_I b \;\text{ iff }\; a \sim_F b.$$

Since $x \sim_I \bot$, there is $y \in F$, with $x \wedge y = \bot$. Because $y \in F$, we get $y \sim_F \top$ and so $y \sim_I \top$; this means that there is $t \le x$, such that $y \vee t = \top \vee t = \top$. Clearly, $y \vee x = \top$, ending the proof. □

In BAs, primeness and maximality for filters and ideals coincide.

Proposition 5.12. *Let B be a Boolean algebra and let A be a proper filter or ideal in B. The following are equivalent :*
(1) *A is prime.*
(2) *For all $x \in L$, either $x \in A$ or $\neg x \in A$.*
(3) *A is maximal.*

Proof. Suppose that A is a filter in B. As usual, the case of ideals is dual. Since for all $x \in B$, $x \vee \neg x = \top \in A$, it is clear that (1) $\Rightarrow$ (2). Now assume (2) and let G be a proper filter containing A. If $x \in G$, then $\neg x$ cannot be in A, since $x \wedge \neg x = \bot$ and G is proper. Consequently, $G \subseteq A$ and A is maximal. (3) $\Rightarrow$ (1) follows from Lemma 4.22.(b). □

Corollary 5.13. *Let B be a Boolean algebra. There is a natural bijective correspondence between the set of ultrafilters in B and the set of BA morphisms from B to $2 = \{\bot, \top\}$, given by $f \mapsto cokerf$.*

Proof. Left to the reader. □

Another characterization of Boolean algebras is given by

Proposition 5.14. *A distributive lattice with $\bot$ and $\top$ is a Boolean algebra iff every prime ideal is maximal.*

Proof. We already know (5.12) that in a BA a filter is prime iff it is maximal. For the converse, suppose $x \in L$ is not clopen. In particular $x \neq \bot, \top$. Set

$$A = \{z \in L : x \wedge z = \bot\}.$$

Then, A is an ideal and $A \cup \{x\}$ has the fup, otherwise x would have a complement. Let

$$F = \{y \in L : x \vee y = \top\} \quad \text{and} \quad I = \mathfrak{i}(A \cup \{x\})\ ^1.$$

Clearly, F is a filter and I an ideal. To see that $I \cap F = \emptyset$, consider what would happen if there was $t \in F \cap I$: in this case, $x \vee t = \top$ and $t \le x \vee z$, for some $z \in A$. But then

$$\top \le (x \vee t) \le x \vee (x \vee z) = x \vee z,$$

and z would be the complement of x. By Theorem 4.24, there is a prime filter P such that $F \subseteq P$ and $P \cap I = \emptyset$. In particular, $x \notin P$ and $P \cap A = \emptyset$. This last equation means that $P \cup \{x\}$ has the fip; thus the proper filter it generates is strictly larger than P, contradicting its maximality. □

We invite the reader to supply a proof of the following preservation result.

[1] $\mathfrak{i}(*)$ is the ideal generated by $*$, as in 3.12.

Proposition 5.15. *The product, fiber product, and reduced product of Boolean algebras are Boolean algebras.*

The statement of the next result should be compared with 4.27.

Theorem 5.16. *Let $C \subseteq B$ be BAs. For every $b \in B - C$, there is an ultrafilter F in C and ultrafilters F_1, F_2 in B, such that*

(i) $F_i \cap C = F$; *(ii) $b \in F_1$ and $\neg\, b \in F_2$.*

Proof. We freely employ the results in 5.7, without explicit mention.

Since $b \notin C$, the ideal $I = b^{\leftarrow} \cap C$ and the filter $G = b^{\rightarrow} \cap C$ are disjoint in C. By Theorem 4.24 (and 5.12) , there is a ultrafilter F *in* C such that $G \subseteq F$ and $F \cap I = \emptyset$.

We now verify that both $F \cup \{b\}$ and $F \cup \{\neg\, b\}$ have the fip (3.13). If $F \cup \{b\}$ did not have the fip, then for some $a \in F$, $a \wedge b = \bot$. Hence, $b \leq \neg\, a$, and so $\neg\, a \in G \subseteq F$, which is impossible, because F is a proper filter. If $F \cup \{\neg\, b\}$ did not have the fip, then for some $a \in F$, we would have $a \leq b$, untenable, because this implies $a \in I$ and F is disjoint from this ideal, by construction.

By Theorem 4.25, there are ultrafilters F_1, F_2 in B, such that $F \cup \{b\} \subseteq F_1$ and $F \cup \{\neg\, b\} \subseteq F_2$. Since F is maximal in C, it is clear that $F_i \cap C = \mathcal{F}$, ending the proof. ☐

The definitions of monic and epic appear in 16.8.

Proposition 5.17. *Let $A \xrightarrow{f} B$ be a BA morphism. Then,*
a) f is monic iff it is injective iff coker $f = \top$.
b) f is epic iff it is surjective.

Proof. a) is straightforward and left to the reader.
b) The argument here is more sophisticated, needing 5.16. All we have to do is verify that epic implies surjective (Exercise 16.38).

Suppose f is epic and there is $b \in B - f(A)$. By 5.6.(c), $f(A)$ is a sub-BA of B. Let F, F_i, $i = 1, 2$ be the ultrafilters in $f(A)$ and B constructed in 5.16. By 5.13, B/F_i may be identified with $2 = \{\bot, \top\}$. Let

$$B \underset{h_1}{\overset{h_2}{\rightrightarrows}} \{\bot, \top\}$$

be the quotient maps induced by F_1, F_2, respectively. For $a \in A$,

$$\begin{aligned} h_1(f(a)) = \top \ &\text{iff}\ \ f(a) \in F_1 \cap f(A) \ \ \text{iff}\ \ f(a) \in F_2 \cap f(A) \\ &\text{iff}\ \ h_2(f(a)) = \top, \end{aligned}$$

which shows that $h_1 \circ f = h_2 \circ f$. But this contradicts the hypothesis that f is epic, since $h_1(b) = \top$, while $h_2(b) = \bot$, that is, $h_1 \neq h_2$. ☐

Exercises

5.18. Recall (2.32, 3.17) that if L is a lattice, $\mathcal{F}(L)$, $\mathcal{I}(L)$ are the poset of filters and ideals in L. Let B be a Boolean algebra.
a) Show that the map $x \mapsto \neg x$ induces an isomorphism from B onto B^{op} *(which is also a BA)*. Show that the above map induces a natural bijective correspondence between filters and ideals in B as follows

$$F \in \mathcal{F}(B) \quad \mapsto \quad \neg\ F = \{b \in B : \neg b \in F\} \in \mathcal{I}(B).$$

Prove that this correspondence originates, in turn, a natural bijection between the congruences generated by filters and ideals, in such a way that for all filters F in B, B/F is naturally isomorphic to $B/\neg\ F$.
b) Show that any **lattice** congruence on B is also a BA congruence, that is, preserves the operation $\neg$. Conclude that any lattice congruence in a Boolean algebra preserves implication, equivalence and symmetric difference.
c) If F is a filter in a BA B, then B/F is a BA and the canonical projection, $B \xrightarrow{\pi_F} B/F$, is a BA morphism. $\square$

5.19. Let B be a Boolean algebra. Show that the map

$$F \in \mathcal{F}(B) \longmapsto \sim_F \in Con(B)$$

is an *isomorphism* between the lattice of filters in B (3.17.(d)) and the lattice of congruences in B. Conclude that the lattice of filters in a BA is a *distributive* lattice. State and prove a similar result for the lattice of ideals in a BA. $\square$

CHAPTER 6

Heyting Algebras

The concept of Heyting algebra is an algebraic encoding of a constructive theory of implication.

DEFINITION 6.1. *A* **Heyting algebra (HA)** *is a distributive lattice with* $\bot$ *and* $\top$, H, *such that for all* $x, y \in H$

$$max\ \{z \in H : z \wedge x \leq y\} \text{ exists in } H.$$

This element of H *is denoted by* $x \to y$ (x implies y). *Hence, for all* x, y, $z \in H$

$$[\to] \qquad x \wedge z \leq y \quad \text{iff} \quad z \leq (x \to y).$$

If H *and* L *are Heyting algebras, a map* $f : H \longrightarrow L$ *is a* **HA-morphism** *iff* f *is a lattice morphism such that*

$$\forall\ x, y \in H, \quad f(x \to y) = fx \to fy.$$

Heyting algebras and their morphisms form a category, written **HA**.

Standard references on HAs are [**3**], [**60**] and [**59**].

EXAMPLE 6.2. The algebra of opens of a topological space T provides one of the fundamental examples of Heyting algebras. The operation $\to$ is given by

$$U \to V = \mathrm{int}((T - U) \cup V).$$

In this Chapter it is recommended that the reader keep in mind the topological example, together with the results in 1.14, that yield geometrical significance to most of the constructions that follow.

A BA, B, is a Heyting algebra because for all $x, y \in B$, $(\neg\, x \vee y)$ is the largest $z \in B$ such that $z \wedge x \leq y$ (Lemma 5.7.(b)).

Chains or linear orders (with first and last element) are also HAs, where

$$x \to y = \begin{cases} \top & \text{if } x \leq y \\ y & \text{otherwise} \end{cases} \qquad \square$$

REMARK 6.3. The class of HAs is equationally definable by adding a binary operation, $\to$, to those of a lattice and adding, to the set of identities defining distributive lattices with $\bot$ and $\top$, the following rules :

[HA 1] : $x \wedge (x \to y) = x \wedge y$.

[HA 2] : $x \wedge (y \to z) = x \wedge ((x \wedge y) \to (x \wedge z))$.

[HA 3] : $z \wedge ((x \wedge y) \to x) = z$.

The reader is referred to Theorem IX.4.1, pg. 177, of [**3**]. In any case, this follows easily from the results to be presented below. $\square$

Some of the basic properties of Heyting algebras are presented in

LEMMA 6.4. *For x, y, z in a Heyting algebra H,*
a) $x \leq y$ *iff* $x \to y = \top$.
b) (Modus Ponens) $x \wedge (x \to y) \leq y$.
c) $x \leq (x \to y) \to y$ *and* $x \wedge (x \to y) = x \wedge y$.
d) $y \leq z$ *implies* $\begin{cases} (x \to y) \leq (x \to z) \\ \quad\quad \text{and} \\ (z \to x) \leq (y \to x). \end{cases}$
e) $(y \vee z) \to x = (y \to x) \wedge (z \to x)$.
f) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$.
g) $(x \to t) \vee (y \to z) \leq (x \wedge y) \to (t \vee z)$.
h) $x \to (y \to z) = (x \wedge y) \to z$.
i) $x \wedge (y \to z) = x \wedge ((x \wedge y) \to (x \wedge z))$.

PROOF. (a) is clear; (b) comes from the fact that $(x \to y)$ is the largest z in H such that $z \wedge x \leq y$. From (b), the definition of $\to$ will immediately yield the first statement in (c), while, taking the meet with x on both sides of (b), will give us the second.
d) Note that $(x \to y) \leq (x \to z)$ is true iff $x \wedge (x \to y) \leq z$. But this follows immediately from (b) and $y \leq z$. Similarly, we have
$$y \wedge (z \to x) \leq z \wedge (z \to x) \leq x,$$
which is equivalent to the second assertion in (d).
e) It follows directly from (d) that the left side of (e) is less than or equal to its right hand side. For the other inequality, note that
$$\begin{aligned}(y \vee z) \wedge (y \to x) \wedge (z \to x) &= \\ &= [y \wedge (y \to x) \wedge (z \to x)] \vee [z \wedge (y \to x) \wedge (z \to x)] \\ &\leq [x \wedge (z \to x)] \vee [x \wedge (y \to x)] \leq x \vee x = x.\end{aligned}$$
Items (f) and (g) are similar and left to the reader.
h) Applying the definition of $\to$ successively, we get
$$\begin{aligned} t \leq [x \to (y \to z)] \quad &\text{iff} \quad x \wedge t \leq (y \to z) \quad \text{iff} \quad y \wedge x \wedge t \leq z \\ &\text{iff} \quad t \leq [(x \wedge y) \to z], \end{aligned}$$
as needed. Item (i) is left to the reader. □

The reader will surely notice the resemblance of the above with some of the usual laws of Logic.

DEFINITION 6.5. *Let H be a Heyting algebra. For each $x \in H$, write*
$$\neg x =_{def} (x \to \bot),$$
called the **pseudo-complement** *of x in H.*

REMARK 6.6. a) If $H \xrightarrow{f} K$ is a HA-morphism, then f preserves pseudo-complement, that is, $f(\neg x) = \neg fx, \ \forall x \in H$.
b) Note that for x in a HA H,

$$\neg\, x \text{ is the largest } z \text{ in } H \text{ such that } z \wedge x = \bot.$$

In particular, $\neg \bot = \top$ and $\neg \top = \bot$. Observe also that if an element x of H has a complement then it must be equal to $\neg\, x$. Hence, this notation is consistent with that adopted for BAs. In fact,

$$H \textit{ is a BA } \textit{ iff } \textit{ for all } x \in H, \quad x \vee \neg\, x = \top. \qquad (*)$$

i.e., $\neg\, x$ is the complement of x (see Exercise 6.26.(a)). ☐

EXAMPLE 6.7. a) In the HA of opens of a topological space T we have

$$\neg\, U = \mathrm{int}(T - U).$$

It is therefore easy to give examples of HAs of this type, in which the only clopen elements are $\bot$ and $\top$.

b) In a chain, $\neg\, x = \bot$ for all $x \neq \bot$. ☐

The basic rules governing pseudo-complementation are described in

LEMMA 6.8. *For all x, y, t, z in a HA H, we have*

a) $y \leq z \;\Rightarrow\; \neg\, z \leq \neg\, y$.
b) $x \leq \neg\neg\, x$ *and* $x \leq y \;\Rightarrow\; \neg\neg\, x \leq \neg\neg\, y$.
c) $t \wedge x = \bot$ *iff* $t \wedge \neg\neg\, x = \bot$.
d) $\neg\, x = \neg\neg\neg\, x$ *and* $x \leq \neg\, y \;\Rightarrow\; \neg\neg\, x \leq \neg\, y$.
e) $\neg\,(y \vee z) = \neg\, y \wedge \neg\, z$.
f) $\neg\,(x \wedge y) = \neg\neg\,(\neg\, x \vee \neg\, y)$.
g) $\neg\neg\,(x \wedge y) = \neg\neg\, x \wedge \neg\neg\, y$.
h) $(\neg\, x \vee y) \leq (x \rightarrow y) \leq \neg\neg\,(\neg\, x \vee y)$.
i) $\neg\,(x \rightarrow y) = \neg\neg\, x \wedge \neg\, y$.
j) $\neg\neg\,(x \rightarrow y) = \neg\neg\, x \rightarrow \neg\neg\, y$.
k) $\neg\neg\, x \leq \neg\neg\,[(x \wedge \neg\, y) \vee y]$.
l) $\neg\,(x \vee \neg\, x) = \bot$; $\neg\neg\,(x \vee \neg\, x) = \top$.

PROOF. Item (a) comes from the second statement in Lemma 6.4.(d), with $x = \bot$. The first part of (b) is an instance of the first inequality in Lemma 6.4.(c), with $y = \bot$, while the second comes from a double application of (a).

c) Clearly, (b) yields $t \wedge \neg\neg\, x = \bot \;\Rightarrow\; t \wedge x = \bot$. On the other hand,

$$t \wedge x = \bot \;\Rightarrow\; t \leq \neg\, x \;\Rightarrow\; \neg\neg\, x \leq \neg\, t \;\Rightarrow\; t \wedge \neg\neg\, x = \bot.$$

The assertions in (d) are direct consequences of (c). Item (e) comes directly from Lemma 6.4.(e), with $x = \bot$.

f) Lemma 6.4.(g), with $t = z = \bot$, yields $\neg\, x \vee \neg\, y \leq \neg\,(x \wedge y)$; from item (d) above, we get $\neg\neg\,(\neg\, x \vee \neg\, y) \leq \neg\,(x \wedge y)$. On the other hand, (e) and a repeated use of (d) yield that $\neg\,(x \wedge y) \wedge x \wedge y = \bot$ implies

$$\neg\,(x \wedge y) \wedge \neg\neg\, x \wedge \neg\neg\, y = \neg\,(x \wedge y) \wedge \neg\,(\neg\, x \vee \neg\, y) = \bot,$$

showing that $\neg\,(x \wedge y) \leq \neg\,(\neg\, x \vee \neg\, y)$, completing the proof of (f). Item (g) follows easily from (e) and (f).

h) Since $x \wedge (\neg\, x \vee y) = x \wedge y$, it follows that $(\neg\, x \vee y) \leq (x \rightarrow y)$. For the other inequality in (h), we again use (d) and (e) :

$$\begin{aligned} x \wedge \neg y \wedge (x \to y) = \bot \;\Rightarrow\; & \neg\neg x \wedge \neg y \wedge (x \to y) \;=\; \bot \\ \Rightarrow\; & \neg(\neg x \vee y) \wedge (x \to y) \;=\; \bot \\ \Rightarrow\; & (x \to y) \leq \neg\neg(\neg x \vee y). \end{aligned}$$

Item (i) is an immediate consequence of (a), (c) and (h).

j) By (g), (b) and *Modus Ponens* (6.4.(b)), we have

$$\neg\neg x \wedge \neg\neg(x \to y) = \neg\neg(x \wedge (x \to y)) \leq \neg\neg y,$$

and the adjunction $[\to]$ in 6.1 yields $\neg\neg(x \to y) \leq \neg\neg x \to \neg\neg y$. For the reverse inequality, from item (i) and *Modus Ponens* comes

$$\begin{aligned} (\neg\neg x \to \neg\neg y) \wedge \neg(x \to y) &= (\neg\neg x \to \neg\neg y) \wedge \neg\neg x \wedge \neg y \\ &\leq \neg\neg y \wedge \neg y = \bot, \end{aligned}$$

and so, by (a), $(\neg\neg x \to \neg\neg y) \leq \neg\neg(x \to y)$, as needed. Items (k) and (l) are left to the reader. $\square$

DEFINITION 6.9. *If H is a HA, define, for $x, y \in H$,*

$$x \leftrightarrow y = (x \to y) \wedge (y \to x),$$

called **the equivalence operation** *in H.*

LEMMA 6.10. *For x, y, z in a HA H,*

a) $z \leq (x \leftrightarrow y)$ iff $x \wedge z = y \wedge z$.

b) $(x \leftrightarrow y) = max\,\{z \in H : z \wedge x = z \wedge y\}$. $\square$

PROPOSITION 6.11. *If L is a distributive lattice, then $\mathcal{F}(L)$ and $\mathcal{I}(L)$* [1] *are Heyting algebras.*

PROOF. Write $\mathcal{F}$ for $\mathcal{F}(L)$. We shall prove the result for $\mathcal{F}$, leaving the corresponding statement for $\mathcal{I}(L)$ to the reader. We start by

Fact. *$\mathcal{F}$ is a distributive lattice.*

Proof. It is sufficient to verify that for $F, K, G \in \mathcal{F}$,

$$F \cap (G \vee K) \subseteq (F \cap G) \vee (F \cap K).$$

Recall that $F \vee G$ is the filter generated by $F \cup G$, as well as that in $\mathcal{F}$, $\bot = \{\top\}$ and $\top = L$.

If $x \in (G \vee K)$, by Lemma 3.13.(a), there are $A \subseteq_f G$ and $B \subseteq_f K$, such that $x \geq \bigwedge A \wedge \bigwedge B$. Thus,

$$x = (x \vee \bigwedge A) \wedge (x \vee \bigwedge B) = [\textstyle\bigwedge_{a \in A} (x \vee a)] \wedge [\bigwedge_{b \in B} (x \vee b)]. \tag{I}$$

If $x \in F$, then $(x \vee a) \in (F \cap G)$, $\forall\, a \in A$. Similarly, if $b \in B$, then $(x \vee b) \in (F \cap K)$. Hence, (I) expresses x as a meet of elements in $(F \cap G) \cup (F \cap K)$, as needed.

The same argument will prove something stronger, namely, that for all $F \in \mathcal{F}$ and $\{G_i : i \in I\} \subseteq \mathcal{F}$,

$$F \cap (\bigvee G_i) = \bigvee (F \cap G_i).$$

To finish the proof, observe that for $F, G, K \in \mathcal{F}$,

$$F \cap K \subseteq G \;\;\Rightarrow\;\; \text{For all } k \in K \text{ and } x \in F, \;\; (x \vee k) \in G.$$

[1] The lattice of filters and ideals in L, as in 2.32 and 3.17.(d).

The set of all $z \in L$ that satisfy this property is easily seen to be a filter in L, that is,

$$F \to G = \{z \in L : \forall\, x \in F,\ (x \vee z) \in G\},$$

and $\mathcal{F}$ is indeed a HA. $\square$

REMARK 6.12. Let L be a distributive lattice. With notation as in Proposition 6.11, if $F \in \mathcal{F}$, then

$$\neg\, F = \{z \in L : \forall\, x \in F, \quad (x \vee z) = \top\}.$$

In the case of $\mathcal{I}(L)$, we have, for $I, J \in \mathcal{I}(L)$

$$\begin{cases} I \to J &= \{z \in L : \exists\, x \in I \text{ such that } x \wedge z \in J\}; \\ \neg\, I &= \{z \in L : \exists x \in I \text{ such that } (x \wedge z) = \bot\}. \end{cases}$$

The lattice of congruences of any lattice is also a HA. We defer the proof of this fact to a later chapter. $\square$

EXAMPLE 6.13. a) If B is a BA and $x, y \in B$, then

$$(x \leftrightarrow y) = (\neg\, x \vee y) \wedge (\neg\, y \vee x) = (x \wedge y) \vee (\neg\, x \wedge \neg\, y).$$

Hence, $(x \leftrightarrow y) = \neg\,(x \bigtriangleup y) = \top \bigtriangleup (x \bigtriangleup y)$, where $\bigtriangleup$ is symmetric difference (5.5).
b) If T is a topological space and $U, V \in \Omega(T)$, then

$$\begin{aligned}(U \leftrightarrow V) &= \mathrm{int}((T - U) \cup V) \cap \mathrm{int}((T - V) \cup U) \\ &= \mathrm{int}\ [(U \cap V) \cup ((T - U) \cap (T - V))].\end{aligned}$$

Thus, $\leftrightarrow$ in $\Omega(T)$ is the interior of $\leftrightarrow$ in 2^T. $\square$

EXAMPLE 6.14. a) If $f : X \longrightarrow Y$ is a continuous map of topological spaces X, Y, in general, $f^* : \Omega(Y) \longrightarrow \Omega(X)$ (4.6) will **not** be a HA-morphism. The same counterexample presented in 4.6 for the non-preservation of regular opens, shows that $\neg$ need not be preserved by f^*. This will, in fact, lead to a change in the concept of morphism when we deal with frames [2]. Nevertheless, the definition of HA-morphism given above has enough importance to merit discussion. It is interesting to find conditions on f such that f^* is a HA-morphism. One such is that f be open.
b) If C and L are chains, then $f : C \longrightarrow L$ is a HA-morphism iff f is strictly increasing on $C -$ coker f. $\square$

PROPOSITION 6.15. *Let H be a HA and F be a filter in H.*
a) The equivalence relation $\sim_F$ generated by F is a congruence with respect to all the operations on H. Furthermore, for $x, y \in H$

$$x \sim_F y \quad \textit{iff} \quad (x \leftrightarrow y) \in F.$$

b) The following are equivalent :
(1) *F is an ultrafilter;*
(2) *F is prime and for all $x \in H$, $\neg\neg\, x \in F \ \Rightarrow\ x \in F$;*
(3) *For all $x \in H$, either $x \in F$ or $\neg\, x \in F$.*
c) If θ is a Heyting algebra congruence on H, then $\sim_{F_\theta} = \theta$. The mapping

[2] Called *complete Heyting algebras* in [**15**] and complete pseudo-Boolean algebras in [**60**].

$$F \in \mathcal{F}(H) \;\;\mapsto\;\; \sim_F \in Con(H)$$

is an isomorphism from $\mathcal{F}(H)$ onto $Con(H)$, the lattice of **HA congruences** *on H.*

d) If f is a HA-morphism, then f is injective iff coker $f = \{\top\}$.

Proof. a) We first verify that

$$x \sim_F y \;\text{ iff }\; (x \leftrightarrow y) \in F.$$

If $x \sim_F y$, then, there is $z \in F$ such that $x \wedge z = y \wedge z$. By 6.10.(a), $z \leq (x \leftrightarrow y)$, and so $(x \leftrightarrow y) \in F$. The converse is consequence of 6.10.(b), that is,

$$x \wedge (x \leftrightarrow y) = y \wedge (x \leftrightarrow y).$$

To complete the proof of (a), it is sufficient to show that

$$x \sim_F y \;\text{ and }\; a \sim_F b \;\Rightarrow\; (x \rightarrow y) \sim_F (a \rightarrow b).$$

Using the result just proven and Exercise 6.23.(b), we get

$$z = [(x \leftrightarrow y) \wedge (a \leftrightarrow b)] \in F \quad\text{and}\quad z \wedge (x \rightarrow y) = z \wedge (a \rightarrow b),$$

and so $\sim_F$ is a congruence with respect to $\rightarrow$.

b) $\underline{(1) \Rightarrow (2)}$: By 4.22, every ultrafilter is prime. If $x \notin F$, then the filter generated by F and $\{x\}$ cannot be proper. By 3.13.(a), there is $a \in F$ such that $a \wedge x = \bot$, that is, $a \leq \neg x$. But then, $\neg x \in F$ and F is not a proper filter. Hence, $x \in F$, as desired.

$\underline{(2) \Rightarrow (3)}$: By 6.8.(l), for all $x \in H$, $\neg\neg(x \vee \neg x) = \top \in F$; therefore, (2) entails $(x \vee \neg x) \in F$ and primeness yields the alternative in (3).

$\underline{(3) \Rightarrow (2)}$: The proof is the same as that for BAs, in 5.12.

c) By 4.12.(c), it is enough to verify that $\theta \subseteq \sim_{F_\theta}$. If $x \;\theta\; y$, applying $(x \rightarrow \cdot)$ to both sides of this relation yields $\top \;\theta\; (x \rightarrow y)$. Similarly, we can show $\top \;\theta\; (y \rightarrow x)$. Hence, $(x \leftrightarrow y) \in F_\theta$, which entails, by item (a), $x \sim_{F_\theta} y$. The remainder of (c), as well as (d), are left to the reader. □

Corollary 6.16. *If F is a filter in a HA H, the quotient H/F is a Heyting algebra and the quotient morphism, $\pi_F : H \longrightarrow H/F$ is a HA-morphism.*

Remark 6.17. a) Note that 6.15.(c) refers only, – as it must –, to *HA-congruences*. In 4.16 there is an example of a *lattice congruence* on the HA $\Omega(T)$ which does not come from a filter or ideal in $\Omega(T)$. By 6.15.(c), that congruence is not a *HA-congruence* on $\Omega(T)$, something that can be checked directly.

Since Proposition 5.14 guarantees that a distributive lattice L is a BA iff all *lattice congruences* on L come from filters, and $\Omega(T)$ *is not* a BA, such examples were certain to exist.

b) In general, the equivalence relation determined by ideals in a HA **will not** preserve $\rightarrow$, i.e., $\sim_I$ is not a HA-congruence. As an example, consider the ideal $I = (0, \infty)^{\leftarrow}$, generated by the open positive axis in the HA of opens of the real line, $\Omega(\mathbb{R})$. Clearly, $(0, \infty) \sim_I \bot$; but $\neg(0, \infty) = (-\infty, 0)$ is not equivalent to $\top$, since there is no $U \in I$ such that $U \cup (-\infty, 0) = \mathbb{R}$. Hence, 6.16 is false for ideals. □

Example 6.18. Let $H = [0, 1]$ be the real unit interval.

a) $D_H = (0, 1]$, because for all $x \neq 0$ in H, $\neg x = 0$.
b) The filters on H are the principal filters and the intervals $(x, \; 1]$, $x \neq 0$, all prime filters. But the only ultrafilter is $D_H = (0, 1]$. This very simple example already displays an important fact : in Heyting algebras, prime filters have greater significance than ultrafilters. $\square$

The next theme is the discussion of a special type of filter in HAs, that will lead to a connection between Heyting and Boolean algebras. The analogy with topological spaces is at the root of

DEFINITION 6.19. *Let H be a Heyting algebra and let x, y be elements of H.*
a) x **is dense in** *y iff $x \leq y \leq \neg\neg x$. Write $D(y)$ for the set of elements dense in y.*
b) x is **dense** *iff $\neg\neg x = \top$. Write D_H for the set of dense elements of H. When H is clear from context it will be omitted from the notation.*
c) x is **regular** *iff $\neg\neg x = x$. Write $Reg(H)$ is the set of regular elements in H.*

PROPOSITION 6.20. *Let H be a HA and $x, y \in H$.*
a) $(x \vee \neg x) \in D$.
b) If $y \in Reg(H)$, then $\begin{cases} x \to y = \neg\neg x \to y; \\ (x \to y) \in Reg(H). \end{cases}$
c) D is a proper filter in H and $Reg(H)$ is closed under meets. Moreover, for all $x, y \in H$, $x \sim_D y$ iff $\neg\neg x = \neg\neg y$.
d) A prime filter in H is maximal iff it contains D.
e) F and G are ultrafilters in H, then
$$F = G \quad \text{iff} \quad F \cap Reg(H) = G \cap Reg(H).$$
f) If $H \xrightarrow{f} P$ is a HA-morphism, then
$$f(D_H) \subseteq D_P \quad \text{and} \quad f(Reg(H)) \subseteq Reg(P).$$

PROOF. Item (a) is a consequence of 6.8.(l).
b) To prove the equality in the statement, it is enough, by 6.4.(d), to show that $(x \to y) \leq (\neg\neg x \to y)$. By the fundamental adjunction $[\to]$ in 6.1, this means
$$\neg\neg x \wedge (x \to y) \leq y. \tag{I}$$
But we have, recalling 6.8.(j) and *Modus Ponens*
$$\begin{aligned} \neg\neg x \wedge (x \to y) &\leq \neg\neg x \wedge \neg\neg (x \to y) = \neg\neg x \wedge (\neg\neg x \to \neg\neg y) \\ &\leq \neg\neg y = y, \end{aligned}$$
establishing (I), as needed. It now clear that if $y \in Reg(H)$, then $\neg\neg (x \to y) = x \to y$, completing the proof of (b).
c) The first assertion in (c) is an immediate consequence of the fact that $\perp \notin D$ and 6.8.(g). For $x, y \in H$, by 6.15.(a) and 6.8.(g)
$$\begin{aligned} x \sim_D y \quad &\text{iff} \quad (x \leftrightarrow y) \in D \quad \text{iff} \quad \neg\neg (x \leftrightarrow y) = \top = (\neg\neg x \leftrightarrow \neg\neg y) \\ &\text{iff} \quad \neg\neg x = \neg\neg y, \end{aligned}$$
as desired. Items (d) and (e) follow from 6.15.(b), while (f) is straightforward. $\square$

Proposition 6.21. *If H is a HA, then H/D is a BA, satisfying the following universal property :*

For all BAs B and all HA-morphisms $H \xrightarrow{f} B$, there is a unique HA-morphism, $H/D \xrightarrow{g} B$, such that $g \circ \pi_D = f$.

$$\begin{tikzcd} H \arrow[r, "\pi_D"] \arrow[d, "f"'] & H/D \arrow[dl, "g"] \\ B & \end{tikzcd}$$

Moreover, the map

$$u \in Reg(H) \mapsto u/D \in H/D$$

is a BA-isomorphism of $Reg(H)$ (6.24) onto H/D.

Proof. We already know that H/D is a HA. Note that for $x \in H$

$$x/D \vee \neg x/D = (x \vee \neg x)/D = \top/D,$$

because $\neg\neg(x \vee \neg x) = \top$ (6.20.(a) and (c)). It now follows from (*) in Remark 6.6.(b) that H/D is a Boolean algebra. If $f : H \longrightarrow B$ is a HA-morphism and $x \in D$, then

$$f(x) = \neg\neg f(x) = f(\neg\neg x) = f(\top) = \top,$$

and so $D \subseteq$ coker f. By 4.13, there is a unique *lattice* morphism, $g : L/D \longrightarrow B$, making the displayed diagram commute. It now follows from 5.6.(b) that g is a BA-morphism. The remaining assertions are left to the reader. $\Box$

The preceding results yield

Corollary 6.22. *Let H be a HA. With notation as in* 6.21*, there are natural bijective correspondences between*

(1) *The ultrafilters in H/D and the ultrafilters in H, given by $F \mapsto \pi_D^{-1}(F)$.*

(2) *The HA-morphisms from H into $2 = \{\bot, \top\}$, and the ultrafilters in H, given by $f \mapsto cokerf$.*

Exercises

6.23. If $*$ is any of the binary operations in a HA H, then, for $x, y, a, b \in H$,

a) $(a * x) \wedge (x \leftrightarrow y) = (a * y) \wedge (x \leftrightarrow y)$.

b) $(a * x) \wedge (x \leftrightarrow y) \wedge (a \leftrightarrow b) = (b * y) \wedge (x \leftrightarrow y) \wedge (a \leftrightarrow b)$. $\Box$

6.24. If H is a HA, define an operation $\vee^*$ on H by

$$a \vee^* b = \neg\neg(\neg\neg a \vee \neg\neg b) = \neg\neg(a \vee b).$$

a) $Reg(H)$ is closed under the operation $\vee^*$ and for $a, b \in Reg(H)$,

$$a \vee^* b = \sup \{a, b\} \text{ in the poset } Reg(H).$$

b) Show that $Reg(H)$ is a BA (the BA of regular elements in H.). □

6.25. Let H be a HA and S be a subset of H. Show that

a) If $\bigvee S$ exists in H, then so does $\bigwedge_{s \in S} \neg s$ and $\neg(\bigvee S) = \bigwedge_{s \in S} \neg s$.

b) Investigate what happens in the case dual to (a). □

6.26. Let H be a HA.

a) The following are equivalent :

(1) H is a BA.
(2) For all $x \in H$, $x \vee \neg x = \top$.
(3) For all $x \in H$, $\neg\neg x = x$.
(4) The only dense element in H is $\top$.
(5) Every prime filter in H is maximal.
(6) For all $x, y \in H$, $(x \to y) = \neg x \vee$ y.

b) The following are equivalent :

(1) For all $x, y \in H$, $\neg(x \wedge y) = \neg x \vee \neg y$.
(2) For all $x \in H$, $\neg x \vee \neg\neg x = \top$. □

CHAPTER 7

Complete Lattices

One could view lattices as **finitely complete** posets. We now introduce their complete counterparts.

Definition 7.1. *A partially ordered set L is* **complete (or a complete lattice)** *if it satisfies the following equivalent conditions :*
$[\bigvee]$: *For all $S \subseteq L$, $\bigvee S$ exists in L;*
$[\bigwedge]$: *For all $S \subseteq L$, $\bigwedge S$ exists in L.*

A subset S of a complete lattice L is a **subbasis** *for L iff every element of L is the join of* **finite** *meets of elements of S. S is a* **basis** *for L if every element of L is the join of elements of S.*

Note that all complete lattices have $\perp$ and $\top$. Among the examples in Chapter 2, the reader will find many complete lattices. Of special importance here are the distributive ones, particularly the algebra of opens of a topological space.

Remark 7.2. Let X be a set and $S \subseteq 2^X$. In view of Lemma 1.8,
(i) S is a subbasis for $\tau(S)$, the topology generated by S on X.
(ii) S is a basis for a topology on X iff every element of $\mathfrak{B}(S)$ is the union of elements of S. In particular, this holds whenever S is *closed under finite intersections*, in which case $\mathfrak{B}(S) = S$. ☐

Let $\mathcal{A}$ be a class of algebras, in which, among others, are defined the (possibly infinitary) **partial** operations $\omega_1, \ldots, \omega_n$. If A, $B \in \mathcal{A}$ and $f : A \longrightarrow B$ is a map, we say that

$$f \text{ is a } [\omega_1, \ldots, \omega_n]\text{-morphism}$$

iff f preserves all the operations $\omega_1, \ldots, \omega_n$, that is, for all $1 \leq j \leq n$, if ω_j is defined in $\overline{a} = \langle\, a_\alpha \,\rangle_{\alpha \in \lambda} \in A^\lambda$ (λ a cardinal), then ω_j is defined in $f(\overline{a}) = \langle\, f(a_\alpha) \,\rangle_{\alpha \in \lambda}$ and

$$f(\omega_j(\overline{a})) = \omega_j(f(\overline{a})).$$

Definition 7.3. *Let L and P be posets. Let $f : L \longrightarrow P$ be a* **morphism of posets**. *Then,*
a) f is a $\bigvee$**-morphism** *iff for all $S \subseteq L$,*

$$\bigvee S \text{ exists in } L \;\Rightarrow\; \bigvee f(S) \text{ exists in } P \text{ and } f(\bigvee S) = \bigvee f(S).$$

We may also write that f preserves joins.
b) f is a $\bigwedge$**-morphism** *iff for all $S \subseteq L$,*

$$\bigwedge S \text{ exists in } L \;\Rightarrow\; \bigwedge f(S) \text{ exists in } P \text{ and } f(\bigwedge S) = \bigwedge f(S).$$

We may also write that f preserves meets.

c) f is a **$[\wedge, \bigvee]$-morphism** *iff f preserves finite meets and all joins.*
d) f is a **complete morphism** *iff f is a $[\bigvee, \bigwedge]$-morphism.*
e) f is a **regular embedding** *iff f is an injective $[\bigvee, \bigwedge]$-morphism.*

A sublattice $L \subseteq P$ is a **regular sublattice** *if the canonical map from L to P is a regular embedding.*

Note that all the notions of morphism introduced above are closed under composition.

Remark 7.4. In the definition of maps preserving infinitary operations, it is not assumed that the posets involved are complete. In particular, for a sublattice to be a regular sublattice, it is necessary and sufficient that any joins or meets existing in the sublattice also exist and be the same in the larger one. As an example, consider the rationals naturally embedded in $\mathbb{R}$. This will be important when we discuss completions. $\square$

One of the basic tools in constructing complete lattices is

Theorem 7.5. (Tarski) *Let $f : L \longrightarrow L$ be a* **poset** *morphism, with L a complete lattice. Then, the set of fixed points of f,*

$$Fix(f) = \{x \in L : fx = x\},$$

with the po induced by L, is a complete lattice. In particular, $Fix(f) \neq \emptyset$. Moreover,
a) If f satisfies $\forall\ x \in L,\ fx \geq x$, then the meet of all $S \subseteq Fix(f)$ are the same as in L, that is, the natural injection of $Fix(f)$ into L is a $\bigwedge$-morphism.
b) If f satisfies $\forall\ x \in L,\ fx \leq x$, then the join of all $S \subseteq Fix(f)$ are the same as in L, that is, the natural injection of $Fix(f)$ into L is a $\bigvee$-morphism.

Proof. Let $S \subseteq Fix(f)$; we show that $\bigwedge S$ exists **in** $Fix(f)$, in the partial order induced by L. Consider

$$T = \{p \in L : \forall\ s \in S\ (p \leq s \quad \text{and} \quad p \leq fp)\}.$$

It is straightforward to check that for $p \in T$, $fp \in T$. It is also clear that $S \subseteq T$.

Let $q = \bigvee T$, this join taken in L. Note that

$$x \in Fix(f) \text{ and } x \leq s, \text{ for all } s \in S \ \ \Rightarrow \ \ x \in T.$$

Therefore, to verify that $q = \bigwedge S$ in Fix(f), it is enough to show that $q \in Fix(f)$. Since $p \leq s$, for all $\langle p, s \rangle \in T \times S$, q is a lower bound for S. Furthermore, for each $p \in T$, $p \leq q$ and so $p \leq fp \leq fq$. This means that fq is an upper bound for T and thus, $q \leq fq$. We have shown that $q \in T$; but then $fq \in T$ and so $fq \leq q$, proving that $q \in Fix(f)$.

Now suppose $fx \geq x$, $\forall\ x \in L$. Let $S \subseteq Fix(f)$ and $t = \bigwedge S$, this meet taken in L. If t is proven to be a fixed point of f, then t will also be the meet of S in $Fix(f)$, as desired. Since f is increasing, we have $ft \leq fs = s$, for all $s \in S$. Thus, ft is a lower bound for S in L. Consequently, $ft \leq t = \bigwedge S$, and so $ft = t$. The case $fx \leq x$ is handled similarly. This ends the proof. $\square$

Example 7.6. In general, $Fix(f)$ **is not** a sublattice of L. One simple reason is that $\perp$ and/or $\top$ might not be in their proper places. We now discuss slightly more elaborate examples.

If T is a topological space we defined in section 1.2 two increasing maps of 2^T into itself, given by $A \mapsto \text{int } A$ and $A \mapsto \overline{A}$, whose basic properties are described in 1.10. Notice that the fixed points of the first are the open sets in T, while the fixed points of the second are the closed sets in T; in both cases, the lattice of fixed points is a sublattice of 2^T with arbitrary unions of opens being open and arbitrary intersections of closed sets being closed, illustrating the parts (b) and (c) of Theorem 7.5.

Now consider $f,\, g : 2^T \longrightarrow 2^T$ given by

$$fA = \overline{\text{int } A} \quad \text{and} \quad gA = \text{int } \overline{A}.$$

Both maps are increasing, with $Fix(f)$ corresponding to the **regular closed sets** i.e., closed sets F such that $F = \overline{\text{int } F}$ and $Fix(g)$ corresponding to the lattice of **regular opens** in T (3.6). If $T = \mathbb{R}$, with its usual topology, then finite meets in $Fix(f)$ and finite joins in $Fix(g)$ are not the same as in $2^{\mathbb{R}}$. Simple examples may be obtained by considering contiguous intervals, for instance [0, 1] and [1, 2] in $Fix(f)$ and (0, 1), (1, 2) in $Fix(g)$. Although $Fix(f)$ is a sub-$\vee$-semilattice and $Fix(g)$ is a sub-semilattice of $2^{\mathbb{R}}$, neither is a **sublattice** of $2^{\mathbb{R}}$.

To construct an example such that neither finite meets nor finite joins are preserved, it is sufficient to consider

$$f \times g : 2^{\mathbb{R}} \times 2^{\mathbb{R}} \longrightarrow 2^{\mathbb{R}} \times 2^{\mathbb{R}}, \text{ given by } (A, B) \mapsto (fA,\, gB),$$

where the product lattice has its natural coordinatewise structure. Clearly, $f \times g$ is increasing and $Fix(f \times g) = Fix(f) \times Fix(g)$. Thus, finite meets and joins are not those induced by $2^{\mathbb{R}} \times 2^{\mathbb{R}}$. □

Theorem 7.5 has many applications. Exercise 7.11 indicates two of these.

In complete lattices, a very general form of associativity holds for both meets and joins, the proof of which is left to the reader :

LEMMA 7.7. *Let L be a complete lattice and S_i, $i \in I$, be a family of subsets of L. Then*
a) $\bigvee_{i \in I} (\bigvee S_i) = \bigvee \{\bigvee_{i \in I} s(i) : s \in \prod_{i \in I} S_i\}$.
b) $\bigwedge_{i \in I} (\bigwedge S_i) = \bigwedge \{\bigwedge_{i \in I} s(i) : s \in \prod_{i \in I} S_i\}$. □

With the infinitary versions of the distributive laws, the situation is quite distinct. In Chapter 8, we discuss this situation in more detail (Proposition 8.7).

We now generalize what was done in Example 4.6.

THEOREM 7.8. *Let L and P be complete lattices. There is a natural bijective correspondence between $\bigvee$-morphisms, $L \xrightarrow{f} P$, and $\bigwedge$-morphisms, $P \xrightarrow{g} L$, defined by the rule :*

$$[ad] \qquad \forall\, u \in L \text{ and } \forall\, v \in P, \quad fu \le v \quad \text{iff} \quad u \le gv.$$

In particular,
a) $f \circ g \le Id_P$ *and* $g \circ f \ge Id_L$.
b) $g \circ f \circ g = g$ *and* $f \circ g \circ f = f$.

PROOF. If $g : P \longrightarrow L$ is a $\bigwedge$-morphism, define $f : L \longrightarrow P$ by

$$fu = \bigwedge \{v \in P : u \leq gv\}.$$

It is immediate that f is increasing and that $u \leq gv \;\Rightarrow\; fu \leq v$. Assume now that $fu \leq v$. Since g is increasing, $g(fu) \leq gv$, while, the fact that g preserves $\bigwedge$ yields

$$g(fu) = g(\bigwedge \{v \in P : u \leq gv\}) = \bigwedge \{gv : u \leq gv\} \geq u.$$

Thus, $u \leq g(fu) \leq gv$ and $[ad]$ is verified.

To prove that f is a $\bigvee$-morphism, let $S \subseteq L$. Since f is increasing we have $\bigvee_{s\in S} fs \leq f(\bigvee S)$. To show equality, it is enough to verify that for $v \in P$, if $fs \leq v$, $\forall\, s \in S$, then $v \geq f(\bigvee S)$. This can be obtained from [ad] as follows :

$$\forall\, s \in S\ (fs \leq v) \;\Rightarrow\; \forall s \in S\ (s \leq gv) \;\Rightarrow\; \bigvee S \leq gv,$$

and so $f(\bigvee S) \leq v$, as desired. Given a $\bigvee$-morphism, $f : L \longrightarrow P$, define $g : P \longrightarrow L$ by dualizing what was done above :

$$gv = \bigvee \{u \in L : fu \leq v\}.$$

Analogous arguments will show that [ad] holds and that g is a $\bigwedge$-morphism. It is straightforward to show that the property [ad] uniquely determines the pair $\langle f, g \rangle$, as well as that it implies items (a) and (b) in the statement. $\square$

A pair $(f,\, g)$ as in Theorem 7.8 will be called an **adjoint pair** with f the **left** adjoint of g and g the **right** adjoint of f.

COROLLARY 7.9. *Let $f : L \longrightarrow P$ and $g : P \longrightarrow L$ be an adjoint pair of maps between complete lattices, with f a $\bigvee$-morphism and left adjoint to g.*
a) The following conditions are equivalent :

(1) *f is onto;* (2) *g is a* **section** *for f, that is, $f \circ g = Id_P$;*

(3) *g is injective;* (4) *$\forall\, v \in P$, $g(v) = max\, f^{-1}(v)$.*

b) The following conditions are equivalent :

(1) *f is injective;* (2) *f is a* **section** *for g, that is $g \circ f = Id_L$;*

(3) *g is onto;* (4) *$\forall\, u \in L$, $f(u) = min\, g^{-1}(u)$.*

PROOF. We give a proof of the equivalence in (a), leaving the dual (b) as an exercise. Recall that the following adjointness condition is verified :

[ad] $$\forall\, (u, v) \in L \times P, \quad fu \leq v \;\text{ iff }\; u \leq gv,$$

as well as relations (a) and (b) in the statement of Theorem 7.8.

For (1) $\Rightarrow$ (2), note that Theorem 7.8.(b) and the fact that f is onto yield, for $v = fu$,

$$f(g(v)) = f(g(f(u))) = fu = v,$$

as needed. (2) $\Rightarrow$ (3) is clear. For (3) $\Rightarrow$ (4), we have, for $v \in P$, $g(f(g(v))) = gv$ (7.8.(b)); the injectivity of g implies $f(g(v)) = v$ and so $gv \in f^{-1}(v)$. It follows directly from [ad] that gv is the largest element in $f^{-1}(v)$. (4) $\Rightarrow$ (1) is immediate and the proof is ended. $\square$

REMARK 7.10. Although in important cases (e.g., Example 6.2) the $\bigvee$-morphism f in Theorem 7.8 is a $[\wedge, \bigvee]$-morphism, g will **not** be a $[\vee, \bigwedge]$-morphism,

that is, g will not be a **lattice** morphism (it is always increasing or a poset morphism). As an example, consider the absolute value function $av : \mathbb{R} \longrightarrow \mathbb{R}_+$. With notation as in Example 6.2 and using the formula given therein for av_*, it is easily checked that
(i) $av_*((0, \infty)) = av_*((-\infty, 0)) = \emptyset$, while (ii) $av_*(\mathbb{R} - \{0\}) = \mathbb{R}_+ - \{0\}$. $\square$

Most authors register the fact that Theorem 7.8 is a consequence of the Adjoint Functor Theorem (see Theorem 16.35).

In [**17**] adjoint pairs are called **Galois connection**; [**17**] contains interesting applications of adjoint pairs, as well as references to the literature.

Exercises

7.11. Show that Theorem 7.5 implies the Cantor-Bernstein Theorem and the possibility of defining functions by transfinite induction over well ordered sets. $\square$

7.12. This exercise consists in showing that the conditions in Theorem 7.8 are equivalent.

Let $f : L \longrightarrow P$ and $g : P \longrightarrow L$ be a $\bigvee$-morphism and a $\bigwedge$-morphism of complete lattices, respectively. The following are equivalent :

(1) $\langle f, g \rangle$ is an adjoint pair;
(2) $f \circ g \leq Id_P$ and $g \circ f \geq Id_L$;
(3) $g \circ f \circ g = g$ and $f \circ g \circ f = f$. $\square$

CHAPTER 8

Frames

We now introduce the algebraic constructs that are fundamental in all that follows.

DEFINITION 8.1. *A complete lattice, L, is a* **frame** *if it satisfies the following distributive law :*

$[\wedge, \bigvee]$ *For all $S \subseteq L$ and all $x \in L$,* $x \wedge \bigvee S \;=\; \bigvee_{s \in S} x \wedge s.$

It is clear that any frame is a distributive lattice.

If L and L' are frames, a map $f : L \longrightarrow L'$ is a **frame morphism** *iff it is a $[\wedge, \bigvee]$-morphism of complete lattices, i.e., f preserves finite meets and arbitrary joins.*

If L is a frame, a subset K of L is a **subframe** *of L if, when endowed with the lattice structure induced by L, the canonical map from K to L is a frame morphism.*

Write **Frame** *for the category of frames and their morphisms.*

EXAMPLE 8.2. If X is a set, 2^X with the usual set-theoretic operations is a frame. Moreover, the subframes of 2^X are exactly the topologies on X. Although topologies are fundamental examples of frames, other important instances of this notion will appear below. □

EXAMPLE 8.3. If $(P, \leq)$ is a poset, let $\mathfrak{U}(P)$ be the topology on P generated by the set $\{x^{\rightarrow} : x \in P\}$ (1.8). Note that :

i) Each $x \in P$ has a smallest open neighborhood, namely $x^{\rightarrow}$.

ii) Opens of the form $x^{\rightarrow}$ are **super compact**, that is, any open covering has an **one element** sub covering.

iii) The frame of opens of this topology on P is an *algebraic frame*, since every open U can be written as $U = \bigcup_{x \in U} x^{\rightarrow}$.

In [**15**], the frame of opens of this topology on P are called Kripke models. □

Induction and the $[\wedge, \bigvee]$ law yields

COROLLARY 8.4. *Let H be a frame and $S_1, \ldots, S_n$ be subsets of H. Then*

$$\bigwedge_{i=1}^{n} \left(\bigvee S_i\right) = \bigvee \left\{\bigwedge_{i=1}^{n} t(i) : T \in \prod S_i\right\}.$$ □

LEMMA 8.5. *A frame L has a natural structure of Heyting algebra, with implication given, for p, $q \in L$, by*

$$p \rightarrow q \;=\; \bigvee \{x \in L : x \wedge p \leq q\},$$

and satisfying the fundamental adjunction $[\rightarrow]$ in Definition 6.1.

PROOF.[1] Fix p, $q \in L$ and set

$$t \;=\; \bigvee \{x \in L : x \wedge p \leq q\}. \tag{1}$$

If S is the subset of L in the right-hand side of (1), the fact that L satisfies the $[\wedge, \bigvee]$-law yields

$$p \wedge t \;=\; p \wedge \bigvee S \;=\; \bigvee_{x \in S} p \wedge x \;\leq\; q,$$

showing that, in fact, $t \;=\; \max\{x \in L : p \wedge x \leq q\}$, as needed. □

If L is a frame, the operation of *implication* in L will always be the canonical operation associated to L as in Lemma 8.5.

It was shown in Lemma 7.7, that a very general form of associativity holds in any complete lattice. On the other hand, infinitary versions of distributivity yield characterizations of some the algebraic objects we have been describing. We start with

DEFINITION 8.6. *Let L be a lattice.*
a) L is a $[\wedge, \bigvee]$-lattice iff it satisfies

$[\wedge, \bigvee]$ *For all $S \subseteq L$ and all $x \in L$, if $\bigvee S$ exists in L, then $\bigvee_{s \in S} (x \wedge s)$ exists in L and $x \wedge \bigvee S = \bigvee_{s \in S} (x \wedge s)$.*

b) L is a $[\vee, \bigwedge]$-lattice iff it satisfies

$[\vee, \bigwedge]$ *For all $S \subseteq L$ and all $x \in L$, if $\bigwedge S$ exists in L, then $\bigwedge_{s \in S} (x \vee s)$ exists in L and $x \vee \bigwedge S = \bigwedge_{s \in S} (x \vee s)$.*

PROPOSITION 8.7. *a) Chains and BAs are $[\vee, \bigwedge]-$lattices.*
b) Every HA is a $[\wedge, \bigvee]$-lattice. In particular, every BA and all chains are $[\wedge, \bigvee]$-lattices.
c) A HA is a frame iff it is complete as a lattice. In particular, complete chains are frames.

PROOF. a) The statement for BAs follows from 5.7.(g). For chains, the verification is straightforward.
b) Let L be a HA, $S \subseteq L$ and suppose that $\bigvee S$ exists in L. Assume that for $t, x \in L$, we have $t \geq x \wedge s$, $\forall\, s \in S$. Then, $s \leq x \rightarrow t$, for all $s \in S$ and so $\bigvee S \leq x \rightarrow t$. But this means that $x \wedge \bigvee S \leq t$. This reasoning proves, in one stroke, that $\bigwedge_{s \in S} (x \wedge s)$ exists in L and is equal to $x \wedge \bigvee S$. Item (c) follows immediately from (b) and Definition 8.1, ending the proof. □

REMARK 8.8. Important references in the subject, [**15**] among them, use the term *complete Heyting algebras* (cHa) for what we here are designating as *frames*. Proposition 8.7 surely describes grounds on which this nomenclature is reasonable. It should be nevertheless registered that a frame and a Heyting algebra, that is complete as a lattice, *are algebras of different types*, although in a certain sense interpretable in one another.

In [**60**], frames and cHas are called *complete pseudo Boolean algebras* . □

[1] See also Exercise 8.27.

Definition 8.9. *A* **complete Boolean algebra (cBa)** *is a BA that is complete as a lattice. The category* **cBa** *of complete Boolean algebras is the category whose objects are cBas and whose morphisms are* $[\wedge, \bigvee]$*-morphisms. By Proposition* 8.7.(b) **cBa** *is a subcategory of* **Frame**.

Example 8.10. It is clear that 2^X is a complete Boolean algebra. By Lemma 1.14 and Example 5.4, if T is a topological space, $Reg(T)$ is a complete Boolean algebra. $\square$

Definition 8.11. *A frame morphism,* $f : L \longrightarrow R$*, is*
a) **implication preserving (ip)** *if*

$$\text{For all } p,\, q \in L, \quad f(p \to q) \quad = \quad fp \to fq.$$

b) **open** *if it is an implication preserving,* $[\bigvee, \bigwedge]$*-morphism, of complete lattices.*

Remark 8.12. Our definition of frame morphism has its origin in Topology (Examples 4.6 and 6.14). Analogy with Topology is also the source of the term "open" for a complete morphism which preserves implication : if $f : X \longrightarrow Y$ is a continuous *open* map of topological spaces, then $f^* : \Omega(Y) \longrightarrow \Omega(X)$ is a complete morphism such that, for all $U, V \in \Omega(Y)$, $f^*(U \to V) = f^*U \to f^*V$.

In [**15**], *open* is used for what we here call a *complete morphism*. Hence, the examples below may be instructive :

1. A regular embedding that does not preserve implication. Let [0, 1] be the closed real unit interval and $L = \Omega((0, 1))$ be the frame of opens of the open unit interval (0, 1). Define $f : [0, 1] \longrightarrow L$ by $r \mapsto (0, r)$ (with $f(0) = \emptyset$); clearly, f is a complete embedding. On the other hand, for all $r < s$ in (0, 1),

$$s \to r = r \text{ in } [0, 1], \text{ while } (0, s) \to (0, r) = (0, r) \cup (s, 1) \text{ in } L.$$

2. An implication preserving frame morphism that is not open. If L is a frame, let

$$D \quad = \quad \{\neg\neg\, x \in L : x \in L\},$$

be the filter of dense opens in L, where as usual $\neg\, x =_{def} x \to \bot$. Corollary 10.5 and Example 8.18, the canonical quotient map from L to L/D is an implication preserving frame morphism, that, in general, does not preserve arbitrary meets.

In view of (1) and (2), it seemed reasonable to reserve the term *open* for those morphisms that had properties more closely resembling the topological case.

Example (1) also shows that a subobject in the category **Frame** may not be a Heyting subalgebra. $\square$

Other examples of frames are described in

Corollary 8.13. *If L is a lattice, then $Con(L)$ is a frame.*

Proof. If θ and $\{\gamma_i\}$ are in $Con(L)$, it is sufficient to show that

$$\theta \wedge (\bigvee \gamma_i) \subseteq \bigvee\, (\theta \wedge \gamma_i).$$

By 4.18, this reduces to showing that for $A \subseteq_f \{\gamma_i\}$ [2], θ distributes over the join of the elements in A. But this comes immediately from 4.19, that guarantees the distributivity of $Con(L)$. $\square$

[2] $\subseteq_f$ means "finite subset of".

There is method in this madness.

PROPOSITION 8.14. *A distributive algebraic lattice satisfies* $[\wedge, \bigvee]$. *In particular, any complete distributive algebraic lattice is a frame.*

PROOF. Let L be a distributive algebraic lattice and $S \subseteq L$ be such that $\bigvee S$ exists in L. We will prove that for all $x \in L$,

$$\bigvee \{x \wedge s : s \in S\} = x \wedge \bigvee S.$$

Fix a set C of compact elements in L such that $x \wedge \bigvee S = \bigvee C$. Let $p \in L$ be such that $p \geq x \wedge s$, for all $s \in S$. For each $c \in C$, $c \leq x$ and $c \leq \bigvee S$ and so, there is a finite subset $\{s_1, \ldots, s_n\} \subseteq S$ such that

$$c \leq x \wedge (\bigvee_{i=1}^{n} s_i) = \bigvee_{i=1}^{n} x \wedge s_i.$$

From $p \geq x \wedge s$, $s \in S$, we get $p \geq c$. Since this holds for all $c \in C$, we conclude that $p \geq x \wedge \bigvee S$, ending the proof. □

In spite of the observation at the end of Remark 8.12, the following simple result will prove useful.

COROLLARY 8.15. *Let H be a HA, P be a frame and $H \xrightarrow{f} P$ be an injective lattice morphism such that $f(H)$ is a basis for P. Then, f is a HA-morphism, that is, for all x, $y \in H$, $f(x \to y) = fx \to fy$.*

PROOF. Since f is increasing and preserves meets, *Modus Ponens* yields

$$fx \wedge f(x \to y) = f(x \wedge (x \wedge y)) \leq f(y),$$

and so the adjunction $[\to]$ in 6.1 entails $f(x \to y) \leq fx \to fy$. To prove equality it is enough to check, again by $[\to]$, that

$$\text{For all } t \in P, \quad t \wedge fx \leq fy \quad \Rightarrow \quad t \leq f(x \to y).$$

Since $f(H)$ is a basis for P, we may write $t = \bigvee_{z \in A} fz$, with $A \subseteq H$. Therefore, the $[\wedge, \bigvee]$-law in P yields

$$t \wedge fx = fx \wedge \bigvee_{z \in A} fz = \bigvee_{z \in A} fx \wedge fz = \bigvee_{z \in A} f(x \wedge z) \leq fy.$$

Hence, for $z \in A$, we have $f(x \wedge z) \leq fy$, or equivalently,

$$f(x \wedge z \wedge y) = f(x \wedge z);$$

since f is injective, this implies $x \wedge z \leq y$ and so $z \leq x \to y$. But then,

$$\text{For all } z \in A, \quad f(z) \leq f(x \to y),$$

that entails $t = \bigvee_{z \in A} fz \leq f(x \to y)$, as desired. □

The behavior of implication and pseudocomplementation with respect to the infinitary operations in a frame is described in

LEMMA 8.16. *Let H be a frame and $S \cup \{x\} \subseteq H$.*

a) $\bigwedge_{s \in S} (x \to s) = x \to \bigwedge S$.

b) $\bigwedge_{s \in S} (s \to x) = \bigvee S \to x$.

c) $\bigvee_{s \in S} (x \to s) \leq x \to \bigvee S$.

d) $\bigvee_{s \in S} (s \to x) \leq \bigwedge S \to x$.

e) $\neg (\bigvee S) = \bigwedge_{s \in S} \neg s$.

f) $\neg (\bigwedge S) \geq \neg\neg (\bigvee_{s \in S} \neg s) = \neg \bigwedge_{s \in S} \neg\neg s$.

g) $\neg\neg \bigwedge_{s\in S} \neg\neg s = \bigwedge_{s\in S} \neg\neg s$.
h) $\neg\neg(\bigvee S) = \neg(\bigwedge_{s\in S} \neg s) = \neg\neg(\bigvee_{s\in S} \neg\neg s)$.
In particular, in a cBa we have the de Morgan laws

$$\neg \bigvee S = \bigwedge_{s\in S} \neg s \quad \text{and} \quad \neg \bigwedge S = \bigvee_{s\in S} \neg s.$$

PROOF. Part (a) is Exercise 8.28.(c). We prove (b), leaving (c) and (d) to the reader. For each $s \in S$,

$$s \wedge (\bigwedge_{s\in S} (s \to x)) \leq s \wedge (s \to x) \leq x,$$

and so $(\bigvee S) \wedge \bigwedge_{s\in S} (s \to x) \leq x$. Thus, $\bigwedge_{s\in S} (s \to x) \leq \bigvee S \to x$. By 6.4.(d), $(\bigvee S \to x) \leq (s \to x)$. Thus, $\bigvee S \to x \leq \bigwedge_{s\in S} (s \to x)$, verifying (b).
e) We have

$$(\bigwedge_{s\in S} \neg s) \wedge \bigvee S = \bigvee \{t \wedge \bigwedge_{s\in S} \neg s : t \in S\} = \bot.$$

Thus, $(\bigwedge_{s\in S} \neg s) \leq \neg(\bigvee S)$. To show equality, assume that

$$x \wedge \bigvee S = \bigvee \{x \wedge s : s \in S\} = \bot.$$

Clearly, we must have, $x \wedge s = \bot$, that is, $x \leq \neg s$, for all $s \in S$. From this, (e) follows immediately.
f) The equality in (f) follows from (e). For the inequality, note that

$$(\bigwedge S) \wedge \bigvee_{s\in S} \neg s = \bigvee \{(\neg t) \wedge \bigwedge_{s\in S} \neg\neg s) : t \in S\} = \bot.$$

This yields $\bigvee_{s\in S} \neg s \leq \neg \bigwedge S$. Therefore, taking the contrapositive (6.8.(a)) twice (or using 6.8.(c)), yields the required result. The remaining items follow straightforwardly from what has been proven. □

REMARK 8.17. It was shown in Lemma 5.7 that the general forms of the de Morgan laws held in a BA, whenever the appropriate joins and meets existed. Hence, if H is a frame and B a cBa,

every $[\wedge, \bigvee]$*-morphism from* B *to* H *is an open morphism.*

In particular, in the category **cBa** all morphisms are open. □

EXAMPLE 8.18. For **finite** subsets S of a HA we have

a) $\neg \bigwedge S = \neg\neg \bigvee_{s\in S} \neg s$ (Lemma 6.8.(e));

b) $\neg\neg \bigwedge S = \bigwedge_{s\in S} \neg\neg s$ (Lemma 6.8.(f)).

However, these formulae do not hold for infinite subsets of a frame. As an example, consider, in $\Omega(\mathbb{R})$, the family $S = \{\mathbb{R} - \{p\} : p \in \mathbb{R}\}$. Then, for all $U \in S$, $\neg\neg U = \mathbb{R}$ and $\bigwedge S = \emptyset$. Thus, $\neg \bigwedge S = \top$, while $\neg\neg \bigvee_{U\in S} \neg U = \emptyset$. Similarly, $\neg\neg \bigwedge S = \emptyset$, while $\neg\neg \bigwedge_{U\in S} \neg\neg U = \mathbb{R}$. Note that $S \subseteq D$, where D is the filter of dense elements in $\Omega(\mathbb{R})$.

The reader is invited to prove that if H is a frame and $S \subseteq H$, then (a) is equivalent to (b).

A modification of the above example shows that $\Omega(\mathbb{R})$ **does not** satisfy$[\vee, \bigwedge]$: let S be as before and $V \neq \mathbb{R}$ be an open set, containing the rationals $\mathbb{Q}$. Then, $V \vee \bigwedge S$ is distinct from $\bigwedge_{U\in S} V \vee U$.

In [**3**] there is a characterization, due to Chang and Horn, of the lattices that satisfy $[\wedge, \bigvee]$ and $[\vee, \bigwedge]$ (Thm. XII.1.7, pg. 230). We have already remarked that BAs and chains satisfy both generalized distributive laws (Proposition 8.7). □

Although Example 8.18 shows that the $[\vee, \bigwedge]$ distributive law does not hold in a frame L, there is a subset of L for which it holds true : the BA, $B(L)$, of clopens in L (Definition 5.3). An important class of distributive lattices is described in

Definition 8.19. *A complete distributive lattice L is* **zero-dimensional** *iff $B(L)$ is a basis for L.*

In Chapter 18 of Part II, we present a whole class of frames, the algebra of opens of totally disconnected spaces, that are significant examples of zero-dimensional frames.

The clopens in a distributive lattice have a number of useful properties, some of which are described in

Lemma 8.20. *Let L be a complete distributive lattice and b an element of $B(L)$.*
a) The laws $[\vee, \bigwedge]$ and $[\wedge, \bigvee]$ hold for b and all $S \subseteq L$, that is
$$b \vee (\bigwedge S) = \bigwedge_{s\in S} (b \vee s) \qquad \text{and} \qquad b \wedge \bigvee S = \bigvee_{s\in S} (b \wedge s).$$
b) If L is a zero-dimensional frame, any element satisfying the $[\vee, \bigwedge]$ law for all $S \subseteq L$ is clopen.
c) If c is compact in L then $b \wedge c$ is compact in L.

Proof. a) For the first equality, it is enough to show that
$$\bigwedge_{s\in S} (b \vee s) \leq b \vee \bigwedge S.$$
Suppose $t \leq (b \vee s)$, s $\in S$; then,
$$(t \wedge \neg b) \leq \neg b \wedge (b \vee s) = \neg b \wedge s,$$
and so $(t \wedge \neg b) \leq \neg b \wedge \bigwedge S$. From $b \vee \neg b = \top$, we get
$$t = (t \wedge b) \vee (t \wedge \neg b) \leq (t \wedge b) \vee (\neg b \wedge \bigwedge S) \leq b \vee \bigwedge S,$$
giving the desired inequality. The proof of the second equation in the statement is similar.
b) If L is zero-dimensional, let $b \in L$ satisfy $[\vee, \bigwedge]$ law, for all $S \subseteq L$. Since $B(L)$ is a basis for L, we have $b = \bigvee_{p\in A} p$, with $A \subseteq B(L)$. For each $p \in$ A, $p \leq b$ entails $b \vee \neg p = \top$. Thus,
$$\top = \bigwedge_{p\in A} (b \vee \neg p) = b \vee \bigwedge_{p\in\ A} \neg p.$$
Since L is a frame, we also have
$$b \wedge \bigwedge_{q\in A} \neg q = \bigvee_{p\in A} (p \wedge \bigwedge \neg q) = \bot,$$
showing that b is complemented in L.
c) Let $S \subseteq L$ be such that $\bigvee S \geq b \wedge c$. Then
$$\neg b \vee \bigvee S \ \geq \ (c \wedge \neg b) \vee (c \wedge b) = c.$$
By compactness, $\neg b \vee \bigvee K \geq c$, with $K \subseteq_f S$. It is clear that $\bigvee K \geq c \wedge b$, and $b \wedge c$ is compact in L. $\square$

Definition 8.21. *An element of a (not necessarily distributive) lattice L is* **linear** *if it satisfies the $[\vee, \bigwedge]$-law for all $S \subseteq L$.*

Remark 8.22. It is clear that the set of linear elements in any lattice is closed under finite sups.

By Lemma 8.20.(a), in distributive lattices, clopens are linear. However, the clopens are, in general, only a proper subset of the linear elements. For instance, in chains all elements are linear, but the only clopens are $\perp$ and $\top$. $\square$

We have already met a setting in which a complete lattice has to be a frame (Proposition 8.14). Another such situation is described in

Proposition 8.23. *A complete zero-dimensional lattice is a frame iff it is distributive.*

Proof. Let L be a complete zero-dimensional distributive lattice and let $S \cup \{x\} \subseteq L$. It is enough to verify that $x \wedge \bigvee S \leq \bigvee_{s\in S} (x \wedge s)$. Let $p \in B(L)$ satisfy $p \leq x \wedge \bigvee S$; then, $p \vee \neg p = \top$ yields

$$\neg p \vee (x \wedge \bigvee S) = (\neg p \vee x) \wedge \bigvee_{s\in S} (\neg p \vee s) = \top,$$

wherefore, $\neg p \vee x = \top$ and $\bigvee_{s\in S} (\neg p \vee s) = \top$. Hence, since $\neg p$ is linear (Lemma 8.20(a)),

$$\begin{aligned}\neg p \vee \bigvee_{s\in S} (x \wedge s) &= \bigvee_{s\in S} \neg p \vee (x \wedge s)\\ &= \bigvee_{s\in S} (\neg p \vee x) \wedge (\neg p \vee s) = \top,\end{aligned}$$

and so $p \leq \bigvee_{s\in S} (x \wedge s)$. Since L is zero-dimensional and p is an arbitrary clopen below $x \wedge \bigvee S$, we obtain the inequality establishing that L is a frame. $\square$

An analysis of the proof of Proposition 8.23 leads to a very interesting result, due to Isbell ([**31**]; also [**70**]), giving a criterion for a complete lattice to be a zero-dimensional frame, which circumvents first verifying that it is distributive. This will later be applied to show that the lattice of congruences in a frame is zero-dimensional. We follow, with minor modifications, the exposition of Isbell's result in [**15**].

We introduce a notion dual to that of basis. Let L be a complete lattice. A subset S of L is a **cobasis** for L iff for all $x, y \in L$,

$$x \leq y \quad \text{iff} \quad \forall s \in S \ (x \vee s = \top \ \Rightarrow \ y \vee s = \top).$$

With these preliminaries we are ready to prove

Theorem 8.24. (Isbell) *A complete lattice is a zero-dimensional frame iff the set linear elements is a cobasis.*

Proof. If L is a zero-dimensional frame, Lemma 8.20.(b) tells us that the set of linear elements is precisely $B(L)$. Since L is distributive, for $p \in B(L)$ and $x \in L$, we have $p \leq x$ iff $\neg p \vee x = \top$. Thus, for all $x, y \in L$,

$$\begin{aligned} x \leq y \quad &\text{iff} \quad \forall\, p \in B(L)\ (p \leq x \Rightarrow p \leq y)\\ &\text{iff} \quad \forall\, q \in B(L)\ (q \vee x = \top \Rightarrow q \vee y = \top)\end{aligned}$$

and so the set of linear elements is a cobasis for L. For the converse, we first verify that L is a frame. For $\{x\} \cup S \subseteq L$, we must prove that $x \wedge \bigvee S \leq \bigvee_{s\in S} (x \wedge s)$. Suppose that z is a linear element in L, such that $z \vee (x \wedge \bigvee S) = \top$. Once it is proven that

$$z \vee \bigvee_{s\in S} (x \wedge s) = \top,$$

the fact that the linear elements form a cobasis will give us the desired inequality. From $z \vee (x \wedge \bigvee S) = \top$ and the linearity of z comes

(*) $$z \vee x = \top = z \vee \bigvee S = \bigvee_{s \in S} (z \vee s).$$

Thus, Lemma 7.7, the linearity of z and (*) yield

$$\begin{aligned} z \vee \bigvee_{s \in S} (x \wedge s) = \bigvee_{s \in S} z \vee (x \wedge s) &= \bigvee_{s \in S} (z \vee x) \wedge (z \vee s) \\ = \bigvee_{s \in S} z \vee s = \top, \end{aligned}$$

showing that L is a frame. It remains to verify that the clopens in L are a basis for L. A path is suggested by 8.20.(b) : prove that linear elements are clopen and so $B(L)$ will be the set of linear elements. For, suppose that this is the case. If t is linear and $x \vee t = \top$, then $\neg t \leq x$ and so

$$t \vee \bigvee \{z \in B(L) : z \leq x\} \geq t \vee \neg t = \top;$$

the linears being a cobasis, we obtain $x \leq \bigvee \{z \in B(L) : z \leq x\}$ and $B(L)$ is a basis for L.

For a linear element $t \in L$, let $z = \bigwedge \{y \in L : t \vee y = \top\}$. By linearity, one has $t \vee z = \top$. To prove that $t \in B(L)$, it is sufficient to show that $t \wedge z = \bot$. Let q be linear, such that $q \vee (t \wedge z) = \top$, that is, $q \vee t = \top$ and $q \vee z = \top$. Since $q \vee t = \top$, we get $z \leq q$. But then, $q = z \vee q = \top$. The above argument shows that for all linear q in L, if $q \vee (t \wedge z) = \top$, then $q = q \vee \bot = \top$. The cobasis property then yields $t \wedge z = \bot$, as needed. □

We end this Chapter with the frame version of "disjointing an union of sets".

Proposition 8.25. *Let $\{p_i : i \in i\}$ be a family of elements in a frame H. Then, there is $\{q_i : i \in I\} \subseteq H$, such that*

(1) *For all $i \in I$, $q_i \leq p_i$;*
(2) *For all $i \neq j$ in I, $q_i \wedge q_j = \bot$;*
(3) *$\bigvee_{i \in I} q_i$ is dense in $\bigvee_{i \in I} p_i$ (6.19.(a)).*

Proof. We assume that I is well-ordered, with first element i_0. By transfinite recursion, define

$*$ $q_{i_0} = p_{i_0}$;
$*$ Having defined q_i, for $i < j$, set $q_j = p_j \wedge \neg \bigvee_{i<j} p_i$.

It is clear that condition (1) is satisfied. If $i \neq j$, we may assume that $i < j$. But then, since $q_i \leq p_i \leq \bigvee_{k<j} p_k$, we obtain

$$q_i \wedge q_j = q_i \wedge p_j \wedge \neg \bigvee_{k<j} p_k = \bot,$$

verifying (2). To establish (3) it is enough to show that for all $j \in J$,

(*) $$p_j \;\leq\; \neg\neg \bigvee_{i \leq j} q_i,$$

which shall be done by transfinite induction; clearly, (*) holds for the first element i_0 of I. Assume that (*) holds for all $i < j$. Note that the induction hypothesis entails

(**) $$\neg \bigvee_{i<j} q_i \;=\; \neg \bigvee_{i<j} p_i.$$

Hence, (**), (e) and (k) in 6.8, together with 8.16.(e), yield

$$\begin{aligned}
\neg\neg \bigvee_{i \le j} q_i &= \neg\neg\Big(q_j \vee \bigvee_{i<j} q_i\Big) \;=\; \neg\Big(\neg\, q_j \wedge \neg \bigvee_{i<j} q_i\Big) \\
&= \neg\Big(\neg\, q_j \wedge \neg \bigvee_{i<j} p_i\Big) \;=\; \neg\neg\Big(q_j \vee \bigvee_{i<j} p_i\Big) \\
&= \neg\neg\Big((p_j \wedge \neg \bigvee_{i<j} p_i) \vee \bigvee_{i<j} p_i\Big) \\
&\geq \neg\neg\, p_j,
\end{aligned}$$

ending the proof. □

The law of double negation in Boolean algebras (5.7.(a)) yields

COROLLARY 8.26. *Let $\{p_i : i \in i\}$ be a family of elements in a* **cBa** *B. Then, there is $\{q_i : i \in I\} \subseteq B$, such that*
(1) *For all $i \in I$, $\;q_i \leq p_i$;*
(2) *For all $i \neq j$ in I, $\;q_i \wedge q_j = \bot$;*
(3) $\bigvee_{i \in I} q_i = \bigvee_{i \in I} p_i$.

Exercises

8.27. Use Theorem 7.8 to construct implication in a frame and to immediately conclude item (a) in Lemma 8.16. □

8.28. a) Show that the class of frames is closed under products and images by frame morphisms.
b) Examine what happens in 7.8 if we assume that L and P are cBas.
c) Let L be a complete lattice and $d : L \longrightarrow L \times L$ be the diagonal map, $dx = \langle x, x \rangle$. Compute the right and left adjoints of d. □

8.29. A distributive lattice is **compact** if $\top$is compact (compare 2.43.(e)). Prove that a complete lattice is a compact zero-dimensional frame iff it is isomorphic to the lattice of ideals of a BA. □

CHAPTER 9

Radical Ideals and Multiplicative Subsets

In general, the lattice of ideals of a commutative ring with identity, R, is not distributive. The first section of this chapter is devoted to the construction of a lattice of ideals in R that is distributive and, in fact, an algebraic frame : the lattice of **radical ideals**. When R is a Gaussian domain or Noetherian, there is another algebraic frame associated to R, that of its **saturated multiplicative subsets**, discussed in section 2. The last section presents the **ring of fractions** generated by a multiplicative subset of R. Most facts we shall use concerning commutative rings, if proofs are not provided, can be found in any standard text in Commutative Algebra, e.g., [**2**].

In what follows, all rings are commutative with identity 1.

1. Radical Ideals

DEFINITION 9.1. *Let R be a ring and $S \subseteq R$.*
a) S is an **ideal** *in R if for all x, $y \in R$*

$$* \; x, y \in S \;\Rightarrow\; x + y \in S; \qquad\qquad * \; x \in R \;\Rightarrow\; xy \in S.$$

An ideal is **proper** *if it is distinct from R.*
b) The **ideal generated** *by S is*

$$(S) = \{t \in R : \begin{array}{c} \exists\, s_1, \ldots, s_n \in S \textit{ and } \lambda_1, \ldots, \lambda_n \in R \\ \textit{such that } t = \sum_{i=1}^{n} \lambda_i s_i. \end{array} \Big\}$$

REMARK 9.2. Let R be a ring and $S \subseteq R$.
a) Clearly, 0 is in any ideal. Moreover, an ideal I is proper iff $1 \notin I$.
b) The intersection of any family of ideals in R is an ideal in R. The union of a up-directed family of ideals in R is an ideal in R.
c) (S) is also the intersection of all ideals that contain S.
d) If I_k, $k \in K$, is a family of ideals in R, then

$$\textstyle\sum_{k \in K} I_k = \{t \in R : \begin{array}{c} \exists\, k_1, \ldots, k_n \in K \text{ and } x_{k_i} \in I_{k_i}, \\ 1 \leq i \leq n, \text{ such that } t = \sum_{i=1}^{n} x_{k_i} \end{array} \Big\}$$

is an ideal, called the **sum** of the I_k. It is straightforward that

$$\textstyle\sum_{k \in K} I_k = \big(\bigcup_{k \in K} I_k\big).$$

e) For ideals I, J in R, their **product** is defined by

$$IJ = \{t \in R : \begin{array}{c} \exists\, a_1, \ldots, a_n \in I \text{ and } b_1, \ldots, b_n \in J, \\ \text{such that, } t = \sum_{i=1}^{n} a_i b_i \end{array} \Big\}$$

f) It is clear that if I, J are ideals in R, then $IJ \subseteq I \cap J$. $\square$

Definition 9.3. *a) An ideal P in R is* **prime** *iff*

$$\text{For all } x, y \in R \ (xy \in P \ \Rightarrow \ x \in P \text{ or } y \in P).$$

$Spec(R)$, the spectrum of R, is the set of proper prime ideals in R.
b) A subset S of R is **multiplicative** *iff $1 \in S$ and*

$$\forall\, x,\, y \in R, \quad x,\, y \in S \quad \Rightarrow \quad xy \in S.$$

S is **proper** *if $0 \notin S$. Write $\mathcal{M}(R)$ for the family of multiplicative subsets of R.*

Lemma 9.4. *Let R be a ring.*
a) If $P \in Spec(R)$, then $P^c \in \mathcal{M}(R)$.
b) Every proper ideal is contained in a maximal ideal [1].
c) All maximal ideals are prime.

Proof. Left to the reader. For (b), one needs Zorn's Lemma. $\square$

The fundamental facts we shall need about prime ideals and multiplicative sets are collected in

Theorem 9.5. *Let R be a ring.*
a) Let I be an ideal in R and S a multiplicative subset of R, such that $S \cap I = \emptyset$. Then, there is a prime ideal P in R such that $I \subseteq P$ and $P \cap S = \emptyset$.
b) If $P_1, \ldots, P_n$ are prime ideals in R and I is an ideal in R, then

$$I \subseteq \bigcup_{i=1}^{n} P_i \quad \Rightarrow \quad \exists\, i \leq n \text{ such that } I \subseteq P_i.$$

c) If $I_1, \ldots, I_n$ are ideals in R and P is a prime ideal in R, then

$$\bigcap_{j=1}^{n} I_j \subseteq P \quad \Rightarrow \quad \exists\, j \leq n \text{ such that } I_j \subseteq P.$$

If $P = \bigcap_{j=1}^{n} I_j$, then, $P = I_j$, for some $1 \leq j \leq n$.

Proof. a) Consider

$$V = \{J \subseteq R : J \text{ is an ideal, } I \subseteq J \text{ and } J \cap S = \emptyset\},$$

partially ordered by inclusion. By Zorn's Lemma (2.20), there is P maximal in V. Clearly, $I \subseteq P$ and $P \cap S = \emptyset$. If P is not prime, there are $u, v \in R$, such that $uv \in P$ and neither u nor v are in P. Let K and J be the ideals generated by $P \cup \{u\}$ and $P \cup \{v\}$, respectively. Then, K and J must have non-empty intersection with S. Hence, there are $x,\, y \in S$, $a, b \in P$ and $\lambda,\, \beta \in R$ such that

$$x = a + \lambda u \quad \text{and} \quad y = b + \beta v.$$

But then $xy = ab + a\beta v + b\lambda u + \beta\lambda uv \in P \cap S$, a contradiction that ends the proof of (a).
b) We proceed by induction on n, the result being obvious for $n = 1$. Assume it holds for $n \geq 1$ and let $P_1, \ldots, P_n$, P_{n+1} be primes in R, such that $I \subseteq \bigcup_{i=1}^{n+1} P_i$. We contend that I is contained in the union of some subset of n elements of the P_i. If not, for each $1 \leq j \leq (n+1)$, there is $x_j \in R$ such that

$$x_j \in (I \cap P_j) - \bigcup_{i \neq j} P_i.$$

i.e., x_j is outside the union of primes with index distinct from j, but in $I \cap P_j$. Let

[1] An ideal is I maximal if for all proper ideals J, $I \subseteq J \Rightarrow I = J$.

$$x = \sum_{j=1}^{n+1} x_1 x_2 \ldots x_{j-1} x_{j+1} \ldots x_{n+1}.$$

It is clear that $x \in I$; on the other hand, $x \notin P_j$, for all $1 \leq j \leq (n+1)$. To see this, fix j between 1 and $(n+1)$; since all terms in x, except the j^{th}, have x_j as a factor, they are in P_j. Consequently, the difference between x and the sum of these terms is in P_j, that is,

$$x_1 x_2 \ldots x_{j-1} x_{j+1} \ldots x_n \in P_j.$$

But this is impossible, because for $i \neq j$, x_i is outside $\bigcup_{k \neq i} P_k \supseteq P_j$. Hence, for some $\alpha \subseteq \{1, \ldots, (n+1)\}$ of cardinality n, $I \subseteq \bigcup_{k \in \alpha} P_k$ and the induction hypothesis entails that $I \subseteq P_i$, $1 \leq i \leq (n+1)$, completing the induction.
c) Suppose $x_j \in I_j - P$, for $1 \leq j \leq n$; then $x = \Pi_{i=1}^{n} x_j \in \bigcap_{j=1}^{n} I_j$, but $x \notin P$, because all x_j are outside P. The remaining statement is clear. □

Closely associated to prime ideals are radical ideals.

Definition 9.6. *An ideal I in R is* **radical** *iff*

For all $x \in R$ and all $n \in \mathbb{N}$ $(x^n \in I \Rightarrow x \in I)$.

Or equivalently, $\forall\, x \in R$, $x^2 \in I$ implies $x \in I$. Write $Rad(R)$ for the set of radical ideals in R.

Note that the complement S of a proper radical ideal is a subset of R satisfying

$$1 \in S \quad \text{and} \quad \text{for all } x \in R \ (x \in S \ \Rightarrow \ \forall\, n \in \mathbb{N},\, x^n \in S).$$

Clearly all primes ideals are radical ideals, but the converse is, in general, false. The next result is straightforward.

Lemma 9.7. *a) $Rad(R)$ is closed under arbitrary intersections.*
b) Partially ordered by inclusion, $Rad(R)$ is a complete lattice. □

Example 9.8. If R is a principal ideal domain [2], an ideal (y) is radical iff y is square free, that is, if p is a prime in R, such that p^n divides y, then $n = 1$. This applies, in particular, to $\mathbb{Z}$ and to rings of polynomials in one indeterminate with coefficients in a field. □

Example 9.9. It is not true that the sum of radical ideals is a radical ideal. To see this, let F be a field in which -1 does not have a square root. In the ring of polynomials in two indeterminates with coefficients in F, $R = F(X, Y)$, we have

Fact. *a) (XY) is a radical ideal.*
b) The polynomial $X^2 + Y^2$ is irreducible.

Proof. It is clear that (X) and (Y) are primes in R. Thus, $(XY) = (X) \cap (Y)$ is a radical ideal (9.7.(a)). Moreover, $X^2 + Y^2 \notin (XY)$. For item (b), suppose that

$$X^2 + Y^2 = p(X, Y)q(X, Y). \tag{I}$$

Substituting 1 for Y in this equation yields

$$p(X, 1)q(X, 1) = X^2 + 1.$$

Since $X^2 + 1$ is irreducible in $F(X)$ – otherwise F would contain a square root of -1 –, we conclude, without loss of generality, that

[2] A commutative ring with 1, in which every ideal is of the form (x).

$$p(X, 1) = u(X^2 + 1) \quad \text{and} \quad q(X, 1) = \frac{1}{u},$$

with $u \neq 0$ in F. It follows that the variable X does not occur in q, that is, $q = q(Y)$. Hence, (I) becomes

$$X^2 + Y^2 = p(X, Y)q(Y). \tag{II}$$

If 1 is substituted for X in (II), we get $1 + Y^2 = p(1, Y)q(Y)$; reasoning as above, we are led to the following alternatives, where w is a non-zero element of F :

$$\begin{cases} \text{(i) } p(1, Y) = w \quad \text{and} \quad q(Y) = \dfrac{(1 + Y^2)}{w} \\ \qquad\qquad \text{or} \\ \text{(ii) } p(1, Y) = \dfrac{(1 + Y^2)}{w} \quad \text{and} \quad q(Y) = w \end{cases}$$

Note that (i) is impossible: it implies $p = p(X)$ and (I) yields $X^2 + Y^2 = p(X)q(Y)$; since the constant term on both sides is zero, $X^2 + Y^2$ would be in (XY). Therefore, we are left with alternative (ii), which guarantees that neither X nor Y occur in q, i.e., q is a constant non-zero polynomial, $q = \frac{1}{u}$, while $p = u(X^2 + Y^2)$, completing the proof of the Fact.

Let $I = (XY)$ and $J = (X^2 + Y^2)$; since R is a Gaussian domain (see 9.32), it follows from item (b) in the Fact that J is a prime ideal in R. Now, observe that

$$(X + Y)^2 = X^2 + Y^2 + 2XY \in (I + J),$$

but a degree argument shows that $(X + Y) \notin I + J$. □

Definition 9.10. *If S is a subset of R, the* **radical ideal generated** *by S is*

$$\sqrt{S} = \bigcap \{I : I \in Rad(R) \text{ and } S \subseteq I\}.$$

When $S = \{x\}$, write $\sqrt{x}$ for the radical of $\{x\}$.

Lemma 9.11. *Let R be a ring and $S \subseteq R$. Let I_k, $k \in K$, be a family of radical ideals in R.*

a) $\sqrt{S} = \sqrt{(S)}$.

b) The sup of the I_ks in the lattice $Rad(R)$ is given by

$$\bigvee I_k = \sqrt{(\textstyle\sum_{k \in K} I_k)}.$$

Proof. Left to the reader. □

Proposition 9.12. *Let I be an ideal in R, S a subset of R and t an element of R.*

a) If $\{t^n : n \in \mathbb{N}\} \cap I = \emptyset$, then there is a prime ideal P in R, such that $I \subseteq P$ and $t \notin P$.

b) $\sqrt{I} = \{t \in R : \exists\, n \in \mathbb{N}$ such that $t^n \in I\}$;

c) $\sqrt{S} = \{t \in R : \exists\, n \in \mathbb{N}$ such that $t^n \in (S)\}$ [3].

d) If I is a proper ideal, then $\sqrt{I} = \bigcap \{P \in Spec(R) : I \subseteq P\}$.

e) If K is a family of ideals in $Rad(R)$, then

[3] (S) is the ideal generated by S, as in 9.1.

$$\bigvee K = \{t \in R : \begin{array}{c} \exists\, m \in \mathbb{N} \textit{ and } I_1, \ldots, I_n \subseteq K, \\ \textit{such that } t^m \in \sum_{i=1}^{n} I_i. \end{array} \Big\}$$

PROOF. For (a), just note that $\{t^n : \geq 0\}$ is a multiplicative set and apply 9.5.(a). The other items are straightforward consequences of (a). $\square$

COROLLARY 9.13. *The intersection of all prime ideals in R – the radical of 0 – is the ideal of nilpotent elements of R.* $\square$

Write η for the ideal of nilpotent elements in R. Note that η is the $\perp$ of the lattice $Rad(R)$.

COROLLARY 9.14. *Let I, J be ideals in R.*
a) The operation of taking radical is satisfies the following properties :
[rad 1] : $I \subseteq \sqrt{I}$;
[rad 2] : $\sqrt{\sqrt{I}} = \sqrt{I}$;
[rad 3] : $\sqrt{(I \cap J)} = \sqrt{I} \cap \sqrt{J}$.
In particular, $\sqrt{\ }$ is increasing, that is, $I \subseteq J$ implies $\sqrt{I} \subseteq \sqrt{J}$.
b) $\sqrt{(IJ)} = \sqrt{I} \cap \sqrt{J} = \sqrt{(\sqrt{I}\sqrt{J})}$.
c) If I, $J \in Rad(R)$, then $\sqrt{IJ} = I \cap J$.

PROOF. a) Properties [rad 1] and [rad 2] are straightforward. For [rad 3], 9.12.(b) yields, for $t \in R$

$$t \in \sqrt{I} \cap \sqrt{J} \quad \text{iff} \quad \exists n,\, m \in \mathbb{N}, \text{ such that } t^n \in I \text{ and } t^m \in J;$$

thus, $t^{n+m} \in I \cap J$, and so $t \in \sqrt{(I \cap J)}$. This show that

$$\sqrt{I} \cap \sqrt{J} \subseteq \sqrt{(I \cap J)},$$

which, in view of [rad 1], implies [rad 3]. Clearly, $\sqrt{\ }$ is increasing.
b) Since $IJ \subseteq I \cap J$, [rad 3] yields $\sqrt{(IJ)} \subseteq \sqrt{I} \cap \sqrt{J}$. Next, we have

$$t \in \sqrt{I} \cap \sqrt{J} \ \Rightarrow\ t^2 \in \sqrt{I}\sqrt{J} \ \Rightarrow\ t \in \sqrt{(\sqrt{I}\sqrt{J})}.$$

Hence, to complete the proof of the stated equality, it remains to check that $\sqrt{(\sqrt{I}\sqrt{J})} \subseteq \sqrt{(IJ)}$. But if P is a prime ideal in R, then 9.45 and 9.12.(d) yield

$$\begin{aligned} IJ \subseteq P \ &\Rightarrow\ I \subseteq P \text{ or } J \subseteq P \ \Rightarrow\ \sqrt{I} \subseteq P \text{ or } \sqrt{J} \subseteq P \\ &\Rightarrow\ \sqrt{I}\sqrt{J} \subseteq P, \end{aligned}$$

and we conclude by 9.12.(d). Item (c) follows immediately from (b). $\square$

REMARK 9.15. Simple examples show that $\sqrt{(IJ)} = \sqrt{I}\sqrt{J}$ is (frequently) false. By Corollary 9.14.(b), this would imply, for instance, that all prime ideals in R are idempotent, that is, $I^2 = I$. There are, however, important examples were many ideals are idempotent, as is the case of **closed left or right ideals** in a C^*-algebra. See [**52**] for more details, further references and examples. $\square$

$Rad(R)$ is yet another example of an algebraic frame :

PROPOSITION 9.16. *Let R be a ring. With notation as above,*
a) For all $x \in R$, $\sqrt{x}$ is a compact element of the lattice $Rad(R)$.
b) For all $I \in Rad(R)$, $I = \bigvee_{x \in I} \sqrt{x}$.
c) $Rad(R)$ is a distributive lattice and an algebraic frame.

d) For $I, J \in Rad(R)$,

$$I \to J = \{r \in R : rI \subseteq J\},$$

where $rI = \{rx : x \in I\}$.

Proof. a) For $x \in R$, suppose that $\sqrt{x} \subseteq \bigvee I_k = \sqrt{(\sum_{k \in K} I_k)}$. Since $x \in \sqrt{x}$, by 9.12.(a) there is $m \in \mathbb{N}$, such that $x^m \in \sum_{k \in K} I_k$. By the definition of sum of ideals (9.2.(d)), there are $k_1, \ldots, k_n \in K$ and $x_{k_i} \in I_{k_i}$, $1 \leq i \leq n$, such that

$$x^m = \textstyle\sum_{i=1}^n x_{k_i}.$$

Thus, $x^m \in \sum_{i=1}^n I_{k_i}$. Hence, 9.12.(a) yields $x \in \bigvee_{i=1}^n I_{k_i}$ and so $\sqrt{x} \subseteq \bigvee_{i=1}^n I_{k_i}$. Item (b) is clear.

c) Let I, J and $K \in Rad(R)$; if $x \in I$ and $x \in (J \vee K)$, then there is $n \in \mathbb{N}$ such that $x^n \in (J + K)$ (9.12.(a)). Hence, $x^n = a + b$, for some $a \in J$ and $b \in K$. Therefore,

$$x^{n+1} = ax + bx \in (I \cap J) + (I \cap K),$$

and so $x \in (I \cap J) \vee (I \cap K)$, verifying distributivity. The remaining assertion in (c) follows from (a), (b) and Proposition 8.14. Item (d) is left to the reader. □

In Commutative Algebra the ideal $(I \to J)$ of 9.16.(d) is written $(J : I)$ and the operation $(\star : \star)$ is called **residuation or ideal quotient**. Some of the properties of this operation are discussed in Exercise 9.46.

2. Multiplicative Subsets

Henceforth, the expression *prime ideal* is synonymous with *proper prime ideal*. The concept of *multiplicative subset of* R is defined in 9.3. Recall (9.3.(a)) that

$$Spec(R) = \{P : P \text{ is a proper prime ideal in } R\}.$$

Definition 9.17. *Let R be a ring.*

a) Write

$$U(R) = \{x \in R : \exists\, y \in R \text{ such that } xy = 1\}$$

for the multiplicative group of **units or invertible** *elements in R.*

b) An element $a \in R$ is a **zero-divisor**, *if there is $y \neq 0$ such that $ya = 0$. Write*

$$nzd(R) = \{x \in R : x \text{ is not } a \text{ zero-divisor in } R\}$$

for the set of non zero-divisors in R.

c) For $T \subseteq R$, set

$$Z_T = \{P \in Spec(R) : P \cap T = \emptyset\}.$$

When $T = \{a\}$ write Z_a for $Z_{\{a\}}$. Hence,

$$Z_a = \{P \in Spec(R) : a \notin P\}.$$

d) A multiplicative subset S of R (9.3.(b)) is **saturated** *if it satisfies*

$$[sat] \qquad \forall\, x, y \in R, \quad xy \in S \;\Rightarrow\; x, y \in S.$$

e) Write $\mathcal{M}_\sigma(R)$ for the set of saturated multiplicative subsets of R.

Example 9.18. Recall that $\mathcal{M}(R)$ is the set of multiplicative subsets of R (9.3.(b)).

a) The smallest element of $\mathcal{M}(R)$ (under inclusion) is $\{1\}$; its largest element is R. In general, the *proper* multiplicative subsets of R have no largest element. An example will be given in 9.20, below. On the other hand, in some important cases, for instance if R is an *integral domain* [4], then

$$R - \{0\} = nzd(R)$$

is the largest *proper* multiplicative subset of $\mathcal{M}(R)$.
b) Note that $U(R)$ is a *saturated* multiplicative subset of R, indeed, the *smallest* element in $\mathcal{M}_\sigma(R)$, since all saturated multiplicative subsets of R must contain $U(R)$. As above, $\mathcal{M}_\sigma(R)$ will not, in general, possess a largest *proper* element. $\square$

LEMMA 9.19. *a) $\mathcal{M}(R)$ and $\mathcal{M}_\sigma(R)$ are closed under arbitrary intersections.*
b) Partially ordered by inclusion, $\mathcal{M}(R)$ and $\mathcal{M}_\sigma(R)$ are complete lattices, whose bottom and top are given by

$$\begin{cases} \bot_{\mathcal{M}(R)} \;=\; \{1\} \qquad \textit{and} \quad \bot_{\mathcal{M}_\sigma(R)} \;=\; U(R); \\ \top_{\mathcal{M}(R)} \;=\; R \;= \top_{\mathcal{M}_\sigma(R)}. \end{cases}$$

c) The union of an up-directed family of proper multiplicative sets is a proper multiplicative set. A similar statement holds for saturated multiplicative sets.

PROOF. Items (a) and (b) are straightforward. Let $\langle I, \leq \rangle$ be an up-directed poset and S_i, $i \in I$, be proper multiplicative subsets of R, such that $i \leq j \Rightarrow S_i \subseteq S_j$. Let $S = \bigcup_{i \in I} S_i$. Clearly, $1 \in S$ and $0 \notin S$. For x, $y \in S$, select i, $j \in I$, with $x \in S_i$ and $y \in S_j$. Since I is up-directed, there is $k \geq i, j$; then, x, $y \in S_k$ and so $xy \in S_k \subseteq S$. The preservation of saturation is clear. $\square$

EXAMPLE 9.20. In general, $\mathcal{M}(R)$ *is not* distributive. Let $R = \mathbb{Z} \times \mathbb{Z}$, with its natural product structure. Then,

$$T_1 = \mathbb{Z} \times \{1, -1\} \quad \text{and} \quad T_2 = \{-1, 1\} \times \mathbb{Z}$$

are multiplicative subsets of R, which are, in fact, saturated [5]. It is easily established that the only subset of R containing $(T_1 \cup T_2)$ and closed under products is R itself (the example promised in 9.18). Hence

$$T_1 \vee T_2 = R,$$

both in $\mathcal{M}(R)$ and $\mathcal{M}_\sigma(R)$. Let

$$S = \{\langle a, a \rangle \in R : a \neq \emptyset\},$$

a proper multiplicative subset of R. Now, note that

$$S \cap T_1 \;=\; S \cap T_2 \;=\; \{\langle a, a \rangle \in R : a = \pm 1\}.$$

Hence, $S \;=\; S \cap (T_1 \vee T_2)$, while

$$(S \cap T_1) \vee (S \cap T_2) \;=\; \{\langle 1, 1 \rangle, \langle -1, -1 \rangle\},$$

proving that $\mathcal{M}(R)$ is not distributive. $\square$

To characterize *joins* in $\mathcal{M}(R)$, we introduce the following

9.21. **Notation.** If $\alpha \subseteq_f R$ is a finite subset of R, write

$$\Pi\ \alpha =_{def} \Pi_{a \in \alpha}\ a$$

[4] $xy = 0 \;\Rightarrow\; x = 0$ or $y = 0$.
[5] In $\mathbb{Z}$, $xy = \pm 1 \;\Rightarrow\; x, y = \pm 1$

for the product of the elements in α. If $\{S_i : i \in I\} \subseteq \mathcal{M}(R)$, define

$$\textstyle\prod_{i\in I} S_i \;\; = \;\; \{\Pi\ \alpha : \alpha \subseteq_f \bigcup_{i\in I} S_i\}.$$

For S, $T \in \mathcal{M}(R)$ we have

$$S \vee T = \{xy : x \in S \text{ and } y \in T\},$$

which is frequently written as $S \cdot T$. $\square$

LEMMA 9.22. *Let $S \in \mathcal{M}_\sigma(R)$ and $\{S_i : i \in I\} \subseteq \mathcal{M}(R)$.*
a) $\prod_{i\in I} S_i$ is the join of the S_i in $\mathcal{M}(R)$.
b) $S \cap \prod_{i\in I} S_i \;\; = \;\; \prod_{i\in I} S \cap S_i$.

PROOF. a) Write S for $\prod_{i\in I} S_i$. It is clear that $1 \in S$ and that any multiplicative subset of R that contains the S_i must contain S. If x, $y \in S$, there are α, $\beta \subseteq_f \bigcup_{i\in I} S_i$, such that $x = \Pi\ \alpha$ and $y = \Pi\ \beta$. Then, $\gamma = \alpha \cup \beta$ is a finite subset of $\bigcup_{i\in I} S_i$ and $xy = \Pi\ \gamma \in S$.
b) It is enough to check that

$$S \cap \textstyle\prod_{i\in I} S_i \;\; \subseteq \;\; \prod_{i\in I} S \cap S_i.$$

Suppose that $x \in S$ and there is $\alpha \subseteq_f \bigcup_{i\in I} S_i$ such that $x = \Pi\ \alpha$. Since S is saturated, it follows that $\alpha \subseteq S$. Hence,

$$\alpha \;\; \subseteq_f \;\; S \cap \textstyle\bigcup_{i\in I} S_i \;\; = \;\; \bigcup_{i\in I} S \cap S_i,$$

and so $x = \Pi\ \alpha \in \prod_{i\in I} S \cap S_i$, as desired. $\square$

Example 9.20 shows the hypothesis of saturation of S in 9.22.(b) is essential.

PROPOSITION 9.23. *Let R be a ring. For $S \in \mathcal{M}(R)$, set*

$$\sigma(S) = \{x \in R : \exists\ y \in R, \text{ such that } yx \in S\}.$$

Then, for $\{S\} \cup \{T_i : i \in I\} \subseteq \mathcal{M}(S)$
a) $\sigma(S)$ is the smallest saturated multiplicative set in R containing S. Moreover,

(1) $0 \in \sigma(S)$ iff $0 \in S$;	*(2) $S \subseteq T \;\Rightarrow\; \sigma(S) \subseteq \sigma(T)$;*
(3) $\sigma(\sigma(S)) = \sigma(S)$;	*(4) $S \in \mathcal{M}_\sigma(R)$ iff $\sigma(S) = S$.*

b) If $T_i \in \mathcal{M}_\sigma(R)$, $i \in I$, then

$$\textstyle\bigvee_{i\in I} T_i \;\; = \;\; \sigma\left(\prod_{i\in I} T_i\right)$$

is the join of the T_i in $\mathcal{M}_\sigma(R)$.

PROOF. a) Clearly, $1 \in \sigma(S)$; if x, $y \in \sigma(S)$, there are s, $t \in R$ such that sx, $ty \in S$. Thus, $(st)(xy) \in S$ and so $xy \in \sigma(S)$. Now suppose that $ab \in \sigma(S)$; then there is $s \in R$ such that $sab \in S$. From

$$(sa)b \in S \quad \text{and} \quad (sb)a \in S$$

we get a, $b \in \sigma(S)$, proving that $\sigma(S) \in \mathcal{M}_\sigma(R)$. It is clear that $\sigma(S)$ is contained in any element of $\mathcal{M}_\sigma(R)$ that includes S. Item $(1) - (4)$ are straightforward and left to the reader.
b) Set $S = \prod_{i\in I} T_i$; since S is the join of the T_i in $\mathcal{M}(R)$, any *saturated* multiplicative set containing the T_i, must contain S. But then, item (a) entails that $\sigma(S)$ is the least element of $\mathcal{M}_\sigma(R)$ containing T_i, as needed. $\square$

Definition 9.24. *a) For $S \in \mathcal{M}(R)$, the saturated multiplicative set $\sigma(S)$, defined in* 9.23*, is the* **saturation** *of S in R.*
b) For $a \in R$, define

$$\sigma(a) \;=\; \sigma(\{a^n : n \geq 0\}).$$

Lemma 9.25. *For $S \in \mathcal{M}_\sigma(R)$ and $a \in R$*
a) $\sigma(a)$ is compact in $\mathcal{M}_\sigma(R)$.
b) $S \;=\; \bigvee_{s \in S} \sigma(s)$.
c) $\mathcal{M}_\sigma(R)$ is a complete algebraic lattice.

Proof. a) For $S_i \in \mathcal{M}_\sigma(R)$, $i \in I$, suppose that $\sigma(a) \subseteq \bigvee_{i \in I} S_i$. By items (a) and (b) in 9.23 this implies

$$a \in \sigma\left(\prod_{i \in I} S_i\right),$$

that is, there is $y \in R$ and $\alpha \subseteq_f \bigcup_{i \in I} S_i$, such that $ya = \Pi\ \alpha$. But then,

$$a \in \sigma\left(\prod_{i \in \alpha} S_i\right),$$

and $\sigma(a) \subseteq \bigvee_{i \in \alpha} S_i$, with α finite in I, as needed.
b) Clearly, $S \subseteq \bigvee_{s \in S} \sigma(s)$; if $x \in \bigvee_{s \in S} \sigma(s) = \sigma(\prod_{s \in S} \sigma(s))$, then there is $y \in R$ and $\alpha \subseteq_f \bigcup_{s \in S} \sigma(s)$, such that $yx = \Pi\ \alpha$. Write

$$\alpha = \{t_1, \ldots, t_n\}.$$

For each $1 \leq j \leq n$, there is $s_j \in S$ with $t_j \in \sigma(s_j)$; hence, there are $z_j \in R$ and integers $k_j \geq 0$ such that for $1 \leq j \leq n$,

$$z_j t_j \;=\; s_j^{k_j}$$

Therefore, if $z = \prod_{j=1}^n z_j$, we have

$$(zy)x \;=\; z\ \Pi\ \alpha \;=\; \prod_{j=1}^n z_j t_j \;=\; \prod_{j=1}^n s_j^{k_j} \;\in\; S,$$

and so $x \in \sigma(S) = S$ (9.23.(a).(4)), as desired. Item (c) is an immediate consequence of (a) and (b). $\Box$

To establish that $\mathcal{M}_\sigma(R)$ is a frame, we must take a closer look at the relationship between saturated multiplicative sets and collections of primes in R. This will also yield *the saturated multiplicative set generated by any subset of R* [6]. With notation as in 9.17, we start with

Lemma 9.26. *For S, $T \subseteq R$ and $a \in R$*
a) $Z_S \;=\; \bigcap_{s \in S} Z_s$.
b) $Z_a = \emptyset \;\Leftrightarrow\; a \in \eta$ [7].
c) S is proper in $\mathcal{M}(R) \;\Leftrightarrow\; Z_S \neq \emptyset$.
d) Set $S \cdot T = \{xy : x \in S$ and $y \in T\}$ [8]*. Then,*

$$1 \in (S \cap T) \;\Rightarrow\; Z_{S \cdot T} = Z_S \cap Z_T.$$

Proof. Item (a) is immediate from the definition, while (b) is a consequence of 9.13. For (c), if $Z_S \neq \emptyset$, then there is a prime P such that $P \cap S = \emptyset$. Hence,

[6] The map σ of 9.23 and 9.24 only works for elements in $\mathcal{M}(R)$.
[7] The ideal of nilpotents in R.
[8] Generalizing notation in 9.21

$0 \notin S$ and S is proper. Conversely, if S is a proper multiplicative subset of R, then $(0) \cap S = \emptyset$, and Theorem 9.5 yields a prime ideal P such that $P \cap S = \emptyset$, that is, $P \in Z_S$. For (d), note that $1 \in (S \cap T)$ implies that $S, T \subseteq S \cdot T$. Hence, any prime disjoint from $S \cdot T$ must be disjoint from S and T, i.e.,

$$Z_{S \cdot T} \ \subseteq \ Z_S \cap Z_T.$$

For the reverse inclusion, observe that

$$x \in P \cap (S \cdot T) \ \Rightarrow \ x = st \in P \ \Rightarrow \ s \in P \cap S \ \text{ or } \ t \in P \cap T.$$

Hence, $Z_S \cap Z_T \subseteq Z_{S \cdot T}$, as desired. □

Proposition 9.27. *a) If $\mathcal{P}$ is a family of prime ideals R, then*

$$M_{\mathcal{P}} \ =_{def} \ \bigcap \{P^c : P \in \mathcal{P}\}$$

is a saturated multiplicative subset of R. Moreover,

$$\mathcal{P} \subseteq \mathcal{Q} \ \Rightarrow \ M_{\mathcal{Q}} \subseteq M_{\mathcal{P}}.$$

b) If S is a multiplicative subset of R, then

$$S \text{ is saturated} \quad \Leftrightarrow \quad S = M_{Z_S} = \bigcap \{P^c : P \cap S = \emptyset\}.$$

c) If $T \subseteq R$, then M_{Z_T} is the least saturated multiplicative set containing T. Moreover, M_T is proper iff $Z_T \neq \emptyset$.

Proof. a) Write M for $M_{\mathcal{P}}$; we may assume that $\mathcal{P} \neq \emptyset$, otherwise $M = R$. Clearly, $1 \in M$ (all P in $\mathcal{P}$ are proper) and $0 \notin M$. Since the elements of $\mathcal{P}$ are prime, M is closed under products. If $xy \in M$, then $xy \notin P$, for all $P \in \mathcal{P}$. Since these are ideals, we obtain $x, y \notin P$ and so $x, y \in M$, establishing saturation. The remaining assertion is immediate.
b) By (a), it is enough to prove ($\Rightarrow$). Moreover, clearly $S \subseteq M_{Z_S}$. For the reverse containment, suppose $b \in R - S$; then $(b) \cap S = \emptyset$. For if there was $y \in R$ such that $yb \in S$, then saturation would imply $b \in S$, contrary to assumption. We now apply Theorem 9.5 to obtain a prime ideal P such that $b \in P$ and $P \cap S = \emptyset$. Hence, $b \notin M_{Z_S}$, as needed.
c) If $S \in \mathcal{M}_\sigma(R)$ contains T, then any prime that is disjoint from S is disjoint from T. Thus, $Z_S \subseteq Z_T$ and so (a) and (b) entail

$$M_{Z_T} \ \subseteq \ M_{Z_S} \ = \ S,$$

as needed. The properness claim is clear. □

Definition 9.28. *If $T \subseteq R$, the* **multiplicative saturation of T** *is*

$$\sigma(T) \ =_{def} M_{Z_T} \ = \ \bigcap \{P^c : P \in Z_T\}.$$

Remark 9.29. Propositions 9.23.(a) and 9.27.(c) guarantee that the map

$$T \in 2^R \ \longmapsto \ M_{Z_T}$$

is an *extension* of the map

$$T \in \mathcal{M}(R) \ \longmapsto \ \{x \in R : \exists\, y \in R \text{ such that } yx \in T\},$$

and we shall employ the same symbol for both, namely σ. □

We now show that if R is a Gaussian domain, then $\mathcal{M}_\sigma(R)$ is a frame. In section 3 of Chapter 2 of [**20**] the reader will find a good exposition of the basic properties of Gaussian domains. We collect the main properties needed in the results that follow.

Theorem 9.30. *Let R be a Gaussian domain.*

a) For $a \in R$, a is irreducible iff a is prime iff (a) is a prime ideal in R.

b) For each $a \in R$, there is $u \in U(R)$ together with a unique and finite [9] *collection of pairs $\{\langle p_k, m_k \rangle : k \in \alpha\}$, where $\{p_k : k \in \alpha\}$ are distinct primes in R and $\{m_k : k \in \alpha\}$ are integers ≥ 1, such that $a = u\ \Pi_{k\in\alpha}\ p_k^{m_k}$.* [10]

c) All pairs of elements in R have a greatest common divisor and a least common multiple.

The following result yields many examples of Gaussian domains.

Theorem 9.31. *a) Every principle ideal domain is Gaussian.*
b) The union of an up-directed family of Gaussian domains is Gaussian.
c) If R is a Gaussian domain, then $R[X]$ [11] *is Gaussian.*

Example 9.32. a) All fields are Gaussian domains.
b) Since $\mathbb{Z}$ and $k[X]$ are principal ideal domains (k a field), they are Gaussian domains.
c) From 9.31.(c) it follows that if k is field, then the ring of polynomials in the variables $X_1, \ldots, X_n$, $k[X_1, \ldots, X_n]$, is a Gaussian domain.
d) To obtain a non-Noetherian example, just apply 9.31.(b) to get that $k[X_1, X_2, \ldots, X_n, \ldots]$, the ring of polynomials in an infinite number of variables with coefficients in a field k, is a Gaussian domain. $\Box$

Proposition 9.33. *Let R be a Gaussian domain.*
a) If $S, T \in \mathcal{M}_\sigma(R)$, then $S \vee T = S \cdot T$.
b) $\mathcal{M}_\sigma(R)$ is an algebraic frame.

Proof. a) Recall that for $x, y \in R$

$$x \textit{ divides } y \quad \text{iff} \quad \exists\, z \in R \text{ such that } zx = y \quad \text{iff} \quad y \in (x).$$

Since $S \vee T = \sigma(S \cdot T)$, it is enough to show that $S \cdot T$ is saturated. Assume that $xy = st \in S \cdot T$. By 9.30.(b) we may write

$$s = u\ \Pi_{i\in\alpha}\ p_i^{m_i} \quad \text{and} \quad t = v\ \Pi_{k\in\beta}\ q_k^{n_k},$$

where p_i and q_k are primes in R and $u, v \in U(R)$. Note that the saturation of S and T guarantees that

(*) For all $i \in \alpha$, $k \in \beta$ and $m \geq 0$, $p_i^m \in S$ and $q_k^m \in T$.

Since an element of p is prime iff for all $x, y \in R$

$$p \text{ divides } xy \quad \text{iff} \quad p \text{ divides } x \quad \text{or} \quad p \text{ divides } y,$$

it follows that

$$\forall \text{ primes } p \in R,\ \ p \text{ divides } x \ \Rightarrow\ p \in \{p_i : i \in \alpha\} \cup \{q_k : k \in \beta\}.$$

Hence, the unique factorization in 9.30.(b) implies that x is the product of a unit in R by powers of some of the p_i and some of the q_k. Since $U(R) \subseteq S \cap T$, (*) then entails that $x \in S \cdot T$. Similarly, one shows that $y \in S \cdot T$, as needed.

[9] Possibly empty; recall that the empty product equals 1.
[10] Because of (a), this may be taken to be the *definition* of Gaussian domain.
[11] The ring of polynomials in the variable X.

b) Since $\mathcal{M}_\sigma(R)$ is an complete algebraic lattice (9.25.(c)), it is enough, by 8.14, to verify that $\mathcal{M}_\sigma(R)$ is distributive. But this follows from 9.22.(b). $\square$

REMARK 9.34. Recall that ring R is **Noetherian** if it satisfies the following equivalent conditions :
[N 1] : If $I_1 \subseteq I_2 \subseteq \ldots \subseteq I_n \subseteq \ldots$ is a sequence of ideals in R, then there is $m \geq 1$ such that $I_m = I_n$, for all $n \geq m$;
[N 2] : Every non-empty set of ideals in R has a maximal element [12];
[N 3] : Every ideal in R is finitely generated, that is, if I is an ideal in R, there are $a_1, \ldots, a_n \in R$ such that

$$I \;=\; \{\textstyle\sum_{i=1}^n c_i a_i : c_i \in R,\, 1 \leq i \leq n\}. \qquad \square$$

THEOREM 9.35. *Let R be a Noetherian ring.*
a) If S, $T \in \mathcal{M}_\sigma(R)$, then $Z_{S \cap T} = Z_S \cup Z_T$.
b) $\mathcal{M}_\sigma(R)$ is an algebraic frame.

PROOF. a) Clearly, $Z_S \cup Z_T \subseteq Z_{S \cap T}$. Now suppose that $P \cap S \cap T = \emptyset$. Then, by 9.27.(b) yields

$$P \;\cap\; \bigcap \{Q^c : Q \in Z_S\} \;\cap\; \bigcap \{R^c : R \in Z_T\} \;=\; \emptyset,$$

that is,

$$P \;\subseteq\; \bigcup \{Q : Q \in Z_S\} \;\cup\; \bigcup \{R : R \in Z_T\}.$$

Since P is finitely generated (R is Noetherian; see [N 3] in 9.34), there are $Q_1, \ldots, Q_n$ in Z_S and $R_1, \ldots, R_m$ in Z_T, such that

$$P \;\subseteq\; \textstyle\bigcup_{i=1}^n Q_i \;\cup\; \bigcup_{j=1}^m R_j.$$

But then, 9.5.(b) implies that either $P \subseteq Q_i$ ($i \leq n$), or $P \subseteq R_j$ ($j \leq m$). If former alternative holds, then $P \in Z_S$, while the latter entails $P \in Z_T$, as needed.
b) Since $\mathcal{M}_\sigma(R)$ is a complete algebraic lattice (9.25.(c)), it is enough to check that it is distributive (8.14). For S, T_1, $T_2 \in \mathcal{M}_\sigma(R)$, item (a), together with 9.26.(d) and 9.27.(b), yields

$$\begin{aligned}
S \cap (T_1 \vee T_2) &= S \cap \sigma(T_1 \cdot T_2) \\
&= \bigcap \{P^c : P \in Z_S\} \;\cap\; \bigcap \{P^c : P \in Z_{T_1 T_2}\} \\
&= \bigcap \{P^c : P \in Z_S\} \;\cap\; \bigcap \{P^c : P \in (Z_{T_1} \cap Z_{T_2})\} \\
&= \bigcap \{P^c : P \in (Z_S \cup (Z_{T_1} \cap Z_{T_2})\} \\
&= \bigcap \{P^c : P \in (Z_S \cup Z_{T_1}) \cap (Z_S \cup Z_{T_2})\} \\
&= \bigcap \{P^c : P \in (Z_{S \cap T_1} \cap Z_{S \cap T_2})\} \\
&= \bigcap \{P^c : P \in Z_{(S \cap T_1) \cdot (S \cap T_2)}\} \\
&= (S \cap T_1) \vee (S \cap T_2),
\end{aligned}$$

ending the proof. $\square$

Although Gaussian domains and Noetherian rings are important classes of commutative rings, we shall construct an even wider class of rings for which the saturated multiplicative sets are an algebraic frame. This will be obtained as an

[12] With respect to inclusion.

application of our development of structures of locally constant functions on a topological space in section 5 of Chapter 24 (see Theorem 24.56).

3. Rings of Fractions

If S is a multiplicative subset of R (9.3.(b)), we can form a new ring

$$RS^{-1},$$

the **ring of fractions** of R by S, by the following procedure :

In the product $R \times S$ define a relation

$$\langle a, s \rangle \sim \langle b, t \rangle \quad \text{iff} \quad \text{there is } w \in S \text{ such that } w(at - bs) = 0.$$

Then, $\sim$ is an equivalence relation in $R \times S$. Let

$$RS^{-1} = \{\frac{a}{s} : a \in R \text{ and } s \in S\}$$

be the set o equivalence classes of $R \times S$ under $\sim$. Note that the class of $\langle a, s \rangle$ by $\sim$ is being written as $\frac{a}{s}$. Hence,

[fraction] $$\frac{a}{s} = \frac{b}{t} \quad \text{iff} \quad \exists\, w \in S, \text{ such that } w(at - bs) = 0.$$

Define operations $+, \cdot$ in RS^{-1} by

$$\frac{a}{s} + \frac{b}{t} = \frac{at + bs}{st} \quad \text{and} \quad \frac{a}{s}\,\frac{b}{t} = \frac{ab}{st}.$$

It is straightforward that this is independent of representatives.

Proposition 9.36. *Let R be a ring and S a multiplicative subset of R.*
a) With the structure introduced above, RS^{-1} is a commutative ring with identity $1 = \frac{1}{1}$.
b) The map $a \in R \longmapsto \iota_S(a) = \frac{a}{1} \in RS^{-1}$ is a ring homomorphism, such that for all $t \in S$,

(1) $t \in S \;\Rightarrow\; \iota_S(t)$ is a unit in RS^{-1}, with inverse $\frac{1}{t}$;

(2) $\iota_S(t) = 0 \;\Rightarrow\; \exists\, s \in S$ such that $ts = 0$.

c) If $R \xrightarrow{f} R'$ is a ring homomorphism such that for all $s \in S$, $f(s)$ is a unit in R', then the map

$$\frac{a}{s} \longmapsto f(a)f(s)^{-1}$$

is the carrier of the unique *ring homomorphism, $g : RS^{-1} \longrightarrow R'$, making the following diagram commutative :*

$$\begin{tikzcd} R \arrow[rr, "\iota_S"] \arrow[dr, "f"'] & & RS^{-1} \arrow[dl, "g"] \\ & R' & \end{tikzcd}$$

Proof. Items (a) and (b) are clear; for (c), we must show that

$$g(a/s) = f(a)f(s)^{-1}$$

is well defined. If $\frac{a}{s} = \frac{b}{t}$, then there is $w \in S$ such that $w(at - bs) = 0$. Whence,

$$f(w)(f(a)f(t) - f(b)f(s)) = 0,$$

and so, since $f(w)$ is a unit in R', we get

$$f(a)f(t) = f(b)f(s),$$

wherefrom it follows that

$$g(\frac{a}{s}) = f(a)f(s)^{-1} = f(b)f(t)^{-1} = g(\frac{b}{t}).$$

It is straightforward that g is a ring homomorphism making the displayed diagram commutative. Uniqueness is clear. $\square$

Remark 9.37. Note that $RS^{-1} = \{0\}$ iff $0 \in S$ [13]. $\square$

Corollary 9.38. *Let R be a ring and $S \subseteq T$ be multiplicative subsets of R.*
a) There is a unique *ring homomorphism*

$$\rho_{ST} : RS^{-1} \longrightarrow RT^{-1}, \quad \rho_{ST}(\frac{a}{s}) = \frac{a}{s},$$

making the following diagram commutative :

$$\begin{tikzcd} R \arrow[rr, "\iota_S"] \arrow[dr, "\iota_T"'] & & RS^{-1} \arrow[dl, "\rho_{ST}"] \\ & RT^{-1} & \end{tikzcd}$$

b) The following conditions are equivalent :
(1) *ρ_{ST} is an isomorphism;* (2) $T \subseteq \sigma(S)$ [14].

Proof. a) Let $\iota_T : R \longrightarrow RT^{-1}$ be the canonical ring homomorphism of 9.36.(b); since for all $s \in S$, $\iota_T(s)$ is a unit in RT^{-1}, existence and uniqueness of ρ_{ST} follow from 9.36.(c). For $\frac{a}{s} \in RS^{-1}$, the proof of 9.36.(c) yields

$$\rho_{ST}(\frac{a}{s}) \ = \ \frac{a}{s},$$

where the right-hand side of the equality is in RT^{-1}.
b) For $x \in R$ and $A \subseteq R$, set $xA = \{xa : a \in A\}$.
<u>(1) $\Rightarrow$ (2)</u> : We first verify that for all $x \in R$

$$(*) \qquad 0 \in xT \ \Rightarrow \ 0 \in xS.$$

Indeed, if $0 \in xT$, then $\iota_T(x) = \frac{x}{1} = 0$ in RT^{-1}. Since $\iota_T = \rho_{ST} \circ \iota_S$ and ρ_{ST} is injective, we get $\frac{x}{1} = 0$ in RS^{-1}, that is, $0 \in xS$.

[13] The ring $\{0\}$ is called the *zero ring.*
[14] The saturation of S in R, as in 9.28.

For $t \in T$, the fact that ρ_{ST} is surjective yields $\frac{a}{s} \in RS^{-1}$, such that, in RT^{-1},

$$\rho_{ST}(\frac{a}{s}) = \frac{a}{s} = \frac{1}{t}.$$

Hence, there is $t' \in T$, such that

$$t'(at - s) = 0.$$

By (*), there is $s' \in S$, satisfying

$$s'(at - s) = 0,$$

that is, $(as')t = s's \in S$, and $t \in \sigma(S)$, as desired.

(2) $\Rightarrow$ (1) : For $t \in T$, select $c \in R$, with $ct \in S$. Then, if $a \in R$, we have

$$\rho_{ST}(\frac{ac}{ct}) \;=\; \frac{ac}{ct} \;=\; \frac{a}{t},$$

in RT^{-1}, showing that ρ_{ST} is surjective. A similar technique establishes injectivity, ending the proof. $\square$

PROPOSITION 9.39. *Let R be a ring and S a multiplicative set in R.*
a) If $P \in Z_S$, then

$$PS^{-1} = \{\frac{a}{s} : a \in P \text{ and } s \in S\}$$

is the unique *prime ideal in RS^{-1}, such that $\iota_S^{-1}(PS^{-1}) = P$.*
b) The map

$$Q \in Spec(RS^{-1}) \;\longmapsto\; \iota_{SZ}(Q) \;=_{def}\; \iota_S^{-1}(Q) \in Spec(R)$$

is a bijection between $Spec(RS^{-1})$ and Z_S, whose inverse is

$$P \in Z_S \longmapsto \beta(P) = PS^{-1} \in Spec(RS^{-1}).$$

Moreover, for $a \in R$ and $s \in S$,

$$\beta^{-1}(Z_{a/s}) = \beta^{-1}(Z_{a/1}) = Z_a \quad \text{and} \quad \iota_{SZ}^{-1}(Z_a) = Z_{a/1}.$$

PROOF. a) It is straightforward that PS^{-1} is a proper ideal in RS^{-1}. To prove primeness, suppose that $\frac{a}{s}\frac{b}{t} = \frac{ab}{st} = \frac{c}{w}$, with $c \in P$; then, there is $u \in S$ such that

$$u(abw - cst) = 0.$$

Hence $(uw)(ab) = c(ust) \in P$. Since $P \cap S = \emptyset$ and $uw \in S$, we obtain $ab \in P$. Hence, $a \in P$ or $b \in P$, and so either $\frac{a}{s}$ or $\frac{b}{t}$ are in PS^{-1}.

Assume that $Q \in Spec(RS^{-1})$ verifies $\iota_{SZ}(Q) = \iota_S^{-1}(Q) = P$. If $\frac{a}{s} \in Q$, then

$$\frac{s}{1} \cdot \frac{a}{s} = \frac{a}{1} = \iota_S(a),$$

and $a \in P = \iota_S^{-1}(Q)$. But then $\frac{a}{s} \in PS^{-1}$, and $Q \subseteq PS^{-1}$. For the reverse inclusion, note that if $a \in P$ and $s \in S$, then

$$\frac{a}{s} = \frac{a}{1} \cdot \frac{1}{s}$$

and so, since Q is an ideal, we must have $\frac{a}{s} \in Q$, as needed.

b) Because every element Q of $Spec(RS^{-1})$ is *proper* prime ideal and s is a unit in RS^{-1}, we cannot have $\iota_{SZ}(Q) \cap S \neq \emptyset$. Hence, the image of ι_{SZ} is contained in $Z_S = \bigcap_{s \in S} Z_s$. It follows from (a) that ι_{SZ} is a bijection between $Spec(RS^{-1})$ and Z_S, whose inverse is the map β in the statement.

For $a \in R$ and $s \in S$, in $Spec(RS^{-1})$, we have
$$Z_{a/s} = Z_a,$$
because s is a unit in RS^{-1}. Moreover, for all $P \in Spec(R)$,
$$a \notin P \quad \text{iff} \quad \frac{a}{1} \notin PS^{-1},$$
wherefrom it is straightforward to conclude that
$$\beta^{-1}(Z_{a/1}) = Z_a \quad \text{and} \quad \iota_{SZ}^{-1}(Z_a) = Z_{a/1},$$
ending the proof. □

COROLLARY 9.40. *Let R be a ring and P a prime ideal in R. If $S = P^c \in \mathcal{M}_\sigma(R)$,*
a) RS^{-1} is a local ring [15]*, whose only maximal ideal is PS^{-1}.*
b) The prime ideals in RS^{-1} correspond bijectively to the prime ideals contained in P.

PROOF. a) If $\frac{x}{s} \notin PS^{-1}$, then $x \notin P$, that is, $x \in S$. Hence, x is a unit in RS^{-1}, the same being therefore true of $\frac{x}{s}$. We have just verified that any element outside PS^{-1} is a unit and so this ideal must be the **only** maximal ideal in RS^{-1}.
b) Just apply 9.39.(b) to $S = P^c$. □

DEFINITION 9.41. *The local ring of* 9.40.(*a*) *is called the* **localization of *R* at the prime *P***, *written R_P. The canonical homomorphism from R to R_P will be written $\iota_P : R \longrightarrow R_P$* [16].

We shall now discuss how rings of fractions behave with respect to ring homomorphisms.

LEMMA 9.42. *Let $f : A \longrightarrow B$ be a homomorphism of rings with identity* [17].
a) If T is a multiplicative set in B, then
(1) *$f^*T =_{def} f^{-1}(T)$ is a multiplicative set in A, which is saturated if the same is true of T. Moreover, f^*T is proper iff T is proper.*
(2) *The map $a/s \longmapsto fa/fs$ is the carrier of the* unique *ring homomorphism, $f_T^* : A(f^*T)^{-1} \longrightarrow BT^{-1}$, making the diagram below-left commutative.*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\iota"'] & B \arrow[d, "\iota"] \\ A(f^*T)^{-1} \arrow[r, "f_T^*"'] & BT^{-1} \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\iota"'] & B \arrow[d, "\iota"] \\ AS^{-1} \arrow[r, "f_*^S"'] & B(f_*S)^{-1} \end{tikzcd}$$

[15] A ring is *local* if it has only one maximal ideal.
[16] Instead of ι_{P^c}.
[17] That is, B is a A-algebra.

b) If S is a multiplicative set in A, then

(1) *$f_*S =_{def} \{fx : x \in S\}$ is a multiplicative set in B, which is proper iff $S \cap ker\, f = \emptyset$.*

(2) *The map $a/s \longmapsto fa/fs$ is the carrier of the* unique *ring homomorphism, $f_*^S : AS^{-1} \longrightarrow B(f_*S)^{-1}$, making the diagram above-right commutative.*

PROOF. a) Clearly, f^*T is multiplicative; and it is proper iff T is proper. If $T \in \mathcal{M}_\sigma(B)$ and $xy \in f^*T$, then $f(xy) = f(x)f(y) \in T$ and so $f(x)$, $f(y) \in T$. Hence, $x, y \in f^*T$, as desired. Existence and uniqueness of f_T^* comes from 9.36.(c), applied to the ring homomorphism $h = \iota_T \circ f : A \longrightarrow BT^{-1}$. The verification of (b) is similar and left to the reader. $\square$

REMARK 9.43. If $f : A \longrightarrow B$ is a ring homomorphism and S is a multiplicative set in A such that $S \cap ker\, f \neq \emptyset$, then $B(f_*S)^{-1} = \{0\}$ is the zero ring (9.37) and $f_*^S(x) = 0$, for all $x \in AS^{-1}$. $\square$

COROLLARY 9.44. *Let $f : A \longrightarrow B$ be a homomorphism of rings with identity. If $P \in Spec(B)$, then $f^*P \in Spec(A)$. Moreover, there is a unique ring homomorphism*

$$f_P^* : A_{f^*P} \longrightarrow B_P, \quad f_P^*(a/x) = f(a)/f(x),$$

making the following diagram commutative :

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\iota_{f^*P}"'] & B \arrow[d, "\iota_P"] \\ A_{f^*P} \arrow[r, "f_P^*"'] & B_P \end{tikzcd}$$

PROOF. Just apply 9.42.(a) to $T = P^c$, using the notational conventions of 9.41. Note that, consistently with the latter, the homomorphism induced by the inverse image of P^c by f is being written f_P^*. $\square$

Exercises

9.45. Let I, J be ideals in R. If P is a prime ideal in R, then $IJ \subseteq P$ iff $I \subseteq P$ or $J \subseteq P$. $\square$

9.46. If I, J, K and K_p, $p \in A$, are ideals in R

a) $I \subseteq (I : J)$.

b) $(I : J)J \subseteq I$.

c) $((I : J) : K) = (I : JK) = ((I : K) : J)$.

d) $\left(\bigcap_{p\in A} K_p : J\right) = \bigcap_{p\in A} (K_p : J)$.

e) $\left(I : \sum_{p\in A} K_p\right) = \bigcap_{p\in A} (I : K_p)$. $\square$

9.47. Show that $nzd(R)$ [18] is a saturated multiplicative subset of R.

9.48. Saturation [19] is a map, $2^R \xrightarrow{\sigma} \mathcal{M}_\sigma(R)$, with the following properties : For T, $S \subseteq R$, and W, $S_i \in \mathcal{M}_\sigma(R)$, $i \in I$,
a) $T \subseteq \sigma(T)$; $T \subseteq W \;\Rightarrow\; \sigma(T) \subseteq W$.
b) $Z_T = Z_{\sigma(T)}$ and $\sigma(\sigma(T)) = \sigma(T)$.
c) $S \subseteq T \;\Rightarrow\; \sigma(S) \subseteq \sigma(T)$.
d) $S \subseteq T \subseteq \sigma(T) \;\Rightarrow\; \sigma(S) = \sigma(T)$.
e) In $\mathcal{M}_\sigma(R)$, $\bigvee_{i\in I} S_i \;=\; \sigma\left(\bigcup_{i\in I} S_i\right)$. □

9.49. If R is a ring, recall (9.47) that $N = nzd(R)$ is a saturated multiplicative set in R. The ring RN^{-1} is called the **total ring of fractions of $\boldsymbol{R}$**.
a) N is the largest multiplicative set S in R for which $\iota_S : R \longrightarrow RS^{-1}$ is injective.
b) Every element of RN^{-1} is either a zero-divisor or a unit.
c) A commutative ring with identity in which every element is either a unit or a zero-divisor is isomorphic to its total ring of fractions. □

[18] Defined in 9.17.(b).
[19] See Definitions 9.24, 9.28 and Remark 9.29.

CHAPTER 10

$\bigvee$-Filters

We now turn to the construction of quotient frames. Our first topic is a characterization of the filters whose quotient map is a frame morphism. In the next chapter we discuss quotients in the category **Frame** using frame congruences, while in Chapter 13, we shall develop an intuitionistically acceptable theory of quotients, applying the fixed point Theorem 7.5 to certain types of operators.

DEFINITION 10.1. *Let H be a frame. A filter F in H is a* $\bigvee$**-filter** *iff for all $S \subseteq H$ and $x \in H$,*

$$\text{If } \ \forall s \in S,\ (s \to x) \in F, \quad \text{then } \ \bigwedge_{s \in S} (s \to x) \in F.$$

For $b \in Reg(H)$, set $D_b = \{x \in H : \neg\neg x \geq b\}$.

By Lemma 6.8.(f), D_b is a filter, which is proper iff $b \neq \bot$. Note that the filter D, of dense elements in H, is precisely $D_\top$. The main properties of $\bigvee$-filters are described in

PROPOSITION 10.2. *Let H, P be frames and let F be a filter in H. Then*
a) For all $x \in H$, $\bigvee x/F = \bigvee \{z \to x : z \in F\}$.
b) The following are equivalent :
(1) *F is a $\bigvee$-filter.*
(2) *For all $x \in H$, $\bigvee \{y \in H : y \sim_F x\} = \bigvee x/F \in x/F$.*
(3) *H/F is a frame and the quotient HA morphism, π_F, is a frame morphism.*
Moreover, for a HA morphism $f : H \longrightarrow P$, a necessary condition for f to be a frame morphism is that coker f be a $\bigvee$-filter. This condition is also sufficient in case f is onto.
c) Principal filters and the filters D_b, $b \in Reg(H)$, are $\bigvee$-filters.
d) The intersection of any family of $\bigvee$-filters is a $\bigvee$-filter.
e) If F is an $\bigvee$-filter in H, then $F \subseteq D_b$, for some $b \in Reg(H)$. Further, the following are equivalent :
(1) *$D = D_\top \subseteq F$;*
(2) *$F = D_c$, for some $c \in Reg(H)$.*

PROOF. a) Recall that for $x, b \in H$, $x \sim_F b$ iff $\exists\, z \in F$ such that $z \wedge x = z \wedge b$. Thus, if $b \sim_F x$ then, for some $z \in F$, $z \wedge b \leq x$, and so $b \leq z \to x$. On the other hand, by *Modus Ponens*, $x \sim_F (z \to x)$, $\forall\, z \in F$. Clearly, (a) follows directly from these observations.

b) $\underline{(1) \Rightarrow (2)}$: For all $x \in H$, $x \leq \bigvee x/F$, and so $(x \to \bigvee x/F)$ is always in F. If it is shown that $[(\bigvee x/F) \to x] \in F$, then $x \sim_F \bigvee x/F$ (6.15.(a)). Now, if $b \in x/F$, then $(b \to x) \in F$; since F is a $\bigvee$-filter, we get

$$\bigwedge \{(b \to x) : b \in x/F\} = [(\bigvee x/F) \to x] \in F,$$

as required.

$\underline{(2) \Rightarrow (1)}$: Initially we make note of two simple facts.

Fact 1. *For $a, b \in H$, $a \to b \in F$ iff $a \wedge b \in a/F$ iff $a \vee b \in b/F$.*

Proof. Corollary 8.13.(a) yields

$$a/F \leq b/F \ \text{ iff } \ \exists\, z \in F \text{ such that } a \wedge z \leq b \ \text{ iff } \ a \to b \in F.$$

Hence, in H/F, $\begin{cases} a/F \vee b/F = (a \vee b)/F = b/F; \\ a/F \wedge b/F = (a \wedge b)/F = a/F. \end{cases}$

Similar considerations will yield

Fact 2. *For $a, b, c \in H$,*

$$a \leq b \leq c \text{ and } a/F = c/F \ \Rightarrow \ a/F = b/F = c/F.$$

Suppose $S \cup \{x\} \subseteq H$ satisfies $(s \to x) \in F$, $\forall\, s \in S$; by Fact 1, $(s \vee x) \in x/F$. Since $x \leq (\bigvee S) \vee x \leq \bigvee x/F$, (2) and Fact 2 yield $(\bigvee S) \vee x \in x/F$. Another application of Fact 1 will give

$$\bigwedge_{s \in S} (s \to x) = (\bigvee S) \to x \in F,$$

showing that F is a $\bigvee$-filter.

$\underline{(1) \Rightarrow (3)}$: We show that H/F is a frame and that π_F is a frame morphism, by proving that for all $S \subseteq H$,

$$\pi_F(\bigvee S) = \bigvee \pi_F(S).$$

Since π_F is increasing, it is enough to prove that

$$x/F \geq s/F, \text{ for all } s \in S \ \Rightarrow \ x/F \geq (\bigvee S)/F = \pi_F(\bigvee S).$$

By Fact 1, $x/F \geq s/F$ means $(s \to x) \in F$; if F is a $\bigvee$-filter, from $x/F \geq s/F$ we get $\bigwedge_{s \in S} (s \to x) = (\bigvee S) \to x \in F$, yielding $(\bigvee S)/F \leq x/F$, as needed.

$\underline{(3) \Rightarrow (1)}$: Since π_F is a HA-morphism (6.16), it is enough to show that the cokernel of any HA-morphism, *which is a frame morphism*, must be a $\bigvee$-filter, and so we discuss this in general.

Let $H \xrightarrow{f} P$ be a HA-morphism, preserving arbitrary joins. Let $S \cup \{x\} \subseteq H$ satisfy $(s \to x) \in \text{coker } f$, $s \in S$. Then, $f(s \to x) = fs \to fx = \top$. Therefore, $fs \leq fx$, $\forall\, s \in S$. Since f preserves joins, $f(\bigvee S) = \bigvee f(S) \leq fx$. Therefore, $(\bigvee S \to x) \in \text{coker } f$, proving the latter to be a $\bigvee$-filter.

If f is onto and $F = \text{coker } f$, P is naturally isomorphic to H/F, with π_F a frame morphism. Since any isomorphism is open, f, as a composition of frame morphisms, will also be a frame morphism.

c) Fix $b \in Reg(H)$ and $x \in H$. By (b), it is sufficient to prove that $(\bigvee x/D_b) \in x/D_b$. Since $D \subseteq D_b$, we have $x \vee \neg x \in D_b$; thus, the equation

$$x \wedge (x \vee \neg x) = \neg\neg x \wedge (x \vee \neg x)$$

guarantees that $x/D_b = \neg\neg x/D_b$. Now, for each $z \in D_b$, we obtain, using Lemmas 6.8.(a) and 6.4.(d)

$$(z \to \neg\neg x) \le (\neg x \to \neg z) \le (\neg\neg z \to \neg\neg x) \le b \to \neg\neg x.$$

By (a), $\bigvee x/D_b = \bigvee \neg\neg x/D_b = b \to \neg\neg x$; from $b \in D_b$ and $b \wedge (b \to \neg\neg x) = b \wedge \neg\neg x$, we conclude that $b \to \neg\neg x \in x/D_b$, completing the proof of (c). Item (d) is straightforward.

e) For the first assertion in (e), if F is a $\bigvee$-filter, let $b = \bigvee \perp/F \in \perp/F$. To show that b is regular, note that

$$b \sim_F \perp \;\Rightarrow\; \neg b \sim_F \top \;\Rightarrow\; \neg\neg b \sim_F \perp,$$

that entails $b = \neg\neg b$. Thus, b, $\neg b \in Reg(H)$ and $\neg b \in F$. Furthermore, if $t \in F$, then $\neg t \in \perp/F$ and so $\neg t \le b$. This immediately implies the inclusion of F in $D_{\neg b}$. To verify the stated equivalence, it is enough to check that (1) $\Rightarrow$ (2). But the argument above shows that if F is a $\bigvee$-filter, then $F \subseteq D_{\neg b}$, where $b = \bigvee \perp/F$ and $\neg b \in F$. If $D \subseteq F$, then for all $x \in H$, $x \sim_F \neg\neg x$. Consequently,

$$x \in D_{\neg b} \;\Rightarrow\; \neg\neg x \ge \neg b \;\Rightarrow\; \neg\neg x \in F \;\Rightarrow\; x \in F,$$

verifying that $F = D_{\neg b}$ and concluding the proof. □

Remark 10.3. It follows immediately from 10.2.(d) that the only $\bigvee$-filters in a cBa are the principal ones. Thus, onto frame morphisms (which are the same as onto open morphisms), originating in a cBa B, are in bijective correspondence with the principal filters in B. □

Remark 10.4. By Proposition 10.2.(d), a $\bigvee$-filter, F, is contained in one of the type D_b, $b \in Reg(H)$. The difference between them might be large : in the frame $[0, 1] \subseteq \mathbb{R}$, **all** filters are $\bigvee$-filters, but the only one of type D_b is $(0, 1]$.

The inclusion $F \subseteq D_b$ is far from sufficient for F to be a $\bigvee$-filter. In $\Omega(\mathbb{R})$, let F be the filter of cofinite sets (every cofinite set is open); note that $F \subseteq D$, but F is not a $\bigvee$-filter : if $U = \mathbb{R} - (\{1/n \,:\, n \ge 1\} \cup \{0\})$, $\bigvee U/F = \mathbb{R} - \{0\} \notin U/F$. In fact, 10.2.(d) requires that $F = D$, which is clearly not the case. □

An useful consequence of Proposition 10.2 is the following

Corollary 10.5. *Let D be the filter of dense elements in a frame H.*
a) The quotient H/D is a cBa and the quotient morphism [1]*, $\pi_D : H \longrightarrow H/D$, preserves all joins in H. Further, we have the following universal property :*

> *For all cBas B and all implication preserving frame morphisms $f : H \longrightarrow B$, there is an unique cBa morphism $g : H/D \longrightarrow B$, such that $g \circ \pi_D = f$.*

[reg]
$$\begin{tikzcd} H \arrow[r, "\pi_D"] \arrow[d, "f"'] & H/D \arrow[dl, "g"] \\ B & \end{tikzcd}$$

[1] Which is a HA-morphism, by 6.16.

b) For $S \subseteq Reg(H)$, define

$$\bigvee^* S = \neg\neg(\bigvee S) \quad \text{and} \quad \bigwedge^* S = \bigwedge_{s \in S} \neg\neg s,$$

where $\bigvee$ and $\bigwedge$ are the join and meet in H. With these operations $Reg(H)$ is a cBa and the map $x \in Reg(H) \mapsto x/D \in H/D$ is an isomorphism of $Reg(H)$ onto H/D.

Proof. a) Propositions 6.21 and 10.2 imply that H/D is a cBa, that π_D is an implication preserving frame morphism and the uniqueness of the BA-morphism g, making the displayed diagram commutative. It remains to verify that g preserves sups (and so all operations, since B and H/D are cBas). For $S \subseteq H$, since both f and π_D are sup preserving, we get

$$\bigvee_{s \in S} g(s/D) = \bigvee_{s \in S} f(s) = f(\bigvee S) = g(\pi_D(\bigvee S)) = g((\bigvee S)/D),$$

as needed.
b) The assertions about meets and joins follow from (g) and (h) in 8.16 [2]. The isomorphism claim follows from (a) and the fact that the map $x \longmapsto x/D$ is the isomorphism of Proposition 6.21.(e). $\square$

Remark 10.6. The universal property in Proposition 6.21 and Corollary 10.5 does not hold for a lattice or a frame morphism from H to a BA B. To see this, consider $i : \Omega(\mathbb{R}) \longrightarrow 2^{\mathbb{R}}$, the canonical injection of the opens in $\mathbb{R}$ into the cBa of parts of $\mathbb{R}$. This map is a lattice morphism and a frame morphism (unions of opens and finite intersections of opens are open); it is not open, since $\neg$ is not preserved. In fact, there is **no lattice morphism** $g : \Omega(\mathbb{R})/D \longrightarrow 2^{\mathbb{R}}$, such that $g \circ \pi_D = i$, because coker $i = \{\top\} \neq D$.

By 8.18, in general, π_D will not preserve arbitrary meets, i.e., it is a frame morphism and a HA-morphism, but **it is not open**. $\square$

Remark 10.7. Note that $\bigwedge$-filters and complete filters, which would originate meet preserving and open quotient maps, respectively, *are principal !* That is, if $F \subseteq H$ is a filter such that for all $S \cup \{x\} \subseteq H$,

$$\forall s \in S \ (x \to s) \in F \ \Rightarrow \ \bigwedge_{s \in S} (x \to s) = x \to \bigwedge S \in F,$$

then F is principal. Indeed, first note that $\forall\, x \in H$, $\top \to x = x$; thus, the above condition implies that $\bigwedge_{x \in F} (\top \to x) = \bigwedge F \in F$ and F is principal. There is a result similar to 10.2 for this situation, whose statement and proof is left to the reader, who is also invited to find interesting examples of this phenomenon. $\square$

Exercises

10.8. a) Construct a theory of frame quotients by ideals. Show in particular that the quotient of a frame by a principal ideal is a frame and the natural quotient map is a frame morphism. Give examples showing that these quotients cannot be described by filters.
b) Give an example of an onto frame morphism $f : H \longrightarrow P$ that is not isomorphic to any quotient produced by filters or ideals. $\square$

[2] That should be compared with the analogous statements in 1.14.

CHAPTER 11

Frame Congruences

In general, when dealing with quotients, we define congruence relations and obtain quotients from these. Taking into account the morphisms we have in the category **Frame**, we set down

DEFINITION 11.1. *Let H be a frame and $R \subseteq H^2$ be an equivalence relation on H. R is a* **frame congruence** *iff for all $a, b, x, y \in H$ and $\{x_i, y_i\}_{i \in I} \subseteq H$ we have*
a) If $\langle a, x \rangle \in R$ and $\langle b, y \rangle \in R$, then $\langle a \wedge b,\, x \wedge y \rangle \in R$.
b) If, for all $i \in I$, $\langle x_i, y_i \rangle \in R$, then $\langle \bigvee x_i,\, \bigvee y_i \rangle \in R$.

Thus, R is a frame congruence iff it preserves finite meets and all joins. We denote by $C(H)$ the poset of all frame congruences on H, partially ordered by inclusion.

We adopt the standard convention that $a\ R\ b$ means $\langle a, b \rangle \in R$.

If R is a frame congruence on H write H/R for the set of equivalence classes, x/R, of elements $x \in H$. Let π_R be the canonical quotient map $H \longrightarrow H/R$.

The basic properties of this construction are described in the following Proposition, whose statement one should be obliged to write out in its entirety only once each lifetime. The proof is straightforward and will be omitted.

PROPOSITION 11.2. *Let H be a frame, R a lattice congruence on H and $x, y, z \in H$.*
a) R is a frame congruence iff for all $S \subseteq H$, $\langle \bigvee S,\, \bigvee_{s \in S} \bigvee (s/R) \rangle$ is in R. In particular, if R is a frame congruence, then $(\bigvee x/R) \in x/R$.
b) If R is a frame congruence then,
(1) *$x\ R\ (x \wedge y)$ iff $y\ R\ (x \vee y)$. The relation*

$$x/R \leq y/R \quad \text{iff} \quad x\ R\ (x \wedge y)$$

is independent of representatives and defines a partial order on H/R, with which π_R is an increasing map.
(2) *The prescriptions*

$$x/R \wedge y/R = (x \wedge y)/R \quad \text{and} \quad x/R \vee y/R = (x \vee y)/R,$$

are independent of representatives and define operations that make H/R into a distributive lattice with $\perp = \perp/R$ and $\top = \top/R$, in the po described in (1).
(3) *With the po described in* (1) *and the operations in* (2), *H/R is a frame and the map π_R is a frame morphism. For $A \subseteq H/R$,*

$$\bigvee A = \pi_R(\bigvee \{x \in H \ :\ x/R \in A\}).$$

c) If $H \xrightarrow{f} P$ *is an onto frame morphism, then*

$$Rf = \{\langle x, y \rangle \in H \times H : fx = fy\}$$

is a frame congruence on H *and there is a* **unique** *isomorphism,* $H/Rf \xrightarrow{g} P$*, such that* $g \circ \pi_{Rf} = f$.

$$\begin{tikzcd} H \arrow[r, "\pi_R"] \arrow[d, "f"'] & H/Rf \arrow[dl, "g"] \\ P & \end{tikzcd}$$

We now consider the structure that can be found in the set of frame congruences on a frame H. Since it is clear that the intersection of any family of frame congruences is a frame congruence, $\boldsymbol{C(H)}$ **is a complete lattice** in the inclusion po. The diagonal of $H \times H$, ΔH, is the least element of $C(H)$ and $H \times H$ is its largest element. The preservation of the property of being a frame congruence by arbitrary intersections allows us to define the **frame congruence generated** by $A \subseteq H \times H$,

$$[A] = \bigcap \{R \in C(H) : A \subseteq R\},$$

the least frame congruence containing A. If $S \subseteq C(H)$ is a set of frame congruences on H, then

$$\bigvee S = [\bigcup S].$$

We set down some special notation for filters and ideals.

Definition 11.3. *If* A *is a filter or ideal in the frame* H *let*

a) $[A]$ = *frame congruence generated by the lattice congruence* $\sim_A$
$= [\sim_A]$.

b) Write μ_A *for the quotient mapping* $H \longrightarrow H/[A]$ *in case* A *is an ideal, and* μ^A *in case* A *is a filter.*

c) For principal filters and ideals, write

$$\begin{cases} E^a = [a^{\rightarrow}] = \{\langle x, y \rangle \in H \times H : a \wedge x = a \wedge y\} \\ \qquad\qquad \textit{and} \\ E_a = [a^{\leftarrow}] = \{\langle x, y \rangle \in H \times H : a \vee x = a \vee y\}, \end{cases}$$

for the frame congruences generated by the filter $a^{\rightarrow}$ *and the ideal* $a^{\leftarrow}$*, respectively. The corresponding quotient morphisms will be indicated by* μ^a *and* μ_a.

Proposition 11.4. *With notation as above, let* H *be a frame.*

a) For all $R \in C(H)$ *and* $a, b \in H$*, the following hold in* $C(H)$ *:*

(1) $E^a \vee R = \{\langle x, y \rangle \in H \times H : \langle a \wedge x, a \wedge y \rangle \in R\}$.

(2) $E_a \vee R = \{\langle x, y \rangle \in H \times H : \langle a \vee x, a \vee y \rangle \in R\}$.

(3) E^a *and* E_a *are linear in* $C(H)$.
(4) $E^a \vee E_a = \top$ *and* $E^a \wedge E_a = \bot$.
(5) $(E_a \wedge E^b) \vee (E^a \vee E^b) = \top$.

b) For all $a, b \in H$ *and* $R \in C(H)$, *are equivalent :*
(1) $\langle a, b \rangle \in R$.
(2) $(E^a \wedge E^b) \vee (E^b \wedge E_a) \subseteq R$.
(3) $R \vee (E_a \vee E^b) = \top$ *and* $R \vee (E^a \vee E^b) = \top$.

Proof. We must take care in our calculations for we do not (yet) know that $C(H)$ is distributive.

a) (1) Let S be the right-hand side of the equality in (1). Since the diagonal of $H \times H$, ΔH, is contained in R, we have $E^a \subseteq S$; since R is a lattice congruence, $\langle x, y \rangle \in R$ implies $\langle a \wedge x,\, a \wedge y \rangle \in R$. Thus, $R \subseteq S$ and so $E^a \vee R \subseteq S$. That S is a frame congruence, comes directly from the $[\wedge, \bigvee]$ law in H. It remains to see that every frame congruence containing E^a and R contains S. Let Θ be a frame congruence, with E^a, $R \subseteq \Theta$. If $x, y \in H$ are such that $\langle a \wedge x,\, a \wedge y \rangle \in R$, then :

(i) $\langle x,\, x \vee (a \wedge y) \rangle \in R$ (take the join with x on both sides)
(ii) $\langle y \vee (a \wedge x),\, y \rangle \in R$ (take the join with y on both sides)
(iii) $\langle y \vee (a \wedge x),\, x \vee (a \wedge y) \rangle \in E^a$, (the meet of both coordinates of this pair with a is $a \vee (x \wedge y)$).

Since Θ is transitive, (i), (ii) and (iii) imply $\langle x, y \rangle \in \Theta$, proving item (1). The proof of (2) is similar (actually, dual) and will be omitted.

a) (3) Let R_i, $i \in I$, be a family of congruences in $C(H)$; using (1) and the fact that the meet in $C(H)$ is set theoretic intersection, we obtain

$$\begin{aligned} E^a \vee \bigwedge R_i &= \{\langle x, y \rangle \in H \times H : \langle a \wedge x,\, a \wedge y \rangle \in \textstyle\bigcap_{i \in I} R_i\} \\ &= \textstyle\bigcap_{i \in I} \{\langle x, y \rangle \in H \times H : \langle a \wedge x,\, a \wedge y \rangle \in R_i\} \\ &= \bigwedge (E^a \vee R_i). \end{aligned}$$

The proof of the linearity (8.21) of E_a is analogous.

a) (4) Since $\langle \bot, a \rangle \in E_a$ and $\langle a, \top \rangle \in E^a$, $\langle \bot, \top \rangle \in E^a \vee E_a$. Any lattice congruence that declares $\bot$ equivalent to $\top$, has to be $H \times H$ (that is, $\top$ in $C(H)$). For the second equation, if $\langle x, y \rangle \in E^a \wedge E_a$, then $a \wedge x = a \wedge y$ and $a \vee x = a \vee y$. Hence,

$$\begin{aligned} x = x \vee (a \wedge y) &= (a \vee x) \wedge (x \vee y) = (a \vee y) \wedge (x \vee y) \\ &= y \vee (a \wedge x) = y, \end{aligned}$$

showing that $E^a \wedge E_a = \Delta H = \bot$ in $C(H)$. (5) follows from (4) and the linearity (8.21) of E^a and E^b.

b) $\underline{(1) \Rightarrow (2)}$: We prove that $E^a \wedge E^b \subseteq R$; by symmetry, $E^b \wedge E_a \subseteq R$ and the statement in (2) follows. Assume $\langle a, b \rangle \in R$ and that x, y satisfy $a \wedge x = a \wedge y$, as well as $b \vee x = b \vee y$. Then,

$$\begin{aligned} (*) \quad y \vee (b \wedge x) &= (y \vee b) \wedge (y \vee x) = (b \vee x) \wedge (y \vee x) \\ &= x \wedge (b \vee y) = x \wedge (b \vee x) = x. \end{aligned}$$

Now, from $\langle a, b \rangle \in R$, we get

(i) $\langle\, a \wedge x,\ b \wedge x \,\rangle \in R$ (taking meet with x on both sides);
(ii) $\langle\, a \wedge y,\ b \wedge x \,\rangle \in R$ (from (i) and $a \wedge x = a \wedge y$);
(iii) $\langle\, y,\ y \vee (b \wedge x) \,\rangle = \langle\, y, x \,\rangle \in R$ (take the join with y on both sides of (ii) and use (*))

showing that $E^a \wedge E^b \subseteq R$. (2) $\Rightarrow$ (3) comes directly from (5) in (a).
(3) $\Rightarrow$ (1) : From (1), (2) in (a) and the hypothesis in (3), we get

$$\begin{aligned} H \times H &= R \vee (E_a \vee E^b) = (R \vee E_a) \vee E^b \\ &= \{\langle\, x, y \,\rangle \in H \times H : \langle\, a \vee (b \wedge x),\ a \vee (b \wedge y) \,\rangle \in R\}. \end{aligned}$$

Thus, taking $x = \bot$ and $y = \top$ yields $\langle\, a,\ a \vee b \,\rangle \in R$. By symmetry, the assumption $R \vee (E^a \vee E^b) = \top$ leads to $\langle\, b,\ b \vee a \,\rangle \in R$ and so transitivity yields $\langle\, a, b \,\rangle \in R$, ending the proof. $\Box$

Proposition 11.4 is the fundamental step towards

THEOREM 11.5. (Isbell) *The lattice of frame congruences on a frame is a zero-dimensional frame.*

PROOF. By Theorem 8.24, it is enough to verify that the linear elements of $C(H)$ form a cobasis. Given R, S in $C(H)$, suppose that

(*) For all linear $z \in C(H)$, $S \vee z = \top \;\Rightarrow\; R \vee z = \top$.

If $\langle\, a, b \,\rangle \in S$, Proposition 11.4.(b) assures that

(**) $S \vee (E^a \vee E^b) = \top$ and $S \vee (E^a \vee E^b) = \top$.

Item (3) in 11.4.(a) guarantees that E^x and E_x are linear. Since linearity is preserved by finite joins, it follows from (*) and (**) that $R \vee (E^a \vee E^b) = \top$ and $R \vee (E_a \vee E^b) = \top$. From this we get $\langle\, a, b \,\rangle \in R$, proving that $S \subseteq R$ and that the linear elements are indeed a cobasis for $C(H)$. $\Box$

By 11.4.(b), $[\langle\, a, b \,\rangle] = (E^a \wedge E^b) \vee (E^b \wedge E_a)$. If one knew that $C(H)$ was distributive, zero-dimensionality would follow immediately from item (4) in 11.4.(a).

Exercises

11.6. a) Try to prove directly that $C(H)$ is a frame. Note that it is sufficient to show that it is distributive (8.23).
b) Prove that $E^a \vee E^b = E^{a \wedge b}$ and $E_a \vee E^b = E_{a \vee b}$. $\Box$

CHAPTER 12

Points and Sober Spaces

One of the main examples of frames are the algebras of opens in topological spaces. However, it is not true that every frame is isomorphic to one of this type. We shall apply the results in Chapters 10 and 11 to exhibit a class of such examples. We shall also develop a characterization of which frames come from Topology.

Recall that if X is a topological space, $\Omega(X)$ is the frame of opens in X and ν_p is the filter of open neighborhoods of the point p in X. This filter has a special property :

$$\text{If } \bigcup U_i \in \nu_p \text{ then, for some } i \in I,\, U_i \in \nu_p.$$

We use this observation to give a general definition of "point".

Definition 12.1. *A* **proper** *filter F in a complete lattice L is a* **point** *in L iff for all $S \subseteq L$, $\bigvee S \in F$ implies $S \cap F \neq \emptyset$. Write $pt(L)$ for the set of points in L.*

For $a \in L$, set $P_a = \{F \subseteq L : F$ is a point in L and $a \in F\}$.

Some authors use **completely prime or pure state** for those filters we here call points [1].

Lemma 12.2. *Let L, R be a complete lattices and $S \subseteq L$ and $a, b \in L$. Then,*
a) $P_a \cap P_b = P_{a \wedge b}$ and $P_{\bigvee S} = \bigcup_{s \in S} P_s$.
b) $P_\top = pt(L)$ and $P_\bot = \bot$.
c) The collection $\{P_b\}_{b \in L}$ is a $T0$ topology on $pt(L)$.
d) If $\lambda : L \longrightarrow R$ is a $[\wedge, \bigvee]$-morphism, then the map

$$pt(\lambda) : pt(R) \longrightarrow pt(L), \text{ defined by } pt(\lambda)(P) = \lambda^{-1}(P),$$

is continuous. The map $\lambda \mapsto pt(\lambda)$ preserves composition and $pt(Id_L) = Id_{pt(L)}$.

Proof. (a) and (b) are immediate from the definitions. For (c), just note that the collection in question contains $\emptyset$ and $pt(L)$, being closed under finite intersections and arbitrary unions. Thus, it is a topology. To check that it is T0 (1.20), note that

$$F \neq G \text{ in } pt(L) \;\Rightarrow\; \exists\, a \in (F - G) \cup (G - F),$$

that is, $P_a \in \nu_F \bigtriangleup \nu_G$. Item (d) is straightforward. $\square$

If L is a complete lattice, write $\Omega(pt(L))$ for the frame of opens of the topological space $pt(L)$, with the topology of 12.2.(c). We have a map

[1] See Exercise 12.14.

$$\alpha_L : L \longrightarrow \Omega(pt(L)), \text{ defined by } \alpha_L(a) = P_a.$$

LEMMA 12.3. *Let L be a complete lattice. With notation as above,*
a) The map α_L is a surjective $[\wedge, \bigvee]$-morphism.
b) If $L = \Omega(X)$, X a topological space, then α_L is a frame isomorphism.

PROOF. (a) is clear from 12.2.(a); (b) comes from the fact that if L is $\Omega(X)$, then

$$\begin{aligned} U \neq V \text{ in } \Omega(X) \quad & \text{iff} \quad \exists\, a \in X \text{ such that } a \in U \bigtriangleup V \\ & \text{iff} \quad \nu_a \in P_U \bigtriangleup P_V, \end{aligned}$$

and so α_L is injective. □

LEMMA 12.4. *Let X be a topological space.*
a) With the topology in 12.2.(c), the map

$$\nu_X : X \longrightarrow pt(\Omega(X)), \text{ given by } x \mapsto \nu_x,$$

is continuous, with $\nu_X^{-1}(U) = P_U$, for all $U \in \Omega(X)$.
b) ν_X is injective iff X is T0.
c) ν_X is a homeomorphism iff it is bijective.

PROOF. Item (a) is clear, while (b) is immediate from the definition of T0 (see 1.20). For (c), note that

$\nu_X(U) = P_U$ iff Every point F in $\Omega(X)$ such that $U \in F$, is in the image of ν_X.

Hence, if ν_X is bijective, it is open and continuous and thus, a homeomorphism. □

EXAMPLE 12.5. As an example, we verify that ν_X is a homeomorphism if X is a Hausdorff space. Since every Hausdorff space is T0, by 12.4.(c), it is enough to verify that ν_X is surjective. This can be accomplished in two stages : let F be a point in $\Omega(X)$.
(i) There is $p \in X$ such that $\nu_p \subseteq F$.
If not, for $p \in X$ select $U_p \in \nu_p - F$, to get get $X = \bigcup_{p \in X} U_p \in F$, contradicting either that F is a point or that it is a proper filter.
(ii) Let $p \in X$ be such that $\nu_p \subseteq F$. Then, $F = \nu_p$.
Assume, to get a contradiction, that there is $U \in F - \nu_p$; because X is Hausdorff, it follows that $U = \bigcup_{V \in \nu_p} U - \overline{V}$. Since F is a point, for some $V \in \nu_p$ we have $U - \overline{V} \in F$. But then, F is not a proper filter. □

The preceding Example leads to

DEFINITION 12.6. *Let T be a topological space and H be a frame.*
a) T is **sober** *if the map $\nu_T : T \longrightarrow pt(\Omega(T))$ is a bijection* [2].
b) H **has enough points** *if the map α_H is a frame isomorphism.*

[2]Or equivalently, a homeomorphism, by 12.4.(c)).

Thus, Hausdorff spaces are sober and frames of opens in topological spaces have enough points. Also, all sober spaces are T0, but T1 and sober are incomparable (see Exercise 12.15).

By Exercise 12.13, there are no points in atomless cBas. For instance, $Reg(\Omega(\mathbb{R}))$ has no points. With notation as in Chapter 11, a whole class of examples comes from

PROPOSITION 12.7. *Let T be a Hausdorff topological space and F be the filter of cofinite sets in T (which are all open). Let $[F]$ be the frame congruence generated by F in $\Omega(T)$. Then, $\Omega(T)/[F]$ is a frame without points.*

PROOF. Write μ for the quotient map from $\Omega(T)$ to $\Omega(T)/[F]$. We know that μ is a $[\wedge, \bigvee]$-morphism (11.2). If $G \subseteq \Omega(T)/[F]$ is a point, it follows from Lemma 12.2.(d) that $\mu^{-1}(G)$ is a point in $\Omega(T)$. T being sober (12.5), there is $p \in T$ such that $\mu^{-1}(G) = \nu_p$. Since $U = T - \{p\} \in F$, we have $\mu(U) = \top$. Thus, $U \in \mu^{-1}(G)$. On the other hand, because $\bigcap_{V \in \nu_p} \overline{V} = \{p\}$, we have $U = \bigcup_{V \in \nu_p} (U - \overline{V})$. But then, for some $V \in \nu_p$, $U - \overline{V} \in \mu^{-1}(G) = \nu_p$, a contradiction. □

Our next step is to give an intrinsic characterization of sobriety. To that end, we introduce

DEFINITION 12.8. *Let X be a topological space, E be a subset of X and p a point in X.*
a) E is **irreducible in X** *if for all closed sets, F_1, F_2, in X*

[irr] $$E \subseteq F_1 \cup F_2 \quad \Rightarrow \quad E \subseteq F_1 \quad \text{or} \quad E \subseteq F_2.$$

b) p is a **generic point of E** *if $p \in E$ and $E \subseteq \overline{\{p\}}$.*

REMARK 12.9. Let X be a topological space and E a *closed set* in X.
a) Because the finite intersection of closed sets is closed, E is irreducible in X iff for all closed sets F_1, F_2 in X

$$E = F_1 \cup F_2 \quad \Rightarrow \quad E = F_1 \quad \text{or} \quad E = F_2.$$

b) A point $p \in E$ is *generic* iff $\overline{\{p\}} = E$. □

LEMMA 12.10. *Let X be a topological space, $\Omega(X)$ be the frame of opens in X and let $E \subseteq X$.*
a) The following conditions are equivalent :
(1) *E is irreducible in X;*
(2) *For all $U \in \Omega(X)$, $U \cap E \neq \emptyset \Rightarrow U \cap E$ is dense in E;*
(3) *For all $U, V \in \Omega(X)$,*
$$U \cap E \neq \emptyset \quad \text{and} \quad V \cap E \neq \emptyset \quad \Rightarrow \quad U \cap V \cap E \neq \emptyset;$$
(4) *For all $y \in E$ and $V \in \nu_y$, $E \cap \overline{V} = E$.*
b) If E is irreducible in X, the same is true of its closure, $\overline{E}$.
c) If X is $T0$ and E has a generic point, then it is unique.

PROOF. a) It is clear that (2), (3) and (4) are equivalent.

(1) $\Rightarrow$ (2) : Assume, to get a contradiction, that E is irreducible, but that there is $U \in \Omega(X)$ such that $U \cap E \neq \emptyset$ and this intersection is not dense in E. Hence, there are $q \in E$ and $V \in \nu_q$ such that

$$(i)\ q \in V \quad \text{and} \quad (ii)\ V \cap U \cap E \ = \ \emptyset.$$

Consider the closed sets $F_1 = U^c$ and $F_2 = V^c$; by (ii) above, we have $E \subseteq F_1 \cup F_2$. On the other hand, because $q \in E - F_2$ and $U \cap E$ is non-empty, we conclude that neither $E \subseteq F_1$, nor $E \subseteq F_2$, contradicting the irreducibility of E.
(2) $\Rightarrow$ (1) : We may assume that $E \neq \emptyset$. Let F_1, F_2 be closed sets in X such that $E \subseteq F_1 \cup F_2$. Hence,

$$E \cap F_1^c \cap F_2^c \ = \ \emptyset.$$

If $E \cap F_1^c = \emptyset$, then $E \subseteq F_1$ and we are done; otherwise, (2) entails $F_1^c \cap E$ is dense in E and thus, since $F_2^c \in \Omega(X)$, we get $F_2^c \cap E = \emptyset$, i.e., $E \subseteq F_2$, establishing the irreducibility of E, as needed.
b) Let U be an open set in X such that $U \cap \overline{E} \neq \emptyset$; then, $U \cap E \neq \emptyset$ and, since E is irreducible, item (2) of the equivalence in (a) yields

$$\overline{E} \ = \ \overline{U \cap E} \ \subseteq \ \overline{U \cap \overline{E}} \ \subseteq \ \overline{E},$$

showing that $U \cap \overline{E}$ is dense in $\overline{E}$. Another application of (a).(2) implies that $\overline{E}$ is irreducible in X, as desired.
c) Let p, $q \in E$ be generic points for E. If $p \neq q$, since X is $T0$, without loss of generality we may assume that there is $V \in \Omega(X)$ such that $p \in V$ and $q \notin V$. But then, $\overline{\{q\}} \subseteq V^c$ and so the closure of q does not contain E, contradicting its genericity and ending the proof. $\square$

Proposition 12.11. *Let L be a complete lattice. Then, in the topological space $pt(L)$, every non-empty irreducible closed set has a generic point.*

Proof. Since P_a, $a \in L$, is a basis for the topology on $pt(L)$, if P, $Q \in pt(L)$ we have

$$(*) \qquad P \in \overline{\{Q\}} \quad \text{iff} \quad P \subseteq Q.$$

Now let $K \neq \emptyset$ be an irreducible closed set in $pt(L)$; we show that $P = \bigcup K$ is a point in L and that $P \in K$. By (*), P will be a generic point for K.

A moment of thought will convince the reader that, in the verification that P is a completely prime filter, the only property that is not immediately forthcoming is that P is closed under finite intersections. Let $a, b \in$ P; note that this means $K \cap P_a \neq \emptyset$ and $K \cap P_b \neq \emptyset$. We wish to show that $a \wedge b \in P$. Since all elements of K are filters, if this were false, then for all $Q \in K$, either $a \notin Q$ or $b \notin Q$. Thus, we could write

$$K = (K \cap P_a^c) \cup (K \cap P_b^c)$$

where $(*)^c$ denotes complementation in $pt(L)$. Since K has been written as the union of two closed sets in $pt(L)$, its irreducibility yields $K \subseteq P_a^c$ or $K \subseteq P_b^c$. But both alternatives lead to a contradiction. Therefore, $a \wedge b$ belongs to P and P is a point in L. It remains to prove that $P \in K$. Since $P = \bigcup K$, for any $c \in P$, $P_c \cap K \neq \emptyset$. Hence, $P \in \overline{K} = K$, as desired. $\square$

In any topological space the closure of a point is an irreducible closed set. To require that in a T0 space all irreducible closed sets be of this type, is tantamount to requiring that it be sober. We have

THEOREM 12.12. *For a topological space T, the following are equivalent :*
(1) *T is sober.*
(2) *T is $T0$ and every non-empty irreducible closed set in T has a generic point.*
(3) *Every non-empty irreducible closed set in T has a* **unique** *generic point.*

PROOF. $\underline{(1) \Rightarrow (2)}$: If T is sober, then $\nu_T : T \longrightarrow pt(\Omega(T))$ is a homeomorphism (12.4.(c)). It then follows from 12.2.(b) and 12.11, that T is T0 and all irreducible closed sets have a generic point.
$\underline{(2) \Rightarrow (3)}$: Immediate from 12.10.(c).
$\underline{(3) \Rightarrow (1)}$: We must show that $\nu = \nu_T : T \longrightarrow pt(\Omega(T))$ is bijective.
$\underline{\nu \text{ is injective :}}$ For any two points in a topological space,

$$\nu_x = \nu_y \quad \text{iff} \quad \overline{\{x\}} = \overline{\{y\}}.$$

Since closure of points are irreducible and irreducible closed sets in T have a unique generic point, we get $\nu_x = \nu_y$ implies $x = y$, proving that ν is injective (or equivalently that T is T0).
$\underline{\nu \text{ is surjective :}}$ Let P be a point in $\Omega(T)$. We must find $x \in T$, such that $P = \nu_x$. Define $K = \{x \in T : \nu_x \subseteq P\}$; we shall prove that K is a non-empty irreducible closed set in T. We start with
Fact 1. *For all $U \in P$, $U \cap K \neq \emptyset$. In particular, $K \neq \emptyset$.*
Proof. If $U \cap K = \emptyset$, for each $y \in U$ we may select $V_y \in \nu_y - P$. Therefore, $U = \bigcup_{y \in U} V_y$, with each $V_y \notin P$. Patently, such a U cannot be in P.
Fact 2. *K is closed.*
Proof. Let $y \in \overline{K}$; then, for all $V \in \nu_y$, $V \cap K \neq \emptyset$ (1.10.(e)). Therefore, for each $V \in \nu_y$, there is $z \in K$ such that $V \in \nu_z \subseteq P$. Thus, $\nu_y \subseteq P$ and $y \in K$.
Fact 3. *K is irreducible.*
Proof. We verify that for $y \in K$ and $V \in \nu_y$, $\overline{V \cap K} = K$. By 12.10, this is assures that K is irreducible. Suppose $z \in K$ and $U \in \nu_z$; since P is a proper filter and $\nu_y \cup \nu_z \subseteq P$, we must have $V \cap U \in P$. Fact 1 then yields $U \cap V \cap K \neq \emptyset$, showing that $V \cap K$ is dense in K.

By hypothesis, we have $K = \overline{\{x\}}$, for some $x \in K$. Note that this yields $\nu_x \subseteq P$; we show that, in fact, $P = \nu_x$. We ask the reader to verify the following
Fact 4. *If T be a topological space, K is a subset of T and $x \in T$, then, $K \subseteq \overline{\{x\}}$ iff $\forall\, y \in K$, $\nu_y \subseteq \nu_x$.*

Let $U \in P$; by Fact 1, $U \cap K \neq \emptyset$ and so $U \in \nu_y$, for some $y \in K$. But then, Fact 4 yields $U \in \nu_x$ and so $P \subseteq \nu_x$, as desired. $\square$

Exercises

12.13. Show that in a cBa B, F is a point iff F is a principal ultrafilter generated by an atom in B. $\square$

12.14. Let L be a complete lattice. A **pure state** in L is a *surjective* $[\wedge, \bigvee]$-morphism from L to $\{\bot, \top\}$. Let $PS(L)$ be the set of pure states in L. For $a \in L$, define

$$W_a = \{f \in PS(L) : fa = \top\}.$$

Show that there is a natural bijective correspondence $pt(L) \xrightarrow{\omega} PS(L)$, such that $\omega(P_a) = W_a$, for all $a \in L$. $\square$

12.15. Let X be an infinite set and let *Cofin* be the filter of cofinite subsets in X. Prove that $\{\emptyset\} \cup$ *Cofin* is a T1 topology on X, which is not sober. $\square$

12.16. Generalize Proposition 12.7 to sober spaces. $\square$

12.17. Let **Top** be the category of topological spaces and continuous maps and **Frame** that of frames and their morphisms.
a) Show that we can consider $\Omega(*)$ as a contravariant functor from **Top** to **Frame**, while the operation of taking 'points', $pt(*)$, induces a contravariant functor from **Frame** to **Top**.
b) Prove that $\Omega(*)$ is left adjoint to $pt(*)$.
c) There is a duality (contravariant equivalence) between the category of sober topological spaces and continuous mappings and that of frames with enough points and frame morphisms. $\square$

12.18. Continuous lattices (2.43) have a natural topology *(the Scott topology)* to be described below. This exercise consists of a proof that the Scott topology is sober. Let L be a continuous lattice.
a) For each $a \in$ L, set $L_a = \{x \in L : a \ll x\}$. Show that

$$L_a \cap L_b = L_{a \vee b}.$$

Thus, $\{L_a\}_{a \in L}$ is a basis for a topology, the **Scott topology** on L.
b) For all $x \in L$, $x = \bigvee \{a \in L : x \in L_a\}$.
c) Show that the Scott topology is sober. $\square$

CHAPTER 13

Constructive Quotients

In our discussion of quotients via frame congruences, we made use of arbitrary choices, at least in proving that the quotient constructed is complete and a frame. We now present a constructive way of discussing quotients, similar to that expounded in [**15**] and [**34**].

To introduce this point of view, let $L \xrightarrow{f} P$ be a surjective $[\wedge, \bigvee]$-morphism of complete lattices. By Corollary 7.9, there is a $\bigwedge$-morphism $g : P \longrightarrow L$ such that

$$\begin{cases} \text{(i) } f \circ g = Id_P & \text{(ii) } \forall\, p \in P,\, gp = \max f^{-1}(p). \\ \text{(iii) } g \circ f \geq Id_L & \text{(iv) } g \circ f \circ g = g. \end{cases}$$

These relations readily imply that $f_{|g(P)} \circ g = Id_P$ and $g \circ f_{|g(P)} = Id_{g(P)}$. Thus, $g(P)$ is isomorphic to P, via $f_{|g(P)}$. Consider $J : L \longrightarrow L$, given by $J = g \circ f$; then J has the following properties, for $x, y \in L$:

$$\begin{cases} * \; Jx \geq x; & * \; J(x \wedge y) = Jx \wedge Jy; \\ * \; J \circ J = J; & * \; Jx = x \text{ iff } x \in g(P), \text{ i.e., } g(P) = Fix(J). \end{cases}$$

Since $f_{|Fix(J)}$ is an isomorphism, we have obtained an isomorphic copy of P, as the lattice of fixed points of a certain type of operator on L. The preceding observations suggest the development that follows.

Definition 13.1. *Let L be a complete lattice. A map $L \xrightarrow{J} L$ is a* **Q-operator** *on L, if it satisfies, for all $x, y \in L$*

$$[Q\ 1] : Jx \geq x. \qquad\qquad [Q\ 2] : J(x \wedge y) = Jx \wedge Jy.$$

If J and K are Q-operators on L, write

$$J \leq K \quad \textit{iff} \quad \textit{for all } x \in L,\ Jx \leq Kx,$$

that is a partial order on the set of Q-operators on L.

An operator, $J : L \longrightarrow L$, is a **projection or idempotent** *iff $J \circ J = J$. A* **nucleus** *on L is an idempotent Q-operator on L.*

If L is a HA, an **implicative operator** *on L* (I-operator) *is a nucleus, J, such that $\forall\, x, y \in L$, $J(x \rightarrow y) = Jx \rightarrow Jy$.*
Write $J(L)$ for the set of nuclei on L. $J(L)$ is a poset, with the pointwise order.

Note that Id_L and the constant function $\top$ on L are, respectively, $\perp$ and $\top$ in the complete lattice $J(L)$. We now state

THEOREM 13.2. *Let L be a complete lattice and let $L \xrightarrow{g} L$ be a Q-operator on L. Then*
a) $Fix(g)$ is a complete lattice with the partial order induced by L. Moreover, meets in $Fix(g)$ are the same as in L.
b) For $x \in L$, define $Gx = \bigwedge \{y \in Fix(g) : y \geq x\}$. Then, G is a nucleus on L, the least nucleus on L such that $G \geq g$ and $Fix(g) = Fix(G)$.
c) G is a $[\wedge, \bigvee]$-morphism of L onto $Fix(g) = Fix(G)$. In particular, if g is a nucleus, then g is a $[\wedge, \bigvee]$-morphism from L to the complete lattice $Fix(g)$.
d) If L is a frame, then $Fix(g)$ is a frame.

PROOF. a) Since g is increasing (by [Q 2]) and satisfies [Q 1], Theorem 7.5 guarantees that $Fix(g)$ is a complete lattice with the po induced by L, as well as that all meets in $Fix(g)$ are the same as those computed in L.
b) It follows from the definition of G and the fact that $\bigwedge$ coincides in L and in $Fix(g)$, that for all $x \in L$

$$Fix(g) \subseteq Fix(G) \quad \text{and} \quad Gx \in Fix(g).$$

If $y \in Fix(g) \cap x^{\rightarrow}$, then $y = gy \geq gx$, and so that $Gx \geq gx \geq x$, for all $x \in L$. These inequalities entail that $Fix(G) = Fix(g)$, as well as that G verifies [Q 1]. Clearly, G is increasing and, for $q \in Fix(g)$ and $x \in L$, $q \geq Gx$ iff $q \geq x$. But then, G must be idempotent. Straightforward calculations, using the general associative laws in Lemma 7.7 and the fact that g preserves $\wedge$, show that G satisfies [Q 2]. Thus, G is a nucleus on L, such that $G \geq g$ and with the same complete lattice of fixed points as g. Let J be a nucleus on L, such that $J \geq g$ and $Fix(J) = Fix(g)$. For $x \in L$, note that Gx and Jx are both fixed points of g, greater than or equal to x. The definition of G immediately entails $G \leq J$.

For the remainder of this proof, to keep notation straight, write $\bigvee^*$ for the join operation in $Fix(g)$.
c) Since G is idempotent, we have $\text{Im } G = Fix(G) = Fix(g)$. By [$Q$ 2], G preserves finite meets. To verify that it preserves joins, it suffices to show that for $S \subseteq L$, $G(\bigvee S) \leq \bigvee^*_{s \in S} Gs$. Let $q \in Fix(G)$ be such that $q \geq Gs$, $s \in S$; [Q 1] yields $q \geq \bigvee S$ (in L) and so $Gq = q \geq G(\bigvee S)$, as needed.
d) For Fix(g) to be a frame, it must be shown that if $S \cup \{x\} \subseteq Fix(g)$, then

$$(*) \qquad x \wedge \bigvee\nolimits^* S \leq \bigvee\nolimits^*_{s \in S} (x \wedge s).$$

Since $\leq$ is that induced by L, (*) is equivalent to

$$(**) \qquad \bigvee\nolimits^* S \leq x \rightarrow \bigvee\nolimits^*_{s \in S} (x \wedge s),$$

in L. Since $x \wedge s \leq \bigvee^*_{s \in S} (x \wedge s)$, we get $s \leq x \rightarrow \bigvee^*_{s \in S} (x \wedge s)$. So, (**) would follow if $x \rightarrow \bigvee^*_{s \in S} (x \wedge s)$ were in $Fix(g)$. We have :
Fact. $z, y \in Fix(g) \;\Rightarrow\; (z \rightarrow y) \in Fix(g)$.
Proof. [Q 2] and $z \wedge (z \rightarrow y) \leq y$ entail $gz \wedge g(z \rightarrow y) \leq gy$, that is,

$$g(z \rightarrow y) \leq gz \rightarrow gy = (z \rightarrow y).$$

But then, [Q 1] implies that $g(z \rightarrow y) = z \rightarrow y$, completing the proof of the Fact and of the Theorem. □

The argument at the beginning of this Chapter yields

Corollary 13.3. *Let $f : L \longrightarrow P$ be a surjective $[\wedge, \bigvee]$-morphism of complete lattices. Then, there is a nucleus J on L such that $f_{|Fix(J)} : Fix(J) \longrightarrow P$ is an isomorphism, making the following diagram commutative :*

$$\begin{tikzcd} L \arrow[r, "J"] \arrow[d, "f"'] & Fix(J) \arrow[dl, "f_{|Fix(J)}"] \\ P & \end{tikzcd}$$

Example 13.4. Quotients by principal filters and ideals give rise to frame morphisms and so it is natural to ask who are the corresponding nuclei. If a is an element of the complete lattice L, the reader can check that the nuclei associated to the filter $a^{\rightarrow}$ and the ideal $a^{\leftarrow}$ are, respectively,

$$\mu^a x = a \rightarrow x \quad \text{and} \quad \mu_a x = a \vee x. \qquad (x \in L)$$

If H is a frame and $b \in Reg(H)$, then $D_b = \{x \in H : \neg\neg\, x \geq b\}$ is a $\bigvee$-filter (10.2). The nucleus associated to D_b is $B^{\neg b}x = (x \rightarrow \neg\, b) \rightarrow \neg\, b$. When $b = \top$, we get the nucleus associated to the filter D of dense elements in H, $B^{\perp}x = \neg\neg\, x$.

In general, for any $a \in H$, the operator $B^a = (x \rightarrow a) \rightarrow a$ yields a nucleus on H, whose fixed points constitute a Boolean algebra. $\Box$

Example 13.5. With notation as in Chapter 9, let R be a commutative ring with identity. Let $\mathcal{I}(R)$ be the complete lattice of ideals in R. By Corollary 9.14.(a), the map $I \mapsto \sqrt{I}$ is a nucleus on $\mathcal{I}(R)$, whose set of fixed points is exactly $Rad(R)$. By Proposition 9.16, $Rad(R)$ is a frame. To justify this special situation, we start with

Fact. *Let I, $J_1, \ldots, J_n$, $\{K_\alpha : \alpha \in A\}$ be ideals in R. Then,*

a) $I(\bigvee_{k=1}^{n} J_k) = \bigvee_{k=1}^{n} IJ_k$.

b) $I\big(\bigvee_{\alpha \in A} K_\alpha\big) = \bigvee_{\alpha \in A} IK_\alpha$.

Proof. We shall employ the notions of sum and product of ideals in 9.2, keeping in mind that $\bigvee$ and $\sum$ denote the same operation in $\mathcal{I}(R)$.

a) It is enough to check that $I(J \vee K) \subseteq IJ \vee IK$. The stated equality will then follow for $n = 2$ and induction will finish the proof. If $x \in I(J \vee K)$, there are $a_1, \ldots, a_n$ in I, $b_1, \ldots, b_n$ in $(I \vee J)$, such that

$$x = \textstyle\sum_{p=1}^{n} a_p b_p.$$

For each $p \leq n$, select $c_{pq} \in J$ and $d_{pq} \in K$, $1 \leq q \leq m$, such that

$$b_p = \textstyle\sum_{q=1}^{m} c_{pq} + d_{pq}.$$

Then,

$$x = \textstyle\sum_{p=1}^{n} a_p \sum_{q=1}^{m}(c_{pq} + d_{pq}) + \sum_{p=1}^{n} \sum_{q=1}^{m} a_p c_{pq} + a_p d_{pq},$$

verifying that $x \in IJ \vee IK$. Item (b) is a straightforward consequence of (a), since any element of the displayed joins must belong to the supremum of a finite subcollection of ideals.

Write $\bigvee^*$ for the sup in $Rad(R)$. Let I, $\{J_\alpha : \alpha \in A\}$ be a radical ideals in R. To verify that $Rad(R)$ is a frame, we compute as follows, using the Fact above, 9.14.(c) and the preservation of joins by the operation $\surd$ (it is a nucleus !) :

$$\begin{aligned} I \cap {\bigvee^*}_{\alpha \in A} J_\alpha &= I \cap \sqrt{(\bigvee_{\alpha \in A} J_\alpha)} = \sqrt{I(\bigvee_{\alpha \in A} J_\alpha)} = \sqrt{(\bigvee_{\alpha \in A} IJ_\alpha)} \\ &= {\bigvee^*}_{\alpha \in A} \sqrt{(IJ_\alpha)} = {\bigvee^*}_{\alpha \in A} (\sqrt{I} \cap \sqrt{J_\alpha}) \\ &= {\bigvee^*}_{\alpha \in A} (I \cap J_\alpha), \end{aligned}$$

as desired. The context of *Quantales* ([**61**]) provides a general framework for the type of reasoning presented above. ☐

If J is a nucleus on a complete lattice L, write L_J for complete lattice $Fix(J)$, calling it **the quotient** of L by J. Write J for the canonical $[\wedge, \bigvee]$-morphism from L onto L_J (abusing notation). One should keep in mind that L_J is contained in L, with arbitrary meets coinciding with those in L. The proof of the next result is left to the reader.

Proposition 13.6. *Let L be a complete lattice. For $\mu \in J(L)$, the map*

$$\delta \in J(L_\mu) \longmapsto (\delta \circ \mu) \in J(L)$$

$$L \xrightarrow{\ \mu\ } L_\mu \xrightarrow{\ \delta\ } L_\mu \subseteq L$$

is a bijection between $J(L_\mu)$ and $\mu^{\rightarrow}$ (in $J(L)$). The map $\gamma = (\delta \circ \mu)_{|L_{\delta \circ \mu}}$, is a lattice isomorphism, making the following diagram commutative :

$$\begin{tikzcd} L \arrow[r, "\delta \circ \mu"] \arrow[d, "\mu"'] & L_{\delta \circ \mu} \arrow[d, "\gamma"] \\ L_\mu \arrow[r, "\delta"'] & (L_\mu)_\delta \end{tikzcd}$$

The relation between frame congruences and nuclei is described in the following Proposition, in reality a generalization of our treatment of $\bigvee$-filters and of the comments leading to the concept of Q-operator.

Proposition 13.7. *Let H be a frame. For $x, y \in H$ and $J \in J(H)$, define*

$$x \; R_J \; y \quad \textit{iff} \quad Jx = Jy.$$

For $R \in C(H)$ and $x \in H$, define

$$J_R x = \bigvee \{y \in H : \langle x, y \rangle \in R\}.$$

Then, R_J is a frame congruence on H, J_R is a nucleus on H and the maps

$$J \in J(H) \longmapsto R_J \in C(H) \quad \textit{and} \quad R \in C(H) \longmapsto J_R \in J(H)$$

are isomorphisms, inverse to one another. Moreover, these isomorphisms are uniquely characterized by the relation

[*cong*] *For all* $x, y \in H$, $y \leq Jx$ *iff* $\langle x \wedge y, y \rangle \in R$.

Proof. If $R \in C(H)$, then $J_R x = \max x/R$, is the largest element in the class of x. In particular, $J_R x \geq x$ and $J_R \circ J_R = J_R$. To verify [Q 2], suppose $z\ R\ (x \wedge y)$; then both $(z \vee x)\ R\ x$ and $(z \vee y)\ R\ y$. Thus,

$$z \leq (z \vee x) \leq J_R x \quad \text{and} \quad z \leq (z \vee y) \leq J_R y,$$

that imply $J_R(x \wedge y) \leq J_R x \wedge J_R y$. In view of [Q 1], this is sufficient to assure [Q 2] and that J_R is a nucleus on H. Clearly, $R \mapsto J_R$ is increasing and so a poset morphism from $C(H)$ to $J(H)$.

Now let J be a nucleus on H. It is straightforward that R_J is a lattice congruence on H; that it is a frame congruence, is a consequence of the fact that J is a **frame morphism** from H onto $H_J = Fix(J)$.

Let J, $K \in J(H)$ with $J \leq K$. If $Jx = Jy$ then, $x \leq Jx = Jy$ yields

$$Kx \leq K(Jx) = K(Jy) \leq K(Ky) = Ky.$$

Similarly, $Ky \leq Kx$, and so $Ky = Kx$ and the map $J \mapsto R_J$ is increasing.

It is clear from the definitions that [*cong*] (in 13.7) holds and in fact, determines, the maps constructed above between $C(H)$ and $J(H)$. It follows easily from [*cong*] that $J = J_{R_J}$ and $R = R_{J_R}$, showing that they are inverse isomorphisms. $\square$

From Proposition 13.7 and Theorem 11.5 comes

Corollary 13.8. $J(H)$ *is a zero dimensional frame.*

By duality, there is an analogous treatment of $[\wedge, \bigvee]$-sublattices of a complete lattice.

Definition 13.9. *Let* L *be a complete lattice. A map* $L \xrightarrow{j} L$ *is a* **conucleus** *if it satisfies, for all* $x, y \in L$

[*con* 1] : $jx \leq x$; [*con* 2] : $j \circ j = j$; [*con* 3] : $j(x \wedge y) = jx \wedge jy$.

Exercises

13.10. Let H be a frame.

a) If F is a filter in H, show that the map $J_F : H \longrightarrow H$, defined by $J_F x = \bigvee x/F$ is a Q-operator. Further, prove that if F is a $\bigvee$-filter, then J_F is an implicative operator.

b) Show that if $h : H \longrightarrow H$ is an I-operator on H, then there is a $\bigvee$-filter F in H such that $h = J_F$.

c) Study the relation between ideals in H and Q-operators. $\square$

13.11. Let L be a complete lattice. If f_i, $i \in I$, is a family of nuclei on L, then $\bigwedge f_i$, defined, for $x \in L$, by

$$(\bigwedge f_i)x = \bigwedge_{i \in I} f_i x,$$

is a nucleus on L. Conclude that $J(L)$ is a **complete lattice**. $\square$

13.12. Let H be a frame. Let $a, b \in H$ and let J be a nucleus on H. Recall that in $J(H)$, $\perp = Id_H$ and that $\top$ is the constant function $\top$. With the notation of 13.4, prove the following relations in $J(H)$:

(1) $\mu^a \vee \mu^b = \mu^{a \wedge b}$ and $\mu^a \wedge \mu^b = \mu^{a \vee b}$.

(2) $\mu_a \vee \mu_b = \mu_{a \vee b}$ and $\mu_a \wedge \mu_b = \mu_{a \wedge b}$.

(3) $\mu^a \vee J = \mu^a \circ J$ and $\mu_a \vee J = \mathrm{J} \circ \mu_a$.

(4) $\mu^a \wedge \mu_a = \perp$ and $\mu^a \vee \mu_a = \top$.

(5) $\mu_a \vee B^a = B^a$.

(6) $\mu^a \vee B^b = B^{a \to b}$. $\square$

13.13. Using Definition 13.9 and the methods of this chapter, establish a theory of $[\wedge, \bigvee]$-sublattices of a complete lattice. Show that if we start with a frame, then we obtain all its subframes as fixed points of a conucleus. $\square$

CHAPTER 14

Regular Completions

We now turn to the discussion of how to obtain the completion of a Heyting algebra. For distributive lattices in general, an example due to Crawley shows that there are distributive lattices that cannot be regularly embedded in any complete distributive lattice. The reader will find this example, together with other valuable comments on this topic, in Chapter XII of [**3**], as well as in [**21**].

We shall focus mainly on Heyting and Boolean algebras. In the exercises at the end of this chapter, we indicate an alternative route to our results, a construction which is a generalization of the Dedekind cuts used to obtain the reals from the rationals. For general posets, this last construction, called **the MacNeille completion**, regularly embeds any poset in a complete lattice. For Heyting and Boolean algebras the outcomes of "completion by cuts" and that obtained as detailed below are isomorphic.

Our first step is to identify certain types of ideals and filters in a distributive lattice.

Definition 14.1. *Let L be a distributive lattice.*
a) An ideal $I \subseteq L$ is **complete** *iff for all $S \subseteq L$,*

$$S \subseteq I \text{ and } \bigvee S \text{ exists in } L \;\Rightarrow\; \bigvee S \in I.$$

Write $c\mathcal{I}(L)$ for the poset of complete ideals in L, partially ordered by inclusion.
b) A filter $F \subseteq L$ is **complete** *iff for all $S \subseteq L$,*

$$S \subseteq F \text{ and } \bigwedge S \text{ exists in } L \;\Rightarrow\; \bigwedge S \in F.$$

Write $c\mathcal{F}(L)$ for the poset of complete filters in L, partially ordered by inclusion.

Clearly, principal ideals and filters are complete. Moreover, the intersection of any family of complete filters or ideals is again a complete filter or ideal. This means, on the one hand, that the posets $c\mathcal{F}(L)$ and $c\mathcal{I}(L)$ are *complete lattices*; on the other, it allows us to define the **complete filter or ideal generated** by $A \subseteq L$:

$$\begin{cases} A^* = \bigcap \{F \in \mathcal{F}(L) : A \subseteq F \text{ and } F \text{ is a complete filter}\} \\ \qquad\qquad\qquad \textit{and} \\ A_* = \bigcap \{I \in \mathcal{I}(L)\colon A \subseteq I \text{ and } I \text{ is a complete ideal}\} \end{cases}$$

When $A = \{a\}$ the above definitions just give back the principal filter/ideal generated by a. To simplify exposition, we shall write a^* for $a^{\rightarrow}$ and a_* shall stand for $a^{\leftarrow}$. We shall be mainly interested in complete ideals, although our constructions could be also carried out with complete filters.

To increase our chances of prevailing over confusion, we use $\wedge$ and $\bigvee$ for the operations in the lattice L and $\cap$, $\bigcap$ and $\bigvee^*$ for the lattice operations in $c\mathcal{I}(L)$. The symbol $\bigcup$ will always be used as set theoretic union.

In general, if I_k, $k \in K$, is a family of complete ideals, the **ideal** generated by $\bigcup I_k$ is not complete, even if the family is directed. Simple examples can be found in the chain [0, 1]. Note also that principal ideals may not be compact in $c\mathcal{I}(L)$. The next Lemma gives a useful description of joins in $c\mathcal{I}(L)$, when L is a $[\wedge, \bigvee]$-lattice (8.6).

LEMMA 14.2. *Let L be a $[\wedge, \bigvee]$-lattice.*
a) If I_α, $\alpha \in A$, is a family of complete ideals in , then
$$\bigvee^* I_\alpha = \{\bigvee \bigcup_{\alpha\in A} S_\alpha : S_\alpha \subseteq I_\alpha \text{ and } \bigvee \bigcup_{\alpha\in A} S_\alpha \text{ exists in } L\}.$$
b) $c\mathcal{I}(L)$ is a frame.

PROOF. a) Let K be the right-hand side of the stated equality. Clearly, any complete ideal containing all the I_α has to contain K. Thus, it suffices to check that K is a complete ideal. Suppose $b \in K$ and $c \le b$; since $b = \bigvee \bigcup S_\alpha$, $S_\alpha \subseteq I_\alpha$, the $[\wedge, \bigvee]$-law yields
$$c = c \wedge b = \bigvee \bigcup_{\alpha\in A} \{c \wedge s : s \in S_\alpha\}$$
and so $c \in K$. If $T \subseteq K$ then, for each $t \in T$,
$$t = \bigvee \bigcup_{\alpha\in A} S_{t\alpha}, \text{ with } S_{t\alpha} \subseteq I_\alpha.$$
Let $T_\alpha = \bigcup_{t\in T} S_{t\alpha}$; if $\bigvee T$ exists in L then $\bigvee T = \bigvee \bigcup_{\alpha\in A} T_\alpha$, and so belongs to K, showing that K is a complete ideal.
b) Let I, J_α, $\alpha \in$ A, be ideals in $c\mathcal{I}(L)$. It must be shown that
$$I \cap (\bigvee^* J_\alpha) \subseteq {\bigvee^*}_{\alpha\in A} I \cap J_\alpha.$$
Let $a \in I \cap (\bigvee^* J_\alpha)$; by (a), there are $S_\alpha \subseteq J_\alpha$ such that $a = \bigvee \bigcup_{\alpha\in A} S_\alpha$. Define
$$a \wedge S_\alpha = \{a \wedge s : s \in S_\alpha\};$$
then $a \wedge S_\alpha \subseteq J_\alpha$ and $a = \bigvee \bigcup (a \wedge S_\alpha)$. Then, item (a) yields $a \in {\bigvee^*}_{\alpha\in A} I \cap J_\alpha$, ending the proof. $\square$

By 14.2 and 8.7.(b), $H_* = c\mathcal{I}(H)$ is a frame, when H is a HA.

LEMMA 14.3. *Let L be a lattice.*
a) The map $(\cdot)_* : H \longrightarrow H_*$, $a \mapsto a_*$ is a lattice embedding of L into $c\mathcal{I}(L)$, that is, an injection such that for all $a, b \in L$
* $\bot_* = \{\bot\}$ *and* $\top_* = H$.
* $(a \vee b)_* = a_* \vee^* b_*$ *and* $(a \wedge b)_* = a_* \cap b_*$.

b) For $I \in c\mathcal{I}(L)$, $I = {\bigvee^*}_{a\in I} a_*$; hence, $\{a_*\}_{a\in H}$ is a basis for H_*.
c) If L is a Heyting algebra, then $a \mapsto a_*$ is a HA-morphism.

PROOF. Items (a), (b) and (c) are straightforward, while (c) follows immediately from Corollary 8.15. $\square$

Theorem 14.4. *Let H be a HA. Notation as above, the diagram $H \xrightarrow{*} H_*$ has the following properties :*

(1) *The image of $*$ is a basis for H_*;*

(2) *$*$ is a regular embedding of H into H_*.*

(3) *If D is a frame and $H \xrightarrow{f} D$ is a $[\wedge, \bigvee]$-morphism, there is a* **unique** *$[\wedge, \bigvee]$-morphism f_*, such that $f_* \circ * = f$.*

(Extension Property)

$$\begin{tikzcd} H \arrow[r, "*"] \arrow[d, "f"'] & H_* \arrow[dl, "f_*"] \\ D & \end{tikzcd}$$

Hence, if Ω is a frame and $H \xrightarrow{h} \Omega$ satisfies conditions (1) *and* (2), *then there is a frame isomorphism h_* from H_* to Ω.*

Proof. That $*$ is an embedding whose image is a basis for $c\mathcal{I}(H)$ follows from 14.3. To show that $*$ is regular, let $S \subseteq H$ and suppose $\bigwedge S = b$ exists in H. Clearly, $b_* \subseteq \bigcap_{s \in S} s_*$. Now, assume that $I \in H_*$ is such that $I \subseteq s_*$, $s \in S$. Then, $a \leq s$, for all $a \in I$ and so $a \in b_*$. Hence, $b_* = \bigcap_{s \in S} s_*$, and $*$ preserves all meets in H. For joins, note that if $b = \bigvee S$ in H, then, since ${\bigvee^*}_{s \in S}\, s_*$ is complete, we have $b_* \in \bigvee^* s_*$ (14.2.(a)). Hence, ${\bigvee^*}_{s \in S}\, s_* = b_* = (\bigvee S)_*$, as needed.

To verify the extension property in (3), fix a $[\wedge, \bigvee]$-morphism, $f : H \longrightarrow D$, with D a frame. The uniqueness of a $[\wedge, \bigvee]$-morphism g such that $g \circ * = f$ follows immediately from the preservation of joins and the fact that the image of $*$ is a basis for H_*. To prove existence, define, for $x \in H_*$,

$$f_*(x) = \sup_D \{fa : a_* \leq x\}.$$

Clearly, $f_*(a_*) = fa$, for $a \in H$, i.e., $f_* \circ * = f$. Since f_* is increasing, to prove it preserves $\wedge$, it suffices to show that $f_*(x \wedge y) \geq f_*x \wedge f_*y$. Corollary 8.4 yields

$$\begin{aligned} f_*(x) \wedge f_*(y) &= (\bigvee \{fa : a_* \subseteq x\}) \wedge (\bigvee \{fb : b_* \subseteq y\} \\ &= \bigvee \{f(a \wedge b) : a_* \subseteq x \text{ and } b_* \subseteq y\} \leq f_*(x \wedge y). \end{aligned}$$

To check that f_* preserves $\bigvee$, let $\{J_\alpha\}_{\alpha \in A} \subseteq H_*$ and $I = \bigvee^* J_\alpha$ in H_*. Again, since f_* is increasing, it suffices to verify $f_*I \subseteq \sup_D f_*(I_\alpha)$. For $a_* \subseteq I$, there is, by 14.2.(a), $S_\alpha \subseteq J_\alpha$ such that $a = \bigvee \bigcup_{\alpha \in A} S_\alpha$. Since f is a $[\wedge, \bigvee]$-morphism, we get

$$fa = \sup_D \{f(s) : s \in \bigcup S_\alpha\} \leq \sup_D f_*(J_\alpha). \tag{I}$$

By definition, $f_*I = \sup_D \{fa : a_* \subseteq I\}$; since (I) holds for all $a_* \subseteq I$, it follows that $f_*(I) \leq \sup_D f_*(J_\alpha)$, as needed. It is straightforward to show that (1) and (2) determine H_*, up to isomorphism. $\square$

The frame H_* is called the **completion** of H. All we will ever need to know about H_* are the properties of the diagram $H \xrightarrow{*} H_*$ in Theorem 14.4.

EXAMPLE 14.5. The statement of 14.4 is best possible : we cannot hope to get a **complete** extension to H_* of a complete morphism $H \longrightarrow D$. Indeed, let $H = [-1, 0) \cup (0, 1] \subseteq \mathbb{R}$, with its natural order. It is easily established that

$$H_* = [-1, 1] = H \cup \{0\}.$$

Let $D = [0, 1] \cup [2, 3]$; define $H \xrightarrow{f} D$ as the gluing of two linear, strictly increasing maps, one from $[-1, 0)$ onto $[0, 1)$, and the other from $(0, 1]$ onto $(2, 3]$. Note that f is a regular embedding of H into D; f has only two extensions to H_* : one taking 0 to 1 and the other, mapping 0 to 2. The only one that is a $[\wedge, \bigvee]$-morphism is the former, but it does not preserve infinite meets. $\square$

For Boolean algebras, Theorem 14.4 yields

COROLLARY 14.6. *If B is a Boolean algebra, then the frame B_* is a cBa and we have*
(1) $B \xrightarrow{*} B_*$ *is a regular embedding.*
(2) *The image of $*$ is a basis for B_*.*
(3) *If A is a* **cBa** *and $B \xrightarrow{f} A$ is a $[\wedge, \bigvee]$-morphism, there is a unique complete morphism, $B_* \xrightarrow{f_*} A$, such that $f_* \circ * = f$.*
Moreover, if A is a cBa and $B \xrightarrow{h} A$ is a diagram satisfying (1) *and* (2)*, then there is a unique isomorphism h_*, from B_* onto A, such that $h_* \circ * = h$.*

REMARK 14.7. One may ask if "BA-morphism" may be substituted for "$[\wedge, \bigvee]$-morphism" in item (3) of Corollary 14.6. The answer is **no**. To explain what is involved, we succinctly comment on free objects in categories of algebras. For more details, consult [**34**] (Chapter I.4) or [**3**], particularly Chapters I and V.

Let $\mathcal{L}$ be a first order language with equality, but possessing no relation symbols other than equality; let τ be the type of $\mathcal{L}$. A model for $\mathcal{L}$ is called an **algebra of type $\boldsymbol{\tau}$**. The concept of morphism is clear : a map between two algebras of the same type that preserve all the operations in $\mathcal{L}$. Clearly, the identity mapping is a morphism and composition of morphisms – defined as the usual composition of maps –, is a morphism. We have a category, $\boldsymbol{Alg_\tau}$, of algebras of type τ. It is clear that $\boldsymbol{Alg_\tau}$ has products for any family of objects.

DEFINITION 14.8. *Let $\mathbb{K}$ be a class of algebras in $\boldsymbol{Alg_\tau}$ and S a set. An algebra F is the* **free algebra** *generated by S over $\mathbb{K}$ if there is a map $S \xrightarrow{f} F$, such that for all algebras B in $\mathbb{K}$ and all maps $S \xrightarrow{g} B$, there is a* **unique** *morphism in $\boldsymbol{Alg_\tau}$, $F \xrightarrow{h} B$, such that $h \circ f = g$.*

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "h"'] & F \arrow[dl, "g"] \\ B & \end{tikzcd}$$

An algebra is **free** *over* $\mathbb{K}$ *if it is the algebra freely generated by some set over* $\mathbb{K}$.

A class $\mathbb{K}$ in $\boldsymbol{Alg_\tau}$ is an **an equational class** iff $\mathbb{K}$ is closed under the following operations :
(i) Taking isomorphic copies of homomorphic images of members of $\mathbb{K}$;
(ii) Taking isomorphic copies of subalgebras of members of $\mathbb{K}$;
(iii) Taking isomorphic copies of direct products of members of $\mathbb{K}$.
$\mathbb{K}$ is **trivial** if it consists of the one element algebras of $\boldsymbol{Alg_\tau}$.

The classes of lattices, distributive lattices, Heyting and Boolean algebras, groups, rings, etc., are all non-trivial equational classes (but not all of the same type). There is an important result of G. Birkhoff guaranteeing that a class of algebras in $\boldsymbol{Alg_\tau}$ is equational iff it consists of all models of the universal closure of a set of atomic formulas in $\mathcal{L}$. For equational classes we can state

Theorem A. (Thm. I.12.4, [**3**]) *Let* $\mathbb{K}$ *be a non-trivial equational class. Then, for each set* S*, the free algebra generated by* S *over* $\mathbb{K}$ *exists and belongs to* $\mathbb{K}$.

Theorem A guarantees the existence of free objects in any of the categories (lattices, HAs, BAs, etc.) mentioned above.

The free BA or the free HA on a set S is a familiar object : they are the algebras of propositions, constructed by taking S as the set of atomic propositions, under the equivalence relation "provably equivalent" from the axioms of the Classical or the Heyting Propositional Calculus.

In contrast to Theorem A, we have (see [**34**] or [**72**])

Theorem B. (Gaifmann; Hales; Solovay) *There is no free* **complete** *Boolean algebra on a countable number of generators.*

Consequently, there is no cBa $B_{\mathbb{N}}$ and a map $\mathbb{N} \xrightarrow{g} B_{\mathbb{N}}$, such that for all cBas D and functions $\mathbb{N} \xrightarrow{f} D$, there is a unique **complete** morphism $B_{\mathbb{N}} \xrightarrow{h} D$, satisfying $h \circ g = f$.

We now see that "$[\wedge, \bigvee]$-morphism" cannot be substituted by "BA-morphism" in Corollary 14.6.(3). If it were possible, then the completion of the free BA generated by $\mathbb{N}$ would be the free cBa generated by $\mathbb{N}$, contradicting Theorem B.

It is possible, however, to extend BA-morphisms with values in a complete Boolean algebra. This result is important and its precise formulation will appear in the next Chapter. $\square$

For frames and their morphisms, the situation is quite distinct from that described by Theorem B in 14.7, as shown by Theorem 14.9 below, due to Benabou.

If X is a set, the family of finite subsets of X, $Fin(X)$, is a poset under inclusion, in fact a lattice with $\perp$ but without $\top$ (unless X is finite). Write Ω_X for the frame of opens of the $\mathfrak{U}$-topology of 8.3 on $Fin(X)$. We have

THEOREM 14.9. *Notation as above, let* $X \xrightarrow{\iota} \Omega_X$ *be given by* $\iota(x) = \{x\}^{\rightarrow}$. *Then, for all frames* L *and maps* $g : X \longrightarrow L$, *there is a* unique *frame morphism,* $\widehat{g} : \Omega_X \longrightarrow L$, *making the following diagram commutative :*

$$\begin{tikzcd} X \arrow[rr, "\iota"] \arrow[dr, "g"'] & & \Omega_X \arrow[dl, "\widehat{g}"] \\ & L & \end{tikzcd}$$

Hence, Ω_X *is the free frame generated by* X.

PROOF. Write $\bigwedge$, $\wedge$, $\bigvee$, $\vee$ for meets and joins in L. If $\alpha \in Fin(X)$, then

$$\alpha^{\rightarrow} = \bigcap_{x\in\alpha} \{x\}^{\rightarrow} = \bigcap_{x\in\alpha} \iota(x).$$

We define

(*) $$\widehat{g}(\alpha^{\rightarrow}) = \bigwedge_{x\in\alpha} gx.$$

Note that for $x \in X$, (*) yields $\widehat{g}(\iota(x)) = gx$, and the displayed diagram in the statement is commutative. If α, $\beta \in Fin(X)$, then

$$\alpha^{\rightarrow} \cap \beta^{\rightarrow} = (\alpha \cup \beta)^{\rightarrow},$$

and so

(**) $$\begin{aligned} \widehat{g}(\alpha^{\rightarrow} \cap \beta^{\rightarrow}) &= \widehat{g}((\alpha \cup \beta)^{\rightarrow}) = \bigwedge_{x\in(\alpha\cup\beta)} \iota(x) \\ &= \bigwedge_{x\in\alpha} \iota(x) \wedge \bigwedge_{y\in\beta} \iota(y) \\ &= \widehat{g}(\alpha^{\rightarrow}) \wedge \widehat{g}(\beta^{\rightarrow}). \end{aligned}$$

It follows immediately from (**) that

(***) $$\alpha^{\rightarrow} \subseteq \beta^{\rightarrow} \;\Rightarrow\; \widehat{g}(\alpha^{\rightarrow}) \leq \widehat{g}(\beta^{\rightarrow}).$$

To extend $\widehat{g}$ to $U \in \Omega_X$, write $U = \bigcup_{i\in I} \alpha_i^{\rightarrow}$ and set

$$\widehat{g}(U) = \bigvee_{i\in I} \widehat{g}(\alpha_i^{\rightarrow}).$$

To see that $\widehat{g}$ is well defined, first note that if $\beta^{\rightarrow} \subseteq U$, then the supercompactness of $\beta^{\rightarrow}$ (8.3.(ii)) entails the existence of $i \in I$ such that $\beta^{\rightarrow} \subseteq \alpha_i^{\rightarrow}$. Hence, (***) yields

$$\widehat{g}(\beta^{\rightarrow}) \leq \widehat{g}(\alpha_i^{\rightarrow}) \leq \widehat{g}(U).$$

It now straightforward that if $U = \bigcup_{j\in J} \beta_j^{\rightarrow}$, then

$$\bigvee_{i\in I} \widehat{g}(\alpha_i^{\rightarrow}) = \bigvee_{j\in J} \widehat{g}(\beta_j^{\rightarrow}),$$

and $\widehat{g}$ is indeed well defined. If $U = \bigcup_{\alpha\in S} \alpha^{\rightarrow}$ and $V = \bigcup_{\beta\in T} \beta^{\rightarrow}$, with $S, T \subseteq Fin(X)$, then 8.4, applied in Ω_X, yields

$$U \cap V \;=\; \bigcup_{\langle \alpha,\beta\rangle \in S\times T} \alpha^{\rightarrow} \cap \beta^{\rightarrow}.$$

But then from (**) and 8.4 in L we get

$$\begin{aligned}\widehat{g}(U \cap V) &= \bigvee_{\langle \alpha,\beta\rangle \in S\times T} \widehat{g}(\alpha^{\rightarrow} \cap \beta^{\rightarrow}) = \bigvee_{\langle \alpha,\beta\rangle \in S\times T} \widehat{g}(\alpha^{\rightarrow}) \wedge \widehat{g}(\beta^{\rightarrow}) \\ &= \widehat{g}(U) \wedge \widehat{g}(V),\end{aligned}$$

showing that $\widehat{g}$ preserves finite meets. The preservation of arbitrary joins is straightforward. Uniqueness is clear from the construction. $\square$

Exercises

14.10. Let H be a HA and D be the filter of dense elements in H. Notation as in 14.4 and 10.5, let H_* be the completion of H and write D_* for the filter of dense elements in H_*.

a) There is a *unique* injective BA-morphism, $* : H/D \longrightarrow H_*/D_*$, making the following diagram commutative :

$$\begin{tikzcd} H \arrow[r, "*"] \arrow[d, "\pi_D"'] & H_* \arrow[d, "\pi_{D_*}"] \\ H/D \arrow[r, "*"'] & H_*/D_* \end{tikzcd}$$

b) H_*/D_* is the completion of H/D [1]. $\square$

The exercises that follow describe the MacNeille completion and its relation to the completions we have presented above. A slightly different account of this same construction, using closure operators, can be found in section 2 of Chapter XII of [**3**], under the name of **normal completion**.

14.11. Let $(P, \leq)$ be a poset. All notation is as in Definition 2.6 and the comments right after it.

A **cut** in P is a pair of subsets of P, (A, B) such that $A \subseteq B^{\leftarrow}$ and $B \subseteq A^{\rightarrow}$. Every element x of P gives rise to a cut, $c(x) = (x^{\leftarrow}, x^{\rightarrow})$. Let cP be the set of all cuts in P, partially ordered by

$$(A, B) \leq (C, D) \quad \text{iff} \quad A \subseteq C \text{ (or equivalently, iff } D \subseteq B).$$

We denote by $c : P \longrightarrow cP$ the map $x \mapsto c(x)$.

a) A cut (A, B) is equal to $c(x)$, for some $x \in P$, iff $A \cap B \neq \emptyset$.

b) cP is a complete lattice and c is a regular embedding of P in cP. $\square$

14.12. Notation as in 14.11, assume that P is lattice.

a) If (A, B) is a cut in P, then A is an ideal and B a filter in P.

b) If P is a HA (BA), then cP is a frame (resp., cBa), isomorphic to the completion P_* of Theorem 14.4 (resp., Corollary 14.6). $\square$

[1] In the notation of 14.6, $H_*/D_* = (H/D)_*$.

CHAPTER 15

Extension of Morphisms

We prove here that BA-morphisms with values in a complete Boolean algebra can be extended to BAs containing its domain, an important result due to S. Sikorski.

If B is a BA, it is readily verified that the intersection of any family of subalgebras of B is again a subalgebra of B. Thus, if S is a subset of B, we can define the **subalgebra generated** by S in B, as

$$[S] \ = \ \bigcap \{B' : B' \text{ is a subalgebra of } B \text{ and } S \subseteq B'\}.$$

Recall that for a, b in B, $a \bigtriangleup b$ (the symmetric difference of a and b) is defined in 5.5 as

$$a \bigtriangleup b = (a \wedge \neg b) \vee (b \wedge \neg a).$$

LEMMA 15.1. *Let B be a BA, B' a subalgebra of B and m an element of B. Then*

a) The subalgebra H generated by B' and m is given by

$$H = \{(a \wedge m) \vee (b \wedge \neg m) \colon a,\ b \in B'\}.$$

b) For all x, y, t, z in B', the following are equivalent :

i) $x \wedge m = t \wedge m$ and $y \wedge \neg m = t \wedge \neg m$.

ii) $(x \wedge m) \vee (y \wedge \neg m) = (t \wedge m) \vee (z \wedge \neg m)$.

iii) $x \bigtriangleup t \leq \neg m$ and $y \bigtriangleup z \leq m$.

PROOF. a) It is straightforward to show that the left hand side of the equation is a subalgebra of B, and so it must the least one containing B' and m.

b) It is immediate that (i) implies (ii), while taking the intersection of the equalities in (ii) with m and $\neg m$ yields (ii).

(i) $\Rightarrow$ (iii) : The first equation in (i) yields $x \wedge m \wedge \neg t = \bot$, and so $x \wedge \neg t \leq \neg m$. Similarly, $t \wedge \neg x \leq \neg m$, and $x \bigtriangleup t \leq \neg m$. The other inequality stated in (iii) is proven similarly.

(iii) $\Rightarrow$ (i) : From $x \bigtriangleup t \leq \neg m$ we get

$$x \wedge \neg t \wedge m = t \wedge \neg x \wedge m = \bot.$$

Thus, $x \wedge m \leq t$ and $t \wedge m \leq x$. But then, $x \wedge m = t \wedge m$. The remaining equality in (i) is obtained analogously. $\square$

With these preliminaries we have

THEOREM 15.2. *Let B be a BA and let B' a subalgebra of B. If $\overline{B}$ is a cBa, any BA-morphism from B' to $\overline{B}$ can be extended to a BA-morphism from B to $\overline{B}$.*

PROOF. We may assume that B' is a proper subalgebra of B. We also fix a BA-morphism f from B' to $\overline{B}$. As a first step, we prove

Fact. *For $m \in B$, let H the BA generated in B by B' and m. Then f can be extended to a BA morphism from H to $\overline{B}$.*

Proof. Let $S = m^{\leftarrow} \cap B'$ and $T = m^{\rightarrow} \cap B'$; clearly every element of S is less than or equal to any element of T. Since f is increasing and $\overline{B}$ is complete, we conclude that

$$\bigvee f(S) \leq \bigwedge f(T).$$

Let w be any element of $\overline{B}$ such that $\bigvee f(S) \leq w \leq \bigwedge f(T)$. Using Lemma 15.1(a), define $g : H \longrightarrow \overline{B}$, by

$$g([a \wedge m] \vee [b \wedge \neg m]) \;=\; (fa \wedge w) \vee (fb \wedge \neg w),$$

where $a, b \in B'$. It must be shown that g is well defined; if x, y, t, z are elements of B' such that

$$(x \wedge m) \vee (y \wedge \neg m) = (t \wedge m) \vee (z \wedge \neg m),$$

then by 15.1.(b) we have $(x \vartriangle t) \leq \neg m$ and $(y \vartriangle z) \leq m$. Thus, $(y \vartriangle z) \in S$ and $\neg(x \vartriangle t) \in T$. Hence, since f is a BA morphism, we get

$$f(y \vartriangle z) = fy \vartriangle fz \leq \mathrm{w} \quad \text{and} \quad w \leq f(\neg(x \vartriangle t)) = \neg f(x \vartriangle t),$$

that is, $fx \vartriangle ft \leq \neg w$. But these are exactly the conditions needed to have

$$(fx \wedge w) \vee (fy \wedge \neg w) \;=\; (ft \wedge w) \vee (fz \wedge \neg w),$$

and g is well defined. Clearly, g is a BA morphism and the Fact is proven.

Consider

$$V = \{H \xrightarrow{g} \overline{B} : \begin{array}{c} H \text{ a subalgebra of } B,\ B' \subseteq H, \\ g \text{ is a BA-morphism and } g_{|B'} = f, \end{array}\Big\}$$

partially ordered by "extension", i.e., $g \leq h$ iff dom $g \subseteq$ dom h and $h_{|domg} = g$. V satisfies the conditions of Zorn's Lemma (2.20), and has a maximal element g. To end the proof, note that if dom g where a proper subalgebra of B, then the Fact would provide a proper extension of g, contradicting its maximality. $\square$

Theorem 15.2 is analogous to many other extension results, like the Hahn-Banach Theorem in Functional Analysis. In fact, there are significant relations between these results. We refer the interested reader to [**8**] and [**18**].

COROLLARY 15.3. *A complete Boolean algebra $\overline{B}$ satisfies the following universal properties :*

$[BA - inj]$ *Let $B' \xrightarrow{f} B$ be a BA monomorphism. Then, for any BA-morphism $B' \xrightarrow{h} \overline{B}$, there is a BA-morphism $B \xrightarrow{g} \overline{B}$, such that $g \circ f = h$.*

$[HA - inj]$ *Let H and H' be Heyting algebras and $H' \xrightarrow{f} H$ be a HA-monomorphism. Then, for any HA-morphism $H' \xrightarrow{h} \overline{B}$, there is a HA-morphism $H \xrightarrow{g} \overline{B}$, such that $g \circ f = h$.*

$$\begin{tikzcd}
H' \arrow[r, "f"] \arrow[d, "h"'] & H \arrow[dl, "g"] \\
\overline{B} &
\end{tikzcd}$$

Proof. Property $[BA - inj]$ follows immediately from Theorem 15.2. For the Heyting algebra case, let D and D' be the filter of dense elements in H and H', respectively; by 6.21, f induces an injective BA-morphism from H'/D' to H/D, while h induces a BA-morphism from H'/D' to $\overline{B}$. By $[BA-inj]$, the map induced by h can be extended to H/D; the required HA-morphism g is the composition of this extension with the quotient map $H \longrightarrow H/D$. $\square$

With 15.3 we can generalize, in one direction, 14.6.(3) :

Corollary 15.4. *Let B be a BA and $B \xrightarrow{*} B_*$ be the completion of B. If $\overline{B}$ is a cBa and $B \xrightarrow{f} \overline{B}$ is a BA morphism, then there exists a BA morphism, $f_* : B_* \longrightarrow \overline{B}$, such that $f_* \circ * = f$. Moreover, if f is a monomorphism, the same is true of f_*.*

Proof. The only statement still to verify is that f_* is monic if f is monic. Let $c \neq d \in B_*$; the injectivity of f_* equivalent to $f_*(c \vartriangle d) \neq \perp$. Since B is a basis for B_*, there is $a \neq \perp$ in B such that $\perp < a_* \leq c \vartriangle d$. But then, from

$$f_*(c) \vartriangle f_*(d) = f_*(c \vartriangle d) \geq f_*(a_*) = fa,$$

and the injectivity of f, it is clear that $f_*(c \vartriangle d) \neq \perp$, as needed. $\square$

In the language of Category Theory (see 16.5), Corollary 15.3 tells us that complete Boolean algebras are **injective objects** both in the category of Boolean algebras and in the category of Heyting algebras. The converse of these statements is also true (see Exercise 15.5). Corollary 15.4 guarantees that the completion of a BA B is the **injective hull** of B, that is, a minimal injective object containing B, as set down in item (d) of Definition 16.36.

Exercises

15.5. Prove that a BA (HA) is injective in the category of BAs (resp., HAs) iff it is a complete Boolean algebra. The definition of *injective* is "satisfies the universal properties $[BA - inj]$ and $[HA - inj]$" in the statement of 15.3. $\square$

Part 2

Category Theory

CHAPTER 16

Categorical Constructions

In this chapter we collect most of the language of Category Theory used in the text. It is not a substitute for the reading of [**44**] or [**53**], but is included as a convenient reference for the reader. Many of the basic constructions are also described in chapter I of [**3**].

1. Categories and Morphisms

DEFINITION 16.1. *A* **category** $\mathcal{A}$ *is a pair* $\langle\, Ob(\mathcal{A}),\ \mathcal{M}(\mathcal{A})\,\rangle$ *where* $Ob(\mathcal{A})$ *is a class, whose elements are* **the objects** *of* $\mathcal{A}$ *and* $\mathcal{M}(\mathcal{A})$ *is a class, which is a disjoint union* (1.5)

$$\mathcal{M}(\mathcal{A}) = \textstyle\coprod_{(A,B)\in Ob(\mathcal{A})}\ [A,B]_{\mathcal{A}},$$

where $[A,B]_{\mathcal{A}}$ *is a (possibly empty)* **set***, called the set of* **morphisms** *from* A *to* B *in* $\mathcal{A}$*. Whenever* $\mathcal{A}$ *is clear from context, we write* $[A,B]$ *in place of* $[A,B]_{\mathcal{A}}$*. Moreover, for each triple of objects of* $\mathcal{A}$, $\langle\, A,B,C\,\rangle$*, we have a map*

$$[A,B]\,\times\,[B,C]\longrightarrow [A,C],\quad \langle\, f,g\,\rangle \mapsto g\circ f,$$

called **composition***, which satisfies the following conditions :*

$[\circ\ 1]$ *: Composition, whenever defined, is associative.*

$[\circ\ 2]$ *: For* $A\in Ob(\mathcal{A})$*, there is* Id_A *in* $[A,A]$*, such that for* $B\in Ob(\mathcal{A})$, $f\in[A,B]$ *and* $g\in[B,A]$,

$$f\circ Id_A = f \quad \text{and} \quad Id_A\circ g = g.$$

The morphism Id_A *is unique, being called the* **identity** *of the object* A*. If* $f\in[A,B]$*, we say that* A *is the* domain *of* f ($dom f\ =\ A$) *and that* B *is the* codomain *of* f ($codom f\ =\ B$)*. We use standard functional notation for morphisms. Hence,*

$$f : A\longrightarrow B,\quad A\xrightarrow{f} B,$$

are synonymous with $f\in[A,B]$.

A category is **small** *if the class of its objects is a set; it is said to be* **set based** *if all its objects are sets and all its morphisms are set-theoretical maps.*

DEFINITION 16.2. *Let* $\mathcal{A}$, $\mathcal{B}$ *be categories.* $\mathcal{B}$ *is a* **subcategory** *of* $\mathcal{A}$ *iff*

$$\begin{cases} \qquad Ob(\mathcal{B})\subseteq Ob(\mathcal{A}) \qquad \text{and} \\ \forall\ A,\ B\in Ob(\mathcal{B}),\quad [A,B]_{\mathcal{B}}\subseteq[A,B]_{\mathcal{A}}. \end{cases}$$

$\mathcal{B}$ *is a* **full** *subcategory of* $\mathcal{A}$ *if* $[A,B]_{\mathcal{B}} = [A,B]_{\mathcal{A}}$*, for all* A, B *in* $Ob(\mathcal{B})$.

Remark 16.3. Here is a list of categories, some of which we have already mentioned, that will be of standard use :
∗ **Set**, the category of sets and mappings;
∗ **Po**, the category of partially ordered sets and increasing maps;
∗ $\boldsymbol{\mathcal{L}}$, the category of lattices and lattice morphisms;
∗ $\boldsymbol{\mathcal{D}}$, the category of distributive lattices and lattice morphisms;
∗ **HA**, the category of Heyting algebras and HA-morphisms;
∗ **BA**, the category of Boolean algebras and BA-morphisms;
∗ **Frame**, the category of frames and frame morphisms;
∗ **cBa**, the category of cBas, with complete morphisms;
∗ **Framep**, the category of frames with enough points and frame morphisms;
∗ **Top**, the category of Topological spaces and continuous maps;
∗ **Htop**, the category of Hausdorff spaces and continuous maps;
∗ **CTop**, the category of compact Hausdorff spaces and continuous maps;
∗ **Sob**, the category of sober spaces and continuous maps.
∗ $\boldsymbol{Alg_\tau}$, the category of algebras of type τ (see 14.7). □

Example 16.4. Let $\langle P, \leq \rangle$ be a poset. We construct a category, also written P, by setting $Ob(P) = P$ and for x, $y \in P$,

$$[x, y] = \begin{cases} \{\langle x, y \rangle\} & \text{if } x \leq y \\ \emptyset & \text{otherwise} \end{cases}$$

Composition is clear: if $x \leq y \leq z$, $\langle y, z \rangle \circ \langle x, y \rangle = \langle x, z \rangle$. Clearly, $\langle x, x \rangle = Id_x$, for all $x \in P$. Hence, every poset has a natural structure of category. □

Example 16.5. To every category $\mathcal{A}$ corresponds a **dual category**, written $\mathcal{A}^{op}$, defined as follows :

∗ $Ob(\mathcal{A}^{op}) = Ob(\mathcal{A})$; ∗ For A, $B \in Ob(\mathcal{A})$, $[A, B]_{\mathcal{A}^{op}} = [B, A]_{\mathcal{A}}$.

Clearly, $(\mathcal{A}^{op})^{op} = \mathcal{A}$. Thus, all theorems about categories embodies two results: if a statement S holds in $\mathcal{A}$, then its dual holds in $\mathcal{A}^{op}$. Moreover, if the hypothesis used to prove S, also hold in $\mathcal{A}^{op}$, then its dual is true in $\mathcal{A} = (\mathcal{A}^{op})^{op}$. □

Example 16.6. Let $\mathcal{A}$ be a category and let A be an object in $\mathcal{A}$. We define two categories, $\mathcal{A}_A$ and $\mathcal{A}^A$, as follows :

[$\mathcal{A}_A$ 1] : The objects of $\mathcal{A}_A$ are morphisms in $\mathcal{A}$, $f : A \longrightarrow B$, with $B \in Ob(\mathcal{A})$;

[$\mathcal{A}_A$ 2] : If $I = (A \xrightarrow{f} B)$ and $J = (A \xrightarrow{g} C)$ are objects in $\mathcal{A}_A$, then a morphism in $\mathcal{A}_A$, $I \xrightarrow{h} J$, is a morphism $h \in [B, C]_{\mathcal{A}}$, such that $h \circ f = g$.

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "g"'] & & B \arrow[dl, "h"] \\ & C & \end{tikzcd} \qquad \begin{tikzcd} B \arrow[rr, "\eta"] \arrow[dr, "\beta"'] & & C \arrow[dl, "\gamma"] \\ & A & \end{tikzcd}$$

$[\mathcal{A}^A\ 1]$: The objects of $\mathcal{A}^A$ are morphisms in $\mathcal{A}$, $f : B \longrightarrow A$, with $B \in Ob(\mathcal{A})$;

$[\mathcal{A}^A\ 2]$: If $I = (B \xrightarrow{\beta} A)$ and $J = (C \xrightarrow{\gamma} A)$ are objects in $\mathcal{A}^A$, then a morphism in $\mathcal{A}^A$, $I \xrightarrow{\eta} J$, is a morphism $\eta \in [B, C]_{\mathcal{A}}$ such that $\gamma \circ \eta = \beta$.

We refer to $\mathcal{A}_A$ as the category of *A-algebras in $\mathcal{A}$*, while $\mathcal{A}^A$ is the category of *$\mathcal{A}$-bundles over A*. ☐

DEFINITION 16.7. *Let $\mathcal{A}$ and $\mathcal{B}$ be categories. The* **product** *of $\mathcal{A}$ and $\mathcal{B}$, written $\mathcal{A} \times \mathcal{B}$, is the category defined as follows :*

* *$Ob(\mathcal{A} \times \mathcal{B}) = Ob(\mathcal{A}) \times Ob(\mathcal{B})$;*

* *If $I = \langle A, B \rangle$, $J = \langle C, D \rangle$ are objects in $\mathcal{A} \times \mathcal{B}$, then*

$$[I, J]_{\mathcal{A}\times\mathcal{B}} = [A, C] \times [B, D].$$

Clearly, this definition can be generalized to any number of components.

DEFINITION 16.8. *Let $f \in [A, B]$ be a morphism in a category $\mathcal{A}$.*

a) f is a **monomorphism** *(or a monic) in $\mathcal{A}$ if for all morphisms α, β in $\mathcal{A}$, satisfying codom α = codom $\beta = A$, we have*

$$f \circ \alpha = f \circ \beta \quad \textit{implies} \quad \alpha = \beta.$$

b) f is an **epimorphism** *(or an epic) in $\mathcal{A}$ if for all morphisms α, β in $\mathcal{A}$, satisfying dom α = dom $\beta = B$, we have*

$$\alpha \circ f = \beta \circ f \quad \textit{implies} \quad \alpha = \beta.$$

Hence, f is a monic in $\mathcal{A}$ iff f is an epic in $\mathcal{A}^{op}$.

c) f is a **retraction** *if there is $B \xrightarrow{g} A$, such that $f \circ g = Id_B$.*

d) f is a **coretraction** *if there is $B \xrightarrow{g} A$ such that $g \circ f = Id_A$. In this case, we say that A is a* retract *of B.*

e) f is an **isomorphism** *if it is both a retraction and a coretraction.*

It is frequent to refer to a monic $A \xrightarrow{f} B$, as a **subobject** *of B. A category is* **locally small** *if for all objects A in $\mathcal{A}$, there is a* **set** *S of subobjects of A, such that, every subobject of A in $\mathcal{A}$ is isomorphic, in $\mathcal{A}^A$ (see 16.6), to one in S.*

DEFINITION 16.9. *Let $\mathcal{A}$ be a category. An object A in $\mathcal{A}$ is an* **initial (final) object** *if for all $B \in Ob(\mathcal{A})$, $[A, B]$ (resp., $[B, A]$) is a singleton. Clearly, initial and final objects are unique, up to isomorphism. Write $\top$ and $\bot$, respectively, for the final and initial objects in $\mathcal{A}$, whenever they exist.*

$\mathcal{A}$ is a **category with zero**, *if it has an object, written 0, which is both initial and final.*

EXAMPLE 16.10. In the category **Set**, $\emptyset$ is the initial object, while $\{\emptyset\}$ (or any other singleton) is the final object. The category **Gr** of groups has a zero, the group whose only element is the identity. ☐

2. Functors and Natural Transformations

DEFINITION 16.11. *Let $\mathcal{A}$, $\mathcal{B}$ be categories. A (covariant)* **functor**, *$\mathcal{A} \xrightarrow{F} \mathcal{B}$, is a rule that associates*

* *To each object A in $\mathcal{A}$, an object $F(A)$ in $\mathcal{B}$;*

$*$ *To each morphism* $f \in [A, B]_{\mathcal{A}}$, *a morphism* $F(f) \in [F(A),\ F(B)]_{\mathcal{B}}$, *such that*

$$(1)\ F(Id_A) = Id_{F(A)}; \qquad\qquad (2)\ F(f \circ g) = F(f) \circ F(g),$$

where $\mathcal{A} \in Ob(\mathcal{A})$ *and* f, g *are morphisms with codom* g = *dom* f. *Notice that for all* A, $B \in Ob(\mathcal{A})$, F *induces a map*

$$[A, B]_{\mathcal{A}} \longrightarrow [F(A),\ F(B)]_{\mathcal{B}}, \quad f \mapsto F(f).$$

F *is* **faithful** *if this map is injective and* **full** *if it is onto.*
A **contravariant functor**, $G : \mathcal{A} \longrightarrow \mathcal{B}$, *is a functor from* $\mathcal{A}^{op}$ *to* $\mathcal{B}$.
If $F : \mathcal{A} \longrightarrow \mathcal{B}$ *and* $G : \mathcal{B} \longrightarrow \mathcal{C}$ *are functors, their* **composition** *is a functor,* $(G \circ F) : \mathcal{A} \longrightarrow \mathcal{C}$, *given by*
$*$ *If* $A \in Ob(\mathcal{A})$, $(G \circ F)(A) = G(F(A))$;
$*$ *If* $A \xrightarrow{f} B$ *is a morphism in* $\mathcal{A}$, $(G \circ F)(f) = G(F(f))$.

EXAMPLE 16.12. If $\mathcal{A}$ is a category, $Id_{\mathcal{A}}$ for the **identity functor** from $\mathcal{A}$ to $\mathcal{A}$, i.e., the functor that associates every object and morphism in $\mathcal{A}$ to itself. $\square$

EXAMPLE 16.13. If $\mathcal{A}$ is a set based category, the **forgetful functor** from $\mathcal{A}$ to **Set** is the functor that associates to every object in $\mathcal{A}$ its underlying set and to each morphism in $\mathcal{A}$ its underlying map. In general, if $\mathcal{A}$ and $\mathcal{B}$ are categories, and $\mathcal{A}$ has "richer" structure than $\mathcal{B}$, there is a **forgetful functor** from $\mathcal{A}$ to $\mathcal{B}$, which forgets the richer structure in $\mathcal{A}$. As examples, we mention:
(1) Let $\boldsymbol{\mathcal{L}}$ be the category of lattices and **Po** be the category of posets. The forgetful functor from $\boldsymbol{\mathcal{L}}$ to **Po** associates to each lattice its underlying poset and to each lattice morphism the same map considered just as an increasing function.
(2) The forgetful functor from the category $\boldsymbol{\mathcal{D}}$, of distributive lattices, to the category $\boldsymbol{\mathcal{L}}$.
(3) The forgetful functor from the category of BAs, **BA**, to the category of HAs, **HA**, associates to each BA its underlying HA structure ($x \to y = \neg x \vee y$) and to each BA-morphism the corresponding map as a HA-morphism. $\square$

EXAMPLE 16.14. Let $\mathcal{A}$ be a category. An object A in $\mathcal{A}$ determines two functors from $\mathcal{A}$ to **Set**, h_A and h^A, defined as follows :
[h_A 1] : For each $B \in Ob(\mathcal{A})$, $h_A(B) = [A, B]$, the set of morphisms from A to B in $\mathcal{A}$.
[h_A 2] : For a morphism $B \xrightarrow{g} C$ in $\mathcal{A}$, $h_A(g) : [A,\ B] \longrightarrow [A,\ C]$ is given by $f \longmapsto g \circ f$.

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "g \circ f"'] & & B \arrow[dl, "g"] \\
& C &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
B \arrow[rr, "g"] \arrow[dr, "f \circ g"'] & & C \arrow[dl, "f"] \\
& A &
\end{tikzcd}$$

$[h^A\ 1]$: For each $B \in Ob(\mathcal{A})$, $h^A(B) = [B, A]$, the set of morphisms from B to A in $\mathcal{A}$.
$[h_A\ 2]$: For a morphism $B \xrightarrow{g} C$ in $\mathcal{A}$, $h^A(g) : [C,\ A] \longrightarrow [B,\ A]$ is given by $f \mapsto f \circ g$.
Note that h_A is covariant, while h^A is contravariant. These functors are called the **morphism functors** with parameter A. □

REMARK 16.15. Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be categories. A functor from $\mathcal{A} \times \mathcal{B}$ to $\mathcal{C}$ is usually called a **bifunctor**; it can have distinct variances in each coordinate. For example, it might be covariant in the first and contravariant in the second. Actually, all possible combinations can occur.

Note that if $F : \mathcal{A} \longrightarrow \mathcal{B}$ and $G : \mathcal{C} \longrightarrow \mathcal{D}$ are functors, then we have a bifunctor,

$$T \equiv F \times G : \mathcal{A} \times \mathcal{B} \longrightarrow \mathcal{C} \times \mathcal{D},$$

which on objects is given by $T(A,\ B) = \langle\, F(A),\ G(B)\, \rangle$, and on morphisms is defined by $T(f,\ g) = \langle\, F(f),\ G(g)\, \rangle$. Clearly, these observations can be generalized to any number of components. □

EXAMPLE 16.16. Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ and $G : \mathcal{B} \longrightarrow \mathcal{A}$ be functors of the same variance. We shall assume, without loss of generality, that they are both covariant. These functors give rise to bifunctors

$${}_Fh,\ h_G : \mathcal{A}^{op} \times \mathcal{B} \longrightarrow \mathbf{Set},$$

defined as follows :
∗ For $A \in Ob(\mathcal{A})$ and $B \in Ob(\mathcal{B})$,

$$\begin{cases} {}_Fh(A,B) &=& [F(A),\ B]_{\mathcal{B}} \\ h_G(A,B) &=& [A,\ G(B)]_{\mathcal{A}}. \end{cases}$$

∗ For a morphism $\langle\, f, g\, \rangle : \langle\, A, B\, \rangle \longrightarrow \langle\, C, D\, \rangle$ in $\mathcal{A}^{op} \times \mathcal{B}$,

$$\begin{cases} {}_Fh(f,g) \ :\ [F(A),B] \longrightarrow [F(C),D], \text{ given by } h \ \mapsto\ g \circ h \circ F(f). \\[1ex] h_G(f,g) \ :\ [A,G(B)] \longrightarrow [C,G(D)], \text{ given by } h \ \mapsto\ G(g) \circ h \circ f. \end{cases}$$

$$\begin{tikzcd} F(A) \arrow[r, "h"] & B \arrow[d, "g"] \\ F(C) \arrow[u, "F(f)"] \arrow[r, "{}_Fh(f,g)"'] & D \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "h"] & G(B) \arrow[d, "G(g)"] \\ C \arrow[u, "f"] \arrow[r, "h_G(f,g)"'] & G(D) \end{tikzcd}$$

We shall return to this example when we discuss adjointness. □

Morphisms of functors are called **natural transformations** :

Definition 16.17. *Let F, $G : \mathcal{A} \longrightarrow \mathcal{B}$ be functors, of the same variance* [1]. *A* **natural transformation**, *$\eta : F \longrightarrow G$, is a family of morphisms in $\mathcal{B}$,*

$$\eta = \{\eta_A \in [F(A), G(A)] : A \in Ob(\mathcal{A})\},$$

such that if $A \xrightarrow{f} B$ is a morphism in $\mathcal{A}$, the following diagram is commutative :

$$\begin{tikzcd} F(A) \arrow[r, "\eta_A"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\ G(B) \arrow[r, "\eta_B"'] & F(B) \end{tikzcd} \qquad \begin{tikzcd} F(B) \arrow[r, "\eta_B"] \arrow[d, "F(f)"'] & G(B) \arrow[d, "G(f)"] \\ G(A) \arrow[r, "\eta_A"'] & F(A) \end{tikzcd}$$

where the diagram on the right is for the case in which F and G are contravariant. When η_A is an isomorphism for each $A \in Ob(\mathcal{A})$, η is said to be a **natural equivalence**. *Natural transformations can be composed, as follows : if $F \xrightarrow{\eta} G$ and $G \xrightarrow{\mu} T$ are natural transformations, then $\pi : F \longrightarrow T$, given by*

$$\pi = \{\mu_A \circ \eta_A : A \in Ob(\mathcal{A})\},$$

is the composition $\mu \circ \eta$. Write $[F, G]$ for the class of natural transformations from F to G.

Definition 16.18. *Let $\mathcal{A}$ be a category. A covariant functor $F : \mathcal{A} \longrightarrow$* **Set** *is* **representable** *if there is $A \in Ob(\mathcal{A})$ such that F is naturally equivalent to h_A.*

A fundamental result concerning representable functors is :

Theorem 16.19. (The Yoneda Lemma) *Let $F : \mathcal{A} \longrightarrow$* **Set** *be a functor, where $\mathcal{A}$ is a category. Then,*
a) For any object A in $\mathcal{A}$, there is a bijection

$$\eta_{A,F} : [h_A, F] \longrightarrow F(A), \quad \eta_{AF}(\eta) = \eta_A(Id_A),$$

which is natural in F and A.
b) If A and B are objects in $\mathcal{A}$, the function

$$\eta_{A,C} : [h_A, h_C] \longrightarrow [C, A], \quad \theta_{A,C}(\eta) = \eta_A(Id_A),$$

is a bijection, which is natural in both A and C.

For a proof, see Lemma in pg. 61 of [**44**] or Lemma 2.1 and Corollary 2.2, pg. 97ff, in [**53**].

3. Adjoint functors and Equivalence of Categories

Definition 16.20. *Notation as* 16.16, *two functors $\mathcal{A} \xrightarrow{F} \mathcal{B}$ and $\mathcal{B} \xrightarrow{G} \mathcal{A}$ are an* **adjoint pair**, *(F, G), if there is a natural equivalence of set-valued bifunctors, $\eta : {}_F h \longrightarrow h_G$. Thus, for each object $\langle A, B \rangle \in Ob(\mathcal{A}^{op} \times \mathcal{B})$, we have a natural bijective correspondence*

[1] I.e., both covariant or both contravariant.

$$\eta_{AB} : [F(A),\, B] \longrightarrow [A,\, G(B)].$$

We say that F is **left adjoint** *to G and G is* **right adjoint** *to F.*

EXAMPLE 16.21. Notation as in 14.7, let $\mathbb{K}$ be an equational class in $\boldsymbol{Alg_\tau}$, considered as a category whose morphisms are the maps preserving the operations in τ. Let $G : \mathbb{K} \longrightarrow \mathbf{Set}$ be the forgetful functor. For each set S, let $F(S)$ be the free algebra in $\mathbb{K}$, generated by S (see Theorem A in 14.7) and let $\iota_S : S \longrightarrow F(S)$ be the canonical injection. We make F into a functor from **Set** to $\mathbb{K}$, by stipulating its value on set-theoretical maps, $S \xrightarrow{f} T$, as follows :

$F(f)$ is the unique morphism in $\mathbb{K}$ such that $\iota_T \circ F = F(f) \circ \iota_S$.

$$\begin{tikzcd} S \arrow[r, "\iota_S"] \arrow[d, "f"'] & F(S) \arrow[d, "F(f)"] \\ T \arrow[r, "\iota_T"'] & F(T) \end{tikzcd}$$

The universal property of free objects guarantees the existence and uniqueness of $F(f)$. We now sketch a proof that F is left adjoint to G. If S is a set and A is an algebra in $\mathbb{K}$, define

$$\eta_{SA} : [F(S),\, A] \longrightarrow [S,\, G(A)] \text{ by } \quad h \mapsto h \circ \iota_S,$$

where this last composition is a map in **Set**. The universal property of free objects implies that η_{SA} is a bijection. Clearly, $\eta = \{\eta_{SA}\}$ is a natural transformation, and so the free algebra construction is left adjoint to the forgetful functor from $\mathbb{K}$ to **Set**. $\square$

We shall later state the Adjoint Functor Theorem (16.35), giving a necessary and sufficient condition for a functor to have a left adjoint. But 16.21 can be profitably generalized :

THEOREM 16.22. *Let $\mathcal{B} \xrightarrow{G} \mathcal{A}$ be a covariant functor. The following are equivalent :*

(1) *G has a left adjoint.*

(2) *For all $A \in Ob(\mathcal{A})$, there is $F(A)$ in $Ob(\mathcal{B})$ and a morphism $\iota_A : A \longrightarrow G(F(A))$ in $\mathcal{A}$, satisfying the following property :*

For all $B \in Ob(\mathcal{B})$ and morphisms $A \xrightarrow{f} G(B)$ in $\mathcal{A}$, there is a **unique** *morphism $f_* : F(A) \longrightarrow B$ in $\mathcal{B}$, making the following diagram commutative :*

$$\begin{tikzcd} A \arrow[rr, "\iota_A"] \arrow[dr, "f"'] & & G(F(A)) \arrow[dl, "G(f_*)"] \\ & G(B) & \end{tikzcd}$$

PROOF. Generalize the argument in Example 16.21. □

DEFINITION 16.23. *Let $\mathcal{A}$ be a subcategory of $\mathcal{B}$. $\mathcal{A}$ is a* **reflective** *subcategory of $\mathcal{B}$ if the natural embedding $\mathcal{A} \longrightarrow \mathcal{B}$ has a left adjoint R, called the* **reflection functor** *from $\mathcal{B}$ to $\mathcal{A}$. $\mathcal{A}$ is a* **coreflective** *subcategory of $\mathcal{B}$ if the natural embedding $\mathcal{A} \longrightarrow \mathcal{B}$ has a right adjoint, called the* **coreflection** *of $\mathcal{B}$ in $\mathcal{A}$.*

EXAMPLE 16.24. a) The following are full reflective categories of **Top** [2] :
* The category of T_i spaces, $i = 1, 2, 3$ (1.20);
* The category of completely regular spaces;
* the category of totally disconnected spaces;

b) The category of normal spaces (1.20) is a reflective subcategory of **Htop**.
c) The full subcategory of torsion-free groups is reflective in **Ab**, the category of Abelian groups.
d) The full subcategory **Ab** of Abelian groups is reflective in the category of groups, **Gr**.
e) The full subcategory of torsion groups in **Ab** is an example of a coreflective subcategory. □

DEFINITION 16.25. *Two categories, $\mathcal{A}$ and $\mathcal{B}$, are* **equivalent** *if there are functors, $F : \mathcal{A} \longrightarrow \mathcal{B}$ and $G : \mathcal{B} \longrightarrow \mathcal{A}$, together with natural equivalences*

$$\eta : (G \circ F) \longrightarrow Id_{\mathcal{A}} \quad and \quad \mu : (F \circ G) \longrightarrow Id_{\mathcal{B}}.$$

The pair (F, G) is called an equivalence between $\mathcal{A}$ and $\mathcal{B}$. A contravariant equivalence is called a **duality**.

An example of duality is presented in 12.17 : that between **Sob** and **Framep**. We shall see other important examples in the chapters ahead.

4. Diagrams and Limits

DEFINITION 16.26. *Let $\mathcal{A}$ be a category and let $\mathcal{D}$ be a* **small** *category. A* **$\mathcal{D}$-diagram** *in $\mathcal{A}$ is a functor $D : \mathcal{D} \longrightarrow \mathcal{A}$.*

If $\mathcal{B}$ is a subcategory of $\mathcal{D}$, we write $D_{|\mathcal{B}}$ for the $\mathcal{B}$-diagram obtained by applying D just to the objects and arrows in $\mathcal{B}$.

The notion of diagram is functorial, that is, if $F : \mathcal{A} \longrightarrow \mathcal{B}$ is a functor and D is a $\mathcal{D}$-diagram in $\mathcal{A}$, then $F \circ D$ is a $\mathcal{D}$-diagram in $\mathcal{B}$.

EXAMPLE 16.27. Let $\mathcal{A}$ be a category.
(1) Let $\mathcal{D}$ be the category consisting of two objects (1) and (2), such that [(1), (2)] has two distinct elements. A $\mathcal{D}$-diagram in $\mathcal{A}$ consists of parallel arrows in $\mathcal{A}$,

$$A \underset{f}{\overset{g}{\rightrightarrows}} B\ ,$$

with A, $B \in Ob(\mathcal{A})$ and f, $g \in [A, B]$.

[2] The category of topological spaces and continuous maps.

(2) Let $\mathcal{D}$ be the category whose objects are a set I, and whose class of morphisms contains only the identity morphism of each object. A $\mathcal{D}$-diagram in $\mathcal{A}$, is simply a family of objects in $\mathcal{A}$, indexed by I.
(3) Let $\langle I, \leq \rangle$ be a poset, considered as a category as in Example 16.4. An I-diagram in $\mathcal{A}$ is a family of objects in $\mathcal{A}$, $\{A_i : i \in I\}$, together with morphisms, $f_{ij} \in [A_i, A_j]$, **whenever $i \leq j$**, such that for all $i, j, k \in I$
$*$ $f_{ii} = Id_{A_i}$; $*$ If $i \leq j \leq k$, the following diagram is commutative :

$$\begin{tikzcd} A_i \arrow[r, "f_{ij}"] \arrow[d, "f_{ik}"'] & A_j \arrow[dl, "f_{jk}"] \\ A_k & \end{tikzcd}$$

If I is up-directed (2.26(ii)), an I-diagram is called an **inductive system** in $\mathcal{A}$, while a I^{op}-diagram in $\mathcal{A}$ is a **projective system** in $\mathcal{A}$. As special cases of (3) :
a) Suppose $I = \{i, j, k\}$, with $i \leq j, k$, while j and k are unrelated. The following diagrams exhibit an I-diagram and an I^{op}-diagram in $\mathcal{A}$, respectively :

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \\ C & \end{tikzcd} \qquad \begin{tikzcd} & B \arrow[d, "f"] \\ C \arrow[r, "g"'] & A \end{tikzcd}$$

b) Suppose I is the BA with four elements, $\{\perp, a, \neg\, a, \top\}$, considered as a category. A I-diagram in $\mathcal{A}$ is a commutative covariant square in $\mathcal{A}$, that is, a diagram of the form

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[d, "k"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

with $k \circ f = g \circ h$. $\square$

DEFINITION 16.28. *Let $D : \mathcal{D} \longrightarrow \mathcal{A}$ be a diagram in $\mathcal{A}$. A* **cone** *C over D is a family $\{M \xrightarrow{f_d} D(d) : d \in Ob(\mathcal{D})\}$, where M is an object in $\mathcal{A}$ and f_d are morphisms in $\mathcal{A}$, such that if $d \xrightarrow{\alpha} d'$ is a morphism in $\mathcal{D}$, then the diagram on the left is commutative :*

$$\begin{tikzcd} M \arrow[r, "f_d"] \arrow[d, "f_{d'}"'] & D(d) \arrow[dl, "D(\alpha)"] \\ D(d') & \end{tikzcd} \qquad \begin{tikzcd} M \arrow[rr, "h"] \arrow[dr, "g_d"'] & & N \arrow[dl, "f_d"] \\ & D(d) & \end{tikzcd}$$

If $C = \langle M, \{f_d\}_{d \in Ob(\mathcal{D})} \rangle$ *and* $K = \langle N, \{g_d\}_{d \in Ob(\mathcal{D})} \rangle$ *are cones over* D*, a* **cone morphism***,* $C \xrightarrow{h} K$*, is a morphism* $M \xrightarrow{h} N$ *in* $\mathcal{A}$*, such that for all* $d \in Ob(\mathcal{D})$*, the diagram above right is commutative.*

It is straightforward that the identity and composition of cone morphisms are cone morphisms and so we have a category, $\boldsymbol{Co(D)}$, of cones over the diagram D. Note that if C is a cone over D and $\mathcal{B}$ is a subcategory of $\mathcal{D}$, we obtain a cone over $D_{|\mathcal{B}}$, by taking the values of D only on the objects and morphism in $\mathcal{B}$.

A **dual cone** over a diagram D in $\mathcal{A}$, is an object M in $\mathcal{A}$, together with a family of morphisms in $\mathcal{A}$, $\{D(d) \xrightarrow{f_d} M : d \in Ob(\mathcal{D})\}$, such that for all morphisms $d \xrightarrow{\alpha} d'$ in $\mathcal{D}$, the diagram below commutes :

$$\begin{tikzcd} D(d) \arrow[r, "D(\alpha)"] \arrow[d, "f_d"'] & D(d') \arrow[dl, "f_{d'}"] \\ M & \end{tikzcd}$$

The reader will have no difficulty in defining morphisms of dual cones and verifying that dual cones over D form a category, written $\boldsymbol{dCo(D)}$. The comment about restrictions to subcategories in the end of the preceding paragraph also holds for dual cones.

The notions of cone and dual cone over a $\mathcal{D}$-diagram D are functorial, that is, if C is a cone or dual cone over D and $F : \mathcal{A} \longrightarrow \mathcal{B}$ is a functor, then applying F to the objects and morphisms in C yields a cone or dual cone over the diagram $(F \circ D)$ in $\mathcal{B}$.

Definition 16.29. *Let* D *be a* $\mathcal{D}$*-diagram in a category* $\mathcal{A}$*. A* **limit** *for* D *in* $\mathcal{A}$ *is a final object in the category* $\boldsymbol{Co(D)}$*. Write* $\varprojlim D$ *for the limit of* D *in* $\mathcal{A}$*.*

A **colimit** *for* D *in* $\mathcal{A}$ *is an initial object in the category* $\boldsymbol{dCo(D)}$*. Write* $\varinjlim D$ *for the colimit of* D *in* $\mathcal{A}$*.*

Clearly, limits and colimits are unique, up to isomorphism. If D *has a limit in* $\mathcal{A}$*, it is common usage to write*

$$\lim_{\leftarrow} D \;=\; (A;\ \{f_d\ :\ d \in Ob(\mathcal{D})\}) \quad or \quad A \;=\; \lim_{\leftarrow} D,$$

where $A \in Ob(\mathcal{A})$ is the vertex of the cone corresponding to $\lim\limits_{\leftarrow} D$. Similar usage applies to colimits.

A functor $F : \mathcal{A} \longrightarrow B$ **preserves limits** *if for all $\mathcal{D}$-diagrams D in $\mathcal{A}$, if $\lim\limits_{\leftarrow} D$ exists in $\mathcal{A}$, then $\lim\limits_{\leftarrow} (F \circ D)$ exists in $\mathcal{B}$ and we have*

$$F(\lim_{\leftarrow} D) \;=\; \lim_{\leftarrow} (F \circ D).$$

Similarly, one defines the preservation of colimits by a functor.

A category is **complete** *if it has limits for all $\mathcal{D}$-diagrams in $\mathcal{A}$; it is* **cocomplete** *if it has colimits for all $\mathcal{D}$-diagrams in $\mathcal{A}$. The finite analogue of completeness is* **finitely complete***, that is, has limits for all $\mathcal{D}$-diagrams, where $\mathcal{D}$ is a finite category. Similarly, one defines* **finite cocompleteness***.*

Example 16.30. a) If $\mathcal{D}$ is the category in item (1) of 16.27, a limit for a $\mathcal{D}$-diagram in a category $\mathcal{A}$ is the [3] **equalizer** of the pair

$$\begin{tikzcd} A \arrow[r, shift left, "g"] \arrow[r, shift right, "f"'] & B \end{tikzcd}$$

A colimit for this diagram is the **coequalizer** of the pair (f, g) in $\mathcal{A}$.
b) If $\mathcal{D}$ is the category associated to a discretely ordered set as in item (2) of 16.27, a limit for a $\mathcal{D}$-diagram in $\mathcal{A}$ is called the **product** of the family $\{A_i : i \in I\}$ in $\mathcal{A}$, while a colimit for this diagram is called a **coproduct** of that family in $\mathcal{A}$.
c) If I is an up-directed poset, a colimit for a I-diagram in $\mathcal{A}$ is called an **inductive limit** of the inductive system of objects and morphism determined by that I-diagram.

A limit for a I^{op}-diagram in $\mathcal{A}$ is called a **projective limit** for the projective system of objects and morphism determined by the I^{op}-diagram.
d) If I is the poset of item 4.(b) in 16.27, a limit for an I-diagram in $\mathcal{A}$ is the **pushout** of its morphisms. A colimit for a I^{op}-diagram in $\mathcal{A}$ is the **pullback or fiber product** of its morphisms. $\square$

The following result, due to P. Freyd, is useful in establishing completeness and cocompleteness of a category :

Theorem 16.31. *A category is complete iff it has final object, equalizers and products. A category is cocomplete iff it has initial object, coequalizers and coproducts. Similar statements hold for finite completeness and cocompleteness.*

One of the most important properties of adjoints is their preservation of limits or colimits :

Theorem 16.32. *If F is a left adjoint, then F preserves all colimits that exist in its domain. Dually, a right adjoint preserves all limits that exist in its domain.*

Proof. See Theorem V.5.1, p. 114 in [**44**]. $\square$

[3] The definite article is justified by uniqueness of limits, up to isomorphism.

For complete categories we have characterizations of adjoint and representable functors. To register these important applications of preservation of limits, we need

Definition 16.33. *Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a functor and $B \in Ob(\mathcal{B})$. A set S_B of objects in $\mathcal{A}$ is a* **solution set** *of B with respect to F, if for each $A \in \mathcal{A}$ and each morphism $B \xrightarrow{f} F(A)$, there is $C \in S_B$, together with morphisms $C \xrightarrow{g} A$ and $B \xrightarrow{h} F(C)$, such that the following diagram is commutative :*

$$\begin{tikzcd} B \arrow[r, "h"] \arrow[d, "f"'] & F(C) \arrow[dl, "F(g)"] \\ F(A) & \end{tikzcd}$$

F has solution sets, if every $B \in Ob(\mathcal{B})$ has solution sets with respect to F.

Theorem 16.34. (Representable Functor Theorem) *Let $\mathcal{A}$ be a complete non-empty category. A functor $F : \mathcal{A} \longrightarrow$* **Set** *is representable iff F preserves limits and $\{\emptyset\}$ has a solution set with respect to F.*

Theorem 16.35. (Adjoint Functor Theorem) *Let $\mathcal{B}$ be a complete, non-empty category. Let $G : \mathcal{B} \longrightarrow \mathcal{A}$ be a covariant functor. Then, G has a left adjoint iff G preserves limits and has solution sets.*

5. Injective and Projective Objects

Definition 16.36. *Let $\mathcal{C}$ be a category and A, $C \in Ob(\mathcal{C})$.*

a) ***C* is projective in *$\mathcal{C}$*** *iff for all A, B in $Ob(\mathcal{C})$, all* **epic** *$A \xrightarrow{f} B$ and all morphisms $C \xrightarrow{g} B$, there is $C \xrightarrow{h} A$, such that $f \circ h = g$.*

$$\begin{tikzcd} & C \arrow[dl, "h"'] \arrow[d, "g"] \\ A \arrow[r, "f"'] & B \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "g"] \arrow[d, "\pi"'] & B \arrow[dl, "f"] \\ A & \end{tikzcd} \quad \text{(cover)}$$

b) ***C* is a projective cover for *A*** *if C is a projective object and there is an epimorphism $C \xrightarrow{\pi} A$, such that for all* **projective** *B in $\mathcal{C}$ and all epimorphisms $B \xrightarrow{f} A$, there is an epimorphism $C \xrightarrow{g} B$ such that $f \circ g = \pi$.*
$\mathcal{C}$ has **enough projectives** *iff for every object A in $\mathcal{C}$, there is a projective C and an epimorphism $C \xrightarrow{f} A$.*

c) $\boldsymbol{C}$ **is injective in** $\boldsymbol{\mathcal{C}}$ *iff for all* A, B *in* $Ob(\mathcal{C})$, *all* **monic** $A \xrightarrow{f} B$ *in* $\mathcal{C}$*and all morphisms* $A \xrightarrow{g} C$, *there is a morphism* $B \xrightarrow{h} C$, *such that* $h \circ f = g$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[dl, "h"] \\ C & \end{tikzcd} \qquad\qquad \begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & C \arrow[dl, "g"] \\ B & \end{tikzcd} \quad \text{(hull)}$$

d) C **is an injective hull of** A *if* C *is an injective object and there is a monomorphism* $A \xrightarrow{i} C$ *such that for all* **injective** B *in* $\mathcal{C}$ *and monomorphisms* $A \xrightarrow{f} B$, *there is a monomorphism* $C \xrightarrow{g} B$, *such that* $g \circ i = f$.
The definition of having **enough injectives** *is dual to that of having enough projectives.*

We have seen that in the category of HAs (or even distributive lattices) monomorphisms are simply injective morphisms and so the above definition of injective corresponds to the conditions [BA - inj] and [HA - inj] in Corollary 15.3. As mentioned earlier, the content of Theorem 15.2 is that complete Boolean algebras are injective objects in the category of BAs. Further, by Corollary 15.4, the completion of a BA is its injective hull.

Exercises

16.37. In a category $\mathcal{A}$
(1) Every retraction is epic and every coretraction is monic.
(2) If $f \circ g$ is monic, so is g. (3) If $f \circ g$ is epic, so is f. $\square$

16.38. a) Let $A \xrightarrow{f} B$ be a map in **Set**. Then,
(1) f is monic iff it is injective. (2) f is epic iff it is surjective.
(3) The statement "f is a retraction iff it is surjective" is equivalent to the Axiom of Choice.
b) In a set-based category (16.1), any injective morphism is monic and any surjective morphism is epic. $\square$

16.39. Let X be a topological space and write Δ for the diagonal of the product $X \times X$. A binary relation E on X is **closed**, if E is a closed subset of $X \times X$, with the product topology.
a) Show that X is Hausdorff iff Δ is closed in $X \times X$.
b) Let $f, g : X \longrightarrow Y$ be continuous maps, with Y Hausdorff. Then, $f = g$ iff they coincide in some dense set in X.
c) If X is Hausdorff and E is a closed equivalence relation on X, then X/E, with the quotient topology, is a Hausdorff space. Recall that the *quotient topology* by a surjective map, $X \xrightarrow{h} Y$, is defined by :

$$U \subseteq Y \text{ is open iff } h^{-1}(U) \text{ is open in } X.$$

Hence, it is clear that the canonical projection $\pi_E : Y \longrightarrow Y/E$ is continuous.
d) Suppose X is Hausdorff and let F be a closed set in X. Let

$$Y = X \times \{0\} \cup X \times \{1\}$$

be a disjoint union (1.5) of two copies X, with direct sum topology, that is, $U \subseteq Y$ is open iff its intersection with each copy of X is open. Define a binary relation E on Y, by the rule :

$$\langle x, i \rangle \; E \; \langle z, j \rangle \quad \text{iff} \quad \begin{cases} x = z \text{ and } i = j \\ \quad\quad \textit{or} \\ x = z \in F \text{ and } i \neq j \end{cases}$$

Show that E is a closed equivalence relation on Y and that Y/E is a Hausdorff space. $\square$

16.40. Let **HTop** be the category of Hausdorff spaces and continuous maps.
a) A morphism in **HTop** is monic iff it is injective.
b) A morphism in **HTop** is epic iff its image is dense in its codomain. *(Hint : 16.39).*
c) The embedding of the rationals into the reals is a monic and an epic in **HTop**, which is not an isomorphism. $\square$

16.41. Let $\mathcal{A}$ be a category and let A be an object in $\mathcal{A}$. Notation is as in Example 16.6. For monics $(\beta : B \longrightarrow A)$ and $(\gamma : C \longrightarrow A)$, define

$$f \leq g \quad \text{iff} \quad \exists\, \eta \in [B, C]_{\mathcal{A}}, \text{ such that } \beta = \gamma \circ \eta.$$

Show that, up to isomorphism in $\mathcal{A}^A$, $\leq$ is a partial order in the class of subobjects of A in $\mathcal{A}$. $\square$

16.42. State and prove an analogue of Theorem 16.22 for the existence of right adjoints. $\square$

The next two exercises describe more applications of Theorem 16.22.

16.43. Let G be a group with identity e, written multiplicatively. Let

$$\mathbb{Z}[G] = \{f : G \longrightarrow \mathbb{Z} : \{g \in G : f(g) \neq 0\} \text{ is finite}\}.$$

Define an operation of addition and multiplication for $f, h \in \mathbb{Z}[G]$ by the following rules, where $g \in G$:

$$\begin{cases} [f \; + \; h](g) \; = \; f(g) \; + h(g); \\ [f \; \cdot \; h](g) \; = \; \sum_{g = g'g''} \; f(g')h(g''), \end{cases}$$

where the sum in the last equation is taken over all pairs $\langle g', g'' \rangle$ with $g'g'' = g$; note that because f and h are zero almost everywhere, the sum over these pairs has at most a finite number of non-zero terms. We write $\underline{0} \in Zh[G]$ for the constant function with value 0 ($\in \mathbb{Z}$). For $g \in G$, define $\breve{g} : G \longrightarrow \mathbb{Z}$ by

$$\breve{g}(g') = \begin{cases} 1 & \text{if } g' = g \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\breve{g} \in \mathbb{Z}[G]$, for all $g \in G$.
a) $\mathbb{Z}[G]$ is a ring, with additive neutral $\underline{0}$ and multiplicative identity $\breve{e}$.

b) The map $g \in G \mapsto \check{g} \in \mathbb{Z}[G]$ is a group homomorphism.
c) The construction outlined above $-$ called the **group ring** construction $-$, is the left adjoint to the forgetful functor from the category of rings to the category of groups. $\square$

16.44. Let X be a completely regular space (1.20). Let

$$\gamma(X) = \{X \xrightarrow{f} [0,1] : f \text{ is continuous}\},$$

where $[0,1]$ is the real unit interval. Define

$$\beta : X \longrightarrow [0,1]^{\gamma(X)}, \text{ by } \beta(x) = \langle\, f(x)\,\rangle_{f\in\gamma(X)}.$$

a) Prove that β is a continuous injective map.

Let $\beta X = \overline{Image\ \beta}$ in $[0,1]^{\gamma(X)}$; clearly, βX is a compact Hausdorff space and we have a continuous injection $X \xrightarrow{\beta} \beta X$.
b) Let A be a set. Any continuous map $h : X \longrightarrow [0,1]^A$ has a **unique** extension h_* to βX along β, that is, the following diagram is commutative :

$$\begin{tikzcd} X \arrow[rr, "\beta"] \arrow[dr, "h"'] & & \beta X \arrow[dl, "h_*"] \\ & {[0,1]^A} & \end{tikzcd}$$

c) If K is a compact Hausdorff space, then any continuous map $X \xrightarrow{h} K$ has a **unique** extension to βX along β.
d) The construction outlined above $-$ called the **Stone-Čech compactification** of X $-$, is the left adjoint to the forgetful functor from the category of compact Hausdorff spaces to the category of completely regular spaces. $\square$

In the exercises that follow, we ask the reader to explore the category of partially ordered sets, **Po**, determining some of the constructions that exist therein.

16.45. a) When is a morphism in **Po** a monomorphism? An epimorphism? An isomorphism? Is isomorphism the same as monomorphism and epimorphism?
b) Are there free posets? $\square$

16.46. a) Let $\{X_i\}$ be a family of posets and $\prod X_i$ be their product. The natural projections $\pi_i : \prod X_i \longrightarrow X_i$ are increasing maps. Verify that $(\prod X_i,\ \{\pi_i\})$ is the product of the X_i in **Po**. Notice that if $I = \emptyset$, one obtains a final object for **Po**, namely $\{\emptyset\}$.
b) What about coproducts and initial objects?
c) What about equalizers and coequalizers ?
d) What about general limits and colimits? $\square$

CHAPTER 17

Limits and Colimits of First-Order Structures

Since most of our applications of limits and colimits are for diagrams of first-order structures over a poset, we include some basic results for this case. We assume that the reader is familiar with the fundamental notions of Logic and Model Theory, in particular of the concept of satisfaction. Classic references are [**38**], [**39**], [**68**]; [**71**] has an interesting approach to first-order logic. It should be said that Theorems 17.10, 17.15 and Corollary 17.18 are part of the folklore of Model Theory.

1. First-Order Languages and Logic

We recall the construction of a first-order language L with equality.

17.1. **Alphabet of *L*.** $*$ A set $\{v_n : n \in \mathbb{N}\}$ of variable symbols;
$*$ The logical symbols $\wedge$ (and), $\vee$ (or), $\rightarrow$ (implies) and $\neg$ (negation);
$*$ The quantifiers $\exists$ (existential) and $\forall$ (universal);
$*$ A binary relation symbol $=$, for equality;
$*$ For each integer $n \geq 1$,
(*i*) A set $rel(n, L)$ of n-ary relation symbols;
(*ii*) A set $op(n, L)$ of n-ary operation symbols;
(*iii*) A set $Ct(L)$ of constants symbols. $\square$

17.2. **Terms.** Defined by induction on complexity, as follows :
$*$ Variables and constant are terms;
$*$ If $n \geq 1$ is an integer, $t_1, \ldots, t_n$ are terms in L and $\omega \in op(n)$, then $\omega(t_1, \ldots, t_n)$ is a term. $\square$

17.3. **Formulas.** Defined by induction on complexity, where $n \geq 1$ is an integer :
$*$ If $R \in rel(n)$ and $t_1, \ldots, t_n$ are terms in L, $R(t_1, \ldots, t_n)$ is a formula, called an **atomic** formula;
$*$ If ϕ, ψ are formulas in L and $\diamond$ is one of the logical symbols $\wedge$, $\vee$ or $\rightarrow$, then $(\phi \diamond \psi)$ is a formula.
$*$ If ϕ is a formula, then $\neg\,\phi$ (the negation of ϕ) is a formula;
$*$ If ϕ is a formula and v_n is a variable, then $\forall\, v_n\, \phi$ and $\exists\, v_n\, \phi$ are formulas. $\square$

17.4. **Basic Notions.** a) To every formula ϕ in L is associated a sequence of formulas, of lesser complexity, that describe the process of constructing ϕ from

the atomic formulas. The formulas in this sequence are called **subformulas** of ϕ. The occurrences of a variable v_n in ϕ are divided into two categories :
(1) **bound**, when v_n occurs in a subformula of ϕ of the type $Q\, v_n\, \psi$, where $Q = \exists, \forall$;
(2) **free**, when the occurrence is not bound.
b) (Substitution) If ϕ is a formula, v_n is a variable and τ is a term, write $\phi(\ulcorner \tau \mid v_n \urcorner)$ for the formula obtained by substituting *all free* occurrences of v_n in ϕ by τ.
c) If t is a term and ϕ is a formula in L, write $t(v_1, \ldots, v_n)$, $\phi(v_1, \ldots, v_n)$ to mean that the free variables in t and ϕ are among the ones being displayed. As usual, write $\overline{v}$ for the sequence $\langle\, v_1, \ldots, v_n\,\rangle$; we may also use, as is standard, x, $\overline{x}$, y, $\overline{y}$ and z, $\overline{z}$ to name variables.
d) A term t **is free for v_n in a formula ϕ** iff in $\phi(\ulcorner t \mid v_n \urcorner)$ (see (b) above) no variable in t becomes bound.
e) A formula ϕ in L is
$*$ A **sentence** if it has no free variables. Let $Sent(L)$ be the set of sentences in L;
$*$ **quantifier-free** if there is no quantifier occurs in ϕ;
$*$ **positive** if implication and negation do not occur in ϕ;
$*$ **existential** if it is of the form $\exists\, \overline{x}\, \psi$, where ψ is quantifier-free;
$*$ **universal** if it is of the form $\forall\, \overline{x}\, \psi$, where ψ is quantifier-free;
$*$ **primitive** if it is of the form $\exists\, \overline{x}\, \phi$, where ϕ is a conjunction of atomic formulas;

$*$ $\forall\,\exists$ if is of the form $\forall\, \overline{x}\, \exists\, \overline{y}\, \psi$, where ψ is quantifier-free;
$*$ **Horn** if it is of the form $\forall\, \overline{x}\ \ [(\psi_1 \wedge \ldots \wedge \psi_n) \to \psi]$, where ψ, $\psi_1, \ldots, \psi_n$ are atomic formulas.
The reader should have no trouble in recognizing the content of expressions like "positive-existential" or "positive $\forall\exists$". $\square$

17.5. **The Intuitionistic Predicate Calculus, $\mathcal{H}$.** The following is a Hilbert style [1] formalization of the Intuitionistic Predicate Calculus[2], written $\mathcal{H}$:

If ϕ, ψ and χ are formulas in L, v is a variable and τ is a term :
1. $\phi \to (\psi \to \phi)$;
2. $(\phi \to \psi) \to ((\phi \to (\psi \to \chi)) \to (\phi \to \chi))$;
3. $\phi \to (\psi \to \phi \wedge \psi)$;
4. $\phi \wedge \psi \to \phi$;
5. $\phi \wedge \psi \to \psi$;
6. $\phi \to (\phi \vee \psi)$;
7. $\psi \to (\phi \vee \psi)$;
8. $(\phi \to \chi) \to ((\psi \to \chi) \to (\phi \vee \psi \to \chi))$;
9. $(\phi \to \psi) \to ((\phi \to \neg\, \psi) \to \neg\, \phi)$;
10. $\neg\, \phi \to (\phi \to \psi)$.

[1] For Gentzen style formalizations, see [**58**], [**39**], [**38**].
[2] Due to A. Heyting.

11. If τ is a term free for v in ϕ [3], $\begin{cases} \text{11.a. } \forall v\, \phi \;\; \to \;\; \phi(\ulcorner \tau \mid v \urcorner); \\ \text{11.b. } \phi(\ulcorner \tau \mid v \urcorner) \;\; \to \;\; \exists v\, \phi, \end{cases}$

where $\phi(\ulcorner \tau \mid v \urcorner)$ denotes substitution of τ for v in ϕ, as in 17.4.(b).

12. **Deduction rules** :

$$\text{Modus Ponens} : \frac{\phi,\, \phi \to \psi}{\psi} \qquad \begin{cases} \forall\text{-rule} & : \dfrac{\phi \to \psi(v)}{\phi \to \forall v\, \psi(v)} \\ \exists\text{-rule} & : \dfrac{\psi(v) \to \phi}{\exists v\, \psi(v) \to \phi,} \end{cases}$$

where in the $\forall$-rule and the $\exists$-rule v must not occur free in ϕ.
The axioms for equality are the usual ones, including the Leibniz substitution rule [L] : If τ is a term in L, free for a variable v in ϕ, then [4]

$$\phi(v) \wedge (v = \tau) \;\; \to \;\; \phi(\ulcorner \tau \mid v \urcorner).$$

The first ten schemata, together with **Modus Ponens** formalize the Intuitionistic Propositional Calculus. To obtain the Classical Calculus, add (or replace axiom 10 by) the rule

$$10^C.\ \neg\neg\, \phi \to \phi.$$

where ϕ is any formula in L. In [**65**] one will find a different and interesting proposal for the formalization of the Intuitionistic Predicate Calculus.

If $\Gamma \cup \{\phi\}$ is a set of formulas in L, a **proof** of ϕ from Γ consists of a sequence of formulas $\psi_1, \ldots, \psi_n$ in L, such that ψ_n is ϕ and satisfying, for $1 \leq k \leq n$,
$*\ \psi_k \in \Gamma$ *or* $*\ \psi_k$ is an axiom *or*
$*\ \psi_k$ comes from earlier formulas in the sequence, through the use of one of the deduction rules.

We write

$$\begin{cases} \Gamma \vdash_{\mathcal{H}} \phi & \text{If } \phi \text{ is a consequence of } \Gamma \text{ in } \mathcal{H}; \\ \Gamma \vdash_{C} \phi & \text{If } \phi \text{ is a consequence of } \Gamma \text{ in the Classical Calculus.} \end{cases}$$

It is clear that $\Gamma \vdash_{\mathcal{H}} \phi \;\; \Rightarrow \;\; \Gamma \vdash_C \phi$.

A **theorem** of $\mathcal{H}$ is a formula ϕ such that $\emptyset \vdash_{\mathcal{H}} \phi$, written $\vdash_{\mathcal{H}} \phi$.

A set of *sentences* Σ is a **theory** in $\mathcal{H}$, if it is closed under deduction that is

$$\Sigma \vdash_{\mathcal{H}} \sigma \;\; \Rightarrow \;\; \sigma \in \Sigma.$$

Similarly, one defines the notion of *theory* in the classical Calculus. □

On the basic results about these systems is

Proposition 17.6. (Deduction Theorem) *Let $\Gamma \cup \{\sigma,\, \phi\}$ be a set of formulas in L. If σ is a sentence in L, then*

$$\Gamma,\, \sigma \vdash_{\mathcal{H}} \phi \quad \textit{iff} \quad \Gamma \vdash_{\mathcal{H}} (\sigma \to \phi).$$

A similar statement holds for the relation $\vdash_C$.

One of the basic discoveries of Lindenbaum and Tarski is that the syntax of formal theories generates interesting algebraic structures. If ϕ, ψ are formulas in L, define a relation

[3] As in 17.4.(d).
[4] It is enough to assume this just for function and relation symbols in L.

$$\phi \equiv \psi \quad \text{iff} \quad \vdash_{\mathcal{H}} (\phi \to \psi) \wedge (\psi \to \phi).$$

This is an equivalence relation on L and the set of its equivalence classes is the **Lindenbaum algebra** of L. For intuitionistic systems $-$ propositional or first-order $-$, the Lindenbaum algebra of L is Heyting algebra. For classical systems, it is a Boolean algebra. The propositional case is important, in particular because their Lindenbaum algebras furnish the **free objects** in the categories of Heyting and Boolean algebras. For more information on this topic, see [**59**] and [**60**].

2. First-order Structures and their Morphisms

Since we shall be constantly using sequences, all the notational conventions in 1.4 will be of constant use.

Let L be a first-order language with equality as in the previous section. A ***L*-structure** is a set M, together with

$*$ For $R \in rel(n, L)$ a subset $R^M \subseteq M^n$, called the interpretation of R in M;

$*$ For $\omega \in op(n, L)$, a map $\omega^M : M^n \longrightarrow M$, the interpretation of ω in M;

$*$ For $c \in Ct(L)$, an element $c^M \in M$, the interpretation of c in M;

$*$ The equality symbol is interpreted as the diagonal of the M^2.

By induction on complexity, a term $t(v_1, \ldots, v_n)$, in the free variables $v_1, \ldots, v_n$, induces a map, $t^M : M^n \longrightarrow M$, called the *interpretation* of t in M.

We shall assume the reader familiar with the concept of interpretation of a formula $\phi(v_1, \ldots, v_n)$ in M (see [**7**]) and the symbol

$$M \models \phi[\overline{a}],$$

read $\boldsymbol{\phi}$ **holds in** $\boldsymbol{M}$ **at** $\boldsymbol{\overline{a} \in M^n}$.

The main notions of morphism of L-structures are recalled in

Definition 17.7. *Let L be a first-order language with equality and let M, N be structures for L. Let $f : M \longrightarrow N$ be a map.*

*a) f is a **L**-morphism if for all integers $n \geq 1$*

(i) If $c \in Ct$, then $f(c^M) = c^N$.

(ii) If $\omega \in op(n)$ and $\overline{x} \in M^n$, $f(\omega^M(\overline{x})) = \omega^N(f(\overline{x}))$;

(iii) If $R \in rel(n)$ and $\overline{x} \in M^n$, then $M \models R[\overline{x}] \;\Rightarrow\; N \models R[f(\overline{x})]$.

*b) f is a **L-embedding** if it is a L-morphism such that for all $n \geq 1$, $R \in rel(n)$ and $\overline{x} \in M^n$ $M \models R[\overline{x}] \;\Leftrightarrow\; N \models R[f(\overline{x})]$.*

*c) f is an **elementary embedding** if for all formulas $\phi(v_1, \ldots, v_n)$ in L and $\overline{x} \in M^n$, $M \models \phi[\overline{x}]$ iff $N \models \phi[f(\overline{x})]$.*

*When clear from context, L will be omitted from the notation. L-structures and L-morphism are a category, written **L-mod**.*

Remark 17.8. (1) Since equality is in $rel(2)$, embeddings are injective.

(2) If f is a L-morphism and $\tau(v_1, \ldots, v_n)$ is a term in L, then

$$\text{For all } \overline{x} \in M^n, \quad f(\tau^M(\overline{x})) = \tau^N(f(\overline{x})).$$

(3) Let $\phi \equiv R(\tau_1(v_1, \ldots, v_n), \ldots, \tau_m(v_1, \ldots, v_n))$ be an atomic formula in L and let f be a map satisfying (i) and (ii) in 17.7.(a). Then, f is a L-morphism iff

$$\text{For all } \overline{x} \in M^n, \quad M \models \phi[\overline{x}] \quad \Rightarrow \quad N \models \phi[f(\overline{x})].$$

Furthermore, f is an embedding iff

$$\text{For all } \overline{x} \in M^n, \quad M \models \phi[\overline{x}] \quad \Leftrightarrow \quad N \models \phi[f(\overline{x})]. \qquad \square$$

17.9. **Products in L-mod.** Let M_i, $i \in I$, be a family of L-structures. Let $M = \prod_{i\in I} M_i$ be their set-theoretic product (1.4). M is endowed with a L-structure, defined coordinatewise as follows, where $n \geq 1$ is an integer :

∗ If $c \in Ct$, $c^M = \langle c^{M_i} \rangle_{i\in I}$;

∗ If $\omega \in op(n)$, and $\langle s_1, \ldots, s_n \rangle \in M^n$, then

$$\omega^M(s_1, \ldots, s_n) = \langle \omega^{M_i}(s_1(i), s_2(i), \ldots, s_n(i)) \rangle_{i\in I};$$

∗ If $R \in rel(n)$ and $\overline{s} \in M^n$, then

$$M \models R[\overline{s}] \quad \text{iff} \quad \forall\, i \in I, \ \ M_i \models R[s_1(i), s_2(i), \ldots, s_n(i)].$$

Note that the canonical projections, $\pi_i : M \longrightarrow M_i$, are L-morphisms. This construction is the product of the M_i in the category **L-mod**.

Induction on complexity yields :

a) If $\tau(v_1, \ldots, v_n)$ is a term in L and $\overline{s} \in M^n$, then

$$\tau^M(\overline{s}) = \langle \tau^{M_i}(s_1(1), \ldots, s_n(i)) \rangle_{i\in I}.$$

b) If $\phi(v_1, \ldots, v_n)$ is an atomic formula in L and $\overline{x} \in M^n$, then

$$M \models \phi[\overline{s}] \quad \text{iff} \quad \forall\, i \in I, \ \ M_i \models \phi[s_1(i), \ldots, s_n(i)]. \qquad \square$$

3. Limits in L-mod

Theorem 17.10. *Let L be a first-order language with equality, containing at least one constant symbol and let $\langle I, \leq \rangle$ be a poset.*

a) The category **L-mod** *is complete.*

b) Let D be an I-diagram in **L-mod** *and let $\varprojlim D$ be its limit. Then,*

(1) *If $J \subseteq I$ is down-cofinal in I, $\varprojlim D_{|J}$ is isomorphic to $\varprojlim D$.*

(2) *Let $\sigma \in Sent(L)$ be logically equivalent to a finite conjunction of sentences of the form $\forall\overline{x}\ (\forall\overline{y}\psi \rightarrow \phi)$, where ψ and ϕ are positive and quantifier free* [5]*. Then,*

$$\{i \in I : D(i) \models \sigma\} \text{ is down-cofinal in } I \ \Rightarrow\ \varprojlim D \models \sigma.$$

Proof. a) We make use of Theorem 16.31, verifying that **L-mod** has final object, equalizers and products; by 17.9, it has products.

<u>Equalizers :</u> Let $D = (A \overset{g}{\underset{f}{\rightrightarrows}} B)$ be L-morphisms. Set

$$E = \{a \in A : f(a) = g(a)\}.$$

For all constants c in L, $c^E =_{def} c^A \in E$. Moreover, if $\omega \in op(n)$ is an n-ary operation symbol in L and $\overline{a} = \langle a_1, \ldots, a_n \rangle \in E^n$, then

$$f(\omega^A(\overline{a})) = \omega^B(f(\overline{a})) = \omega^B(g(\overline{a})) = g(\omega^A(\overline{a})),$$

and E is closed under the restriction of the interpretations of all operation symbols in A. For a n-ary relation symbol R in L, set $R^E = R \cap E^n$. Then, the canonical injection, $\eta : E \longrightarrow A$ is a L-embedding. Moreover, since $f \circ \eta = g \circ \eta$,

[5] Defined in 17.4.

$(E; \{\eta, f \circ \eta\})$ is a cone over D in **L-mod**. We now prove that this cone is the equalizer of (f, g), that is, it is $\varprojlim D$.

Let $h : C \longrightarrow A$ be a L-morphism, such that $g \circ h = f \circ h$. Note that this equation implies that for all $x \in C$, $h(x) \in E$. Define $\beta : C \longrightarrow E$ by $x \mapsto h(x)$. Clearly, β is the unique map such that $h = \eta \circ \beta$. It is straightforward to check that β is a L-morphism and so $(E; \{\eta, f \circ \eta\})$ is the equalizer of (f, g) in **L-mod**.

$$\begin{tikzcd}
C \arrow[r, "\beta"] \arrow[d, "h"'] & E \arrow[dl, "\eta"] \\
A \arrow[r, shift left, "g"] \arrow[r, shift right, "f"'] & B
\end{tikzcd}$$

Final object : Let $\mathbf{1} = \{\emptyset\}$, where all relations are interpreted as $\mathbf{1}^n$, all functions are interpreted as projections onto the first coordinate and all constants are assigned value $\emptyset$. Then, $\mathbf{1}$ is the final object in **L-mod**.

b) (1) Let $D = (D(i); \{f_{ij} : \mathrm{i} \leq \mathrm{j}\})$ be a I-diagram in **L-mod**. By (a), D and $D_{|J}$ have limits in **L-mod**. Let

$$\varprojlim D = (\varprojlim D; \{\lambda_i : i \in I\}) \quad \text{and} \quad \varprojlim D_{|J} = (\varprojlim D_{|J}; \{\beta_j : j \in J\}),$$

be presentations of these limits. Since $(\varprojlim D; \{\lambda_j : j \in J\})$ is a cone over $D_{|J}$, there is a unique cone morphism $f : \varprojlim D \longrightarrow \varprojlim D_{|J}$. Now, if $J \subseteq I$ is down-cofinal in I, define, for $i \in I$, $\beta_i : \varprojlim D_{|J} \longrightarrow D(i)$, by the following rule :

Choose $k \in J$ such that $k \leq i$ and set $\beta_i = \beta_k \circ f_{ki}$.

$$\begin{tikzcd}
\varprojlim D_{|J} \arrow[rr, "\beta_k"] \arrow[dr, "\beta_i"'] & & D(k) \arrow[dl, "f_{ki}"] \\
& D(i) &
\end{tikzcd}$$

It is straightforward to verify that the choice of $k \leq i$ is immaterial in the definition β_i and that, for $j \in J$, we get the same L-morphism originally attached to $\varprojlim D_{|J}$. Hence, $(\varprojlim D_{|J}; \{\beta_i : i \in I\})$ is a cone over D and there is a unique cone morphism, $g : \varinjlim D_{|J} \longrightarrow \varinjlim D$. We omit the straightforward verification that f and g are inverse L-isomorphisms.

(2) Because of the isomorphism between $\varprojlim D$ and $\varprojlim D_{|J}$, when J is down-cofinal in I, to prove the preservation of the sentences described in (2), it is enough to verify that,

$$\text{If } \sigma \text{ is true in all } D(i), \text{ then it is true in } \Lambda =_{def} \varprojlim D.$$

Let Σ and Π be the sets of formulas $\phi(v_1, \ldots, v_n)$ in L defined by the following rules :

$$\phi \in \Sigma \quad \text{iff} \quad \forall \overline{s} \in \Lambda^n, \ (\Lambda \models \phi[\overline{s}] \ \text{ iff } \ D(i) \models \phi[\lambda_i(\overline{s})]);$$
$$\phi \in \Pi \quad \text{iff} \quad \forall \overline{s} \in \Lambda^n, \ (D(i) \models \phi[\lambda_i(\overline{s})] \ \Rightarrow \ \Lambda \models \phi[\overline{s}]).$$

One then verifies that

$*$ All atomic formulas are in Σ; $\qquad *$ $\psi, \phi \in \Sigma \ \Rightarrow \ \psi \wedge \phi \in \Sigma$;

$*$ $\psi \in \Pi$, $\phi \in \Sigma \ \Rightarrow \ \forall x \, \psi \in \Pi$ and $\psi \to \phi \in \Pi$.

Details are left to the reader. Clearly, the desired result follows from the above statements. $\square$

REMARK 17.11. Since L-structures satisfy the laws of Classical Logic, the sentence $\exists x(x = x)$ forces them to be non-empty. If the language L has no constants, there are examples where the set of points where two morphisms, with same domain and codomain, are equal is empty. For these languages, **L-mod** does not have equalizers. Hence, the statement of Theorem 17.10 is sharp. $\square$

The following characterization of limits of I-diagrams is useful :

COROLLARY 17.12. *Let $D = (D(i), \ \{f_{ij} \ : \ i \leq j\})$ be an I-diagram in* **L-mod***, where I is a poset. Suppose $Ct(L) \neq \emptyset$. Let $\Lambda = \langle \Lambda; \ \{\lambda_i\}_{i \in I} \rangle$ be a cone in* **L-mod** *over D and let $\lambda = \prod \lambda_i$ be the unique map from Λ to $\prod D(i)$, such that $\pi_i \circ \lambda = \lambda_i$, $i \in I$. Then, Λ is (isomorphic to) $\varprojlim D$ iff*

[*lim* 1] *: The image of λ in $\prod D(i)$ is the set*

$$\{m \in \textstyle\prod D(i) : \forall \, i, j \in I \ (i \leq j \ \Rightarrow f_{ij}(\pi_i(m)) = \pi_j(m)\}.$$

[*lim* 2] *: For all atomic formulas $\phi(v_1, \ldots, v_n)$ in L and $\overline{s} \in \Lambda^n$,*

$$\Lambda \models \phi[\overline{s}] \quad \textit{iff} \quad \textit{for all } i \in I, \ D(i) \models \phi[\lambda_i(\overline{s})].$$

We now discuss limits for morphisms of I-diagrams in **L-mod**.

DEFINITION 17.13. *Let $\langle I, \leq \rangle$ be a poset and*

$$D = (D(i); \{f_{ij} : i \leq j\}) \quad \textit{and} \quad E = (E(i); \{g_{ij} : i \leq j\})$$

be I-diagrams in **L-mod**. *A* **morphism** *$h : D \longrightarrow E$ consists of a family of L-morphisms, $h_i : D(i) \longrightarrow E(i)$, $i \in I$, such that for $i \leq j$ in I, the following diagram is commutative :*

$$\begin{tikzcd} D(i) \arrow[r, "f_{ij}"] \arrow[d, "h_i"'] & D(j) \arrow[d, "h_j"] \\ E(i) \arrow[r, "g_{ij}"'] & E(j) \end{tikzcd}$$

Clearly, $Id_D = (Id_{D(i)})_{i \in I}$ is a morphism; if $D \xrightarrow{h} E \xrightarrow{k} G$ are morphisms of I-diagrams, then

$$k \circ h \;\; = \;\; (k_i \circ h_i)_{i \in I}$$

is a morphism of I-diagrams. Hence, I-diagrams in **L-mod** *constitute a category, written $\mathcal{D}(I,$* **L-mod***).*

PROPOSITION 17.14. *Let $h : D \longrightarrow E$ be a morphism of I-diagrams in* **L-mod** *and assume that $\varprojlim D$ and $\varprojlim E$ exist in* **L-mod***. Then, there is a* **unique** *L-morphism, $\varprojlim h : \varprojlim D \longrightarrow \varprojlim E$, such that for all $i \in I$, the following diagram is commutative :*

$$\begin{tikzcd}
\varprojlim D \arrow[r, "d_i"] \arrow[d, "\varprojlim h"'] & D(i) \arrow[d, "h_i"] \\
\varprojlim E \arrow[r, "e_i"'] & E(i)
\end{tikzcd}$$

where d_i and e_i are the L-morphisms that come with the limit construction.

PROOF. Clearly, $\langle \varprojlim D;\ \{h_i \circ d_i : i \in I\}\,\rangle$ is a cone over E; now the universal property of limits guarantee the existence and uniqueness of $\varprojlim h$. $\square$

4. Colimits in L-mod

THEOREM 17.15. *Let $\langle I, \leq \rangle$ be up-directed poset and let L be a first-order language with equality.*
a) Every I-diagram, D, in **L-mod** *has a colimit. If $J \subseteq I$ is up-cofinal in I then, $\varinjlim D$ is naturally isomorphic to $\varinjlim D_{|J}$.*
b) Let σ be a sentence logically equivalent to a finite disjunction of sentences of the form $\forall \overline{x}\ (\psi \to \exists \overline{y} \phi)$, where ψ and ϕ are positive and quantifier free [6]. Then,

$$\{i \in I : D(i) \models \sigma\} \text{ is up-cofinal in } I \quad \Rightarrow \quad \varinjlim D \models \sigma.$$

PROOF. a) Let $A = \coprod_{i \in I} D(i)$ be the disjoint union of the $D(i)$. We have canonical maps $w_i : D(i) \longrightarrow A$, $x \mapsto \langle x, i \rangle$. Since I is up-directed, the prescription

$$\langle x, i \rangle \equiv \langle y, j \rangle \quad \text{iff} \quad \exists\ k \geq i, j \text{ such that } f_{ik}(x) = f_{jk}(y),$$

defines an equivalence relation $\equiv$ on A. Let

$$G = \{\langle x, i \rangle / \!\equiv\, : \langle x, i \rangle \in A\}$$

be the set of equivalence classes of A by $\equiv$. Note that for a constant c in L we have $\langle c^{D(i)}, i \rangle \equiv \langle c^{D(j)}, j \rangle$. We interpret L in G as follows : for $n \geq 1$ and $\overline{x} \in G^n$,

$$\overline{x} = \langle \langle x_1, i_1 \rangle, \ldots, \langle x_n, i_n \rangle \rangle,$$

i) If $R \in rel(n)$, then $G \models R[\overline{x}]$ iff

$$\exists\ i \geq i_1, \ldots, i_n, \text{ such that } D(i) \models R[f_{i_1 k}(x_1), \ldots, f_{i_n k}(x_n)].$$

[6] As in 17.4.

ii) If $\omega \in op(n, L)$, take $k \geq i_1, \ldots, i_n$ and define $\omega^G(\overline{x})$ as the equivalence class of the pair $\langle \omega^{D(k)}(f_{i_1 k}(x_1), \ldots, f_{i_n k}(x_n)), k \rangle$.
iii) If c is a constant in L, $c^G = \langle c^{D(i)}, i \rangle / \equiv$.
Since I is up-directed, all of the above definitions are independent of representatives and of the choice of indices appearing therein. Further, the composition of the quotient map $A \longrightarrow G$ with the mappings w_i, defines L-morphisms α_i from $D(i)$ to G, making $(G, \{\alpha_i : i \in I\})$ a dual cone over D. If $(N; \{g_i : i \in I\})$ is a dual cone over D, define $h : G \longrightarrow N$ by $h(\langle x, i \rangle / \equiv) = g_i(x)$; the reader can check that h is unique and so, $(G; \{\alpha_i : i \in I\}))$ is $\varinjlim D$. The proof of (b) can be obtained by the same method used for limits in 17.10. □

Remark 17.16. Theorems 17.10 and 17.15 imply that limits for posets, and colimits over up-directed sets, preserve the properties of being a group, ring with identity, Boolean algebra, etc. □

Proposition 17.17. *Let $D = \langle D(i); \{f_{ij} : i \leq j\} \rangle$ be a I-diagram in* **L-mod**, *with $\langle I, \leq \rangle$ an up-directed poset. Let $\langle M; \{f_i : i \in I\} \rangle$ be a presentation of $\varinjlim D$. Then,*
a) If f_{ij} is an embedding for all $i \leq j$ in I, then f_i is an embedding, for all $i \in I$.
b) (Tarski) *If f_{ij} is an elementary embedding for all $i \leq j$ in I, then f_i is an elementary embedding, for all $i \in I$.*

Proof. Item (a) is straightforward diagram chasing. For (b), use induction on the complexity of formulas. □

The results corresponding to 17.12 and 17.14 for colimits follow; their proofs are left to the reader.

Corollary 17.18. *Let $D = (D(i), \{f_{ij} : i \leq j\})$ be an I-diagram in* **L-mod**, *where I is an up-directed poset. A dual cone in* **L-mod** *over D, $\langle G; \{\alpha_i\}_{i \in I} \rangle$, is (isomorphic to) $\varinjlim D$ iff it satisfies the following conditions* [7]*:*
[colim 1] : $G = \bigcup \{\alpha_i(D(i)) : i \in I\}$.
[colim 2] : If $\phi(v_1, \ldots, v_n)$ is an atomic formula in L, $\langle k_1, \ldots, k_n \rangle$ is an n-tuple in I and $s_{k_p} \in D(k_p)$, $1 \leq p \leq n$, then
$$G \models \phi[\alpha_{k_1}(s_{k_1}), \ldots, \alpha_{k_n}(s_{k_n})]$$
$$\text{iff}$$
$$\begin{cases} \exists\, k \in I \text{ such that } k \geq k_1, \ldots, k_n \quad \text{and} \\ D(k) \models \phi[\alpha_{k_1 k}(s_{k_1}), \ldots, \alpha_{k_n k}(s_{k_n})]. \end{cases}$$
□

Proposition 17.19. *Let $h : D \longrightarrow E$ be a morphism of I-diagrams in* **L-mod** *and assume that $\varinjlim D$ and $\varinjlim E$ exist in* **L-mod**. *Then, there is a unique L-morphism, $\varinjlim h : \varinjlim D \longrightarrow \varinjlim E$, such that for all $i \in I$, the following diagram is commutative :*

[7] See also Exercise 17.24.

$$\begin{tikzcd} D(i) \arrow[r, "d_i"] \arrow[d, "h_i"'] & \varinjlim D \arrow[d, "\varinjlim h"] \\ E(i) \arrow[r, "e_i"'] & \varinjlim E \end{tikzcd}$$

where d_i and e_i are the L-morphisms that come with the colimit construction. Moreover, if each h_i is a L-monic, the same is true of $\varinjlim h$.

5. Quotients in L-mod

Definition 17.20. *Let A be a L-structure. An equivalence relation on A is a* **congruence** *if it is a congruence with respect to all L-operations* [8].

Proposition 17.21. *Let A be a L-structure and θ a congruence on A. Then, the set A/θ of equivalence classes by θ can be made into a L-structure, as follows :*
(1) *For all $R \in rel(n, L)$ and $\overline{x} \in A^n$,*

$$A/\theta \models R[\overline{x}/\theta] \qquad \text{iff} \qquad \begin{cases} \exists\ \overline{y} \in A^n \text{ such that } y_k\ \theta\ x_k,\ 1 \le k \le n \\ \qquad\qquad \text{and} \quad A \models R[\overline{y}], \end{cases}$$

where $\overline{x}/\theta = \langle x_1/\theta, \ldots, x_n/\theta \rangle$.
(2) *For $\omega \in op(n, L)$ and $\overline{x} \in A^n$, $\omega^{A/\theta}(\overline{x}) = \omega^A(\overline{x})/\theta$.*
(3) *For $c \in Ct(L)$, $c^{A/\theta} = c^A/\theta$.*
With this L-structure, the canonical quotient map $\pi_\theta : A \longrightarrow A/\theta$ is a L-morphism. Moreover, if $f : A \longrightarrow M$ is a L-morphism and $\theta \subseteq \theta_f$ (17.23), then there is a unique L-morphism $\widehat{f} : A/\theta \longrightarrow M$, making the following diagram commutative :

$$\begin{tikzcd} A \arrow[rr, "\pi_\theta"] \arrow[dr, "f"'] & & A/\theta \arrow[dl, "\widehat{f}"] \\ & M & \end{tikzcd}$$

Proof. The maps in (2) are well defined because θ is a congruence with respect to all L-operations on A. Moreover, (2) implies that π_θ preserves all L-operations on A, while (3) entails that the interpretation of constants is also preserved. Condition (1) is phrased so as to guarantee that the interpretation of L-relations are preserved. Hence, π_θ is a L-morphism.

If M is a L-structure and $f : A \longrightarrow M$ is a L-morphism, define, for $x \in A$,

[8] As in Example 2.49.

$$\widehat{f}(x/\theta) = fx.$$

Since $\theta \subseteq \theta_f$ (17.23), $\widehat{f}$ is well defined. If $R \in rel(n)$, assume that $A/\theta \models R[\overline{x}/\theta]$. Then, there is $\overline{y} \in A^n$ such that $A \models R[\overline{y}]$ and $x_k \; \theta \; y_k$, $1 \le k \le n$. Because f is a L-morphism, we conclude that

$$M \models R[f(\overline{y})].$$

Now $x_k \; \theta \; y_k$ and $\theta \subseteq \theta_f$ imply that $fx_k = fy_k$, $1 \le k \le n$, and so

$$M \models R[\widehat{f}(\overline{x}/\theta)]$$

as needed. It is clear that for all constants c in L, $\widehat{f}(c^{A/\theta}) = c^M$. For operations symbols, the argument is analogous. The uniqueness of $\widehat{f}$ is immediate. $\square$

The L-structure A/θ in 17.21 is the **quotient** of A by the congruence θ.

Exercises

17.22. This exercise is related to 22.15 and 22.37. Let H be a HA and F be a (proper) filter in H. For $a \in H$, let $H_a = H/a^{\rightarrow}$ be the quotient HA by the filter $a^{\rightarrow}$. Write $\pi_a : H \longrightarrow H_a$ for the quotient morphism (instead of $\pi_{a^{\rightarrow}}$).
a) F is an up-directed subset of H^{op} [9].
b) For $a \le b$ in H there is a HA-morphism, $h_{ba} : H_b \longrightarrow H_a$, making the following diagram commutative :

$$\begin{tikzcd} H \arrow[rr, "\pi_b"] \arrow[dr, "\pi_a"'] & & H_b \arrow[dl, "h_{ba}"] \\ & H_a & \end{tikzcd}$$

Moreover, whenever $a \le b \le c$, $h_{aa} = Id_{H_a}$ and $h_{ca} = h_{ba} \circ h_{cb}$. Conclude that $\mathcal{F} = \langle\, H_a; \{h_{ba} : a \le b \text{ in } F\}\,\rangle$ is an inductive system of HAs and HA-morphisms.
c) $H/F = \varinjlim \mathcal{F}$. $\square$

17.23. If $M \xrightarrow{f} N$ is a L-morphism, $\theta_f = \{\langle\, x, y\,\rangle \in M^2 : f(x) = f(y)\}$ is a L-congruence on M. $\square$

17.24. Let $D = (D(i),\ \{f_{ij} \ : \ i \le j\})$ be an I-diagram in **L-mod**, where I is an up-directed poset. A dual cone in **L-mod** over D, $\langle\, G; \{\alpha_i\}_{i \in I}\,\rangle$, is (isomorphic to) $\varinjlim D$ iff it satisfies the following conditions :
[colim $1'$] : For all $i, j \in I$, $x \in D(i)$ and $y \in D(j)$

$$\alpha_i(x) = \alpha_j(y) \quad \text{iff} \quad \exists\, k \ge i, j \text{ such that } f_{ik}(x) = f_{jk}(y).$$

[colim $2'$] : For all n-ary relations R in L and $\overline{\xi} \in G^n$,
$M \models R[\overline{\xi}] \;\Leftrightarrow\; \exists\, k \in I$ and $\overline{x} \in D(k)^n$ such that $\overline{\xi} = \alpha_k(\overline{x})$ and $D(k) \models R[\overline{x}]$. $\square$

[9] Recall (2.5) that H^{op} is H with the opposite order it originally had.

Part 3

Spectral Spaces

CHAPTER 18

Boolean Spaces

In this Chapter the results in section 1.2 may be used without explicit reference.

Recall that $\Omega(X)$, $B(X)$ and $Reg(X)$ (3.6) denote the frame of opens, the BA of clopens and the cBa of regular opens, respectively, of the topological space X.

Definition 18.1. *A* T0 *topological space* X *is* **totally disconnected** *if* $B(X)$ *is a basis of opens in* X*, or equivalently,* $\Omega(X)$ *is a zero-dimensional frame* (8.19). *A* **Boolean space** *is a compact totally disconnected space.*

A totally disconnected space X is *Hausdorff*. For if $x \neq y$ in X, the T0 property (1.20) furnishes an open set U in X, such that $x \in U$ and $y \notin U$ (or the other way around). Since $B(X)$ is a basis of opens in X, we might as well assume that U is clopen, and so x and y have disjoint clopen neighborhoods in X.

A subspace of a totally disconnected space, with the induced topology, is also totally disconnected, i.e., the property of being totally disconnected is *hereditary*. A *closed* subspace of a Boolean space is a Boolean space in its own right.

The following Example describes an important way of obtaining totally disconnected spaces.

Example 18.2. Let S_i, $i \in I$, be a family of sets, considered as topological spaces with the discrete topology (all points are open). Then, the product $S = \prod S_i$, with the product topology, is totally disconnected. Although a basis for S appears in 1.25, we introduce new notation that is convenient for dealing with products of discrete spaces. Recall that $\subseteq_f$ stands for "finite subset of".
Let $S_\omega = \bigcup \{\prod_{j \in J} S_j : J \subseteq_f I\}$. For $s \in S_\omega$, define

$$V_s = \{f \in S : f_{|doms} = s\},$$

that is, V_s is the set of extensions of s in S. Then,

$$\mathcal{V} = \{V_s : s \in S_\omega\},$$

is a *basis of clopens* for the product topology on S.

When each S_i is **finite**, the product topology is compact, by Tychonoff's Theorem 1.28. In particular, for all A, 2^A is a Boolean space, where $2 = \{0, 1\}$; the space $2^{\mathbb{N}}$ is called the **Cantor space**, being homeomorphic to the Cantor "extract the middle third" space, a classical construction on the real closed unit interval.

For **separable metric spaces**, i.e., those possessing a countable dense set, two spaces of this type are fundamental. A result due to Kuratowski asserts that

(1) Every complete separable metric space is the continuous image of $\mathbb{N}^{\mathbb{N}}$ (the Baire space), by a map whose inverse image of each point is at most countable;
(2) The same holds for *compact* metric spaces, with $2^{\mathbb{N}}$ in place of $\mathbb{N}^{\mathbb{N}}$. $\square$

LEMMA 18.3. *Let X be a totally disconnected space. Then, X is homeomorphic to a subspace of 2^γ, where γ is the weight of X (1.16.(b)). If X is Boolean, then X is homeomorphic to a* closed *subspace of 2^γ.*

PROOF. Since the set of cardinals α such that X has a basis of cardinal α has a least element, fix a basis $\mathcal{B} = \{U_\beta : \beta \in \gamma\}$ of cardinal γ, which we may assume to be contained in $B(X)$. Define, for $x \in X$ and $\beta \in \gamma$,

$$f : X \longrightarrow 2^\gamma, \text{ by } \quad f(x)(\beta) = \begin{cases} 1 & \text{if } x \in U_\beta \\ 0 & \text{otherwise.} \end{cases}$$

If $x \neq y$ in X, there is $\beta \in \gamma$, such that $x \in U_\beta$ and $y \notin U_\beta$. Thus, $f(x) \neq f(y)$ and f is injective; for continuity, let $J \subseteq_f \gamma$ and $s \in 2^J$; with notation as in 18.2, V_s is a basic clopen in 2^γ, and

$$f^{-1}(V_s) = \textstyle\bigcap_{\beta \in J} U_\beta,$$

that is clopen in X. To check that f is open (1.18), let $\beta \in \gamma$ and set

$$V_\beta = \{t = \langle\, t_\beta \,\rangle \in 2^\gamma : t_\beta = 1\},$$

a clopen in 2^γ. It is immediate from the definition of f that

$$\text{For all } \beta \in \gamma,\ f(U_\beta) = V_\beta \cap \mathrm{Im} f. \tag{I}$$

For an open set U in X, let $K \subseteq \gamma$ be such that $U = \bigcup_{\beta \in K} U_\beta$. Then, since f is a bijection between X and its image, (I) yields

$$f(U) = \textstyle\bigcup_{\beta \in K} V_\beta \cap \mathrm{Im} f,$$

and $f : X \longrightarrow \mathrm{Im} f$ is a homeomorphism. If X is compact, the results in Lemma 1.24 imply that $\mathrm{Im} f$ is a closed subset of 2^γ. $\square$

DEFINITION 18.4. *A Hausdorff space X is* **dyadic** *if it is the continuous image of 2^I, for some set I.*

Clearly, all dyadic spaces are compact. By the observations (1) and (2) at the end of 18.2, every compact metric space is dyadic. The class of dyadic spaces is important in the study of functional analytic questions in Banach spaces of continuous functions and in general measure theory.

Lemma 18.3 justifies a closer look at spaces of the type 2^A. To deal with topological and combinatorial questions in these spaces, we set down some notation, expanding that already described in 18.2. The posets $pF_\omega(A, 2)$, introduced in Example 2.13 will be of constant use.

Let A be a set; for each $s \in pF_\omega(A, 2)$, as in 18.2,

$$V_s = \{f \in 2^A : f_{|doms} = s\}.$$

When $s = \{\langle\, a, 1 \,\rangle\}$ or $s = \{\langle\, a, 0 \,\rangle\}$, $a \in A$, we may write V_s as $V_{(a,1)}$ or $V_{(a,0)}$, respectively. Clearly,

$$V_s = \bigcap \{V_{(a,s(a))} : a \in \mathrm{dom}\ s\},$$

a *finite* intersection, since *dom s* is finite. The collection

$$\mathcal{B} = \{V_s : s \in pF_\omega(A,2)\}$$

is a clopen basis for the product topology on 2^A. Note that $V_\emptyset = 2^A$. For $f \in 2^A$, a fundamental system of neighborhoods of f is given by :

$$\nu_f = \{V_s : s = f_{|B},\ B \subseteq_f A\}.$$

Recall (2.33, 2.36) that $s, t \in pF_\omega(A,2)$ are *up-compatible* iff

$$s_{|dom s \,\cap\, dom t} = t_{|dom s \,\cap\, dom t};$$

otherwise, s and t are *up-incompatible* ($s \perp^* t$). Up-compatibility is equivalent to the existence of $x \in pF_\omega(A,2)$ such that $s, t \leq x$. That is, s and t are compatible iff they have a common extension in $pF_\omega(A,2)$. We also have that $s \perp^* t$ iff $t \perp^* s$.

For $s \in pF_\omega(A,2)$, define

$$\widehat{s} : dom\, s \longrightarrow 2, \text{ by } \quad \widehat{s}(b) = 1 \quad \text{iff} \quad s(b) = 0.$$

Clearly, $\widehat{\widehat{s}} = s$. If $s, t \in pF_\omega(A,2)$, then

$[\perp^*]$ $$s \perp^* t \quad \text{iff} \quad \exists\, \emptyset \neq B \subseteq dom\, s \cap dom\, t, \text{ such that } \widehat{s_{|B}} \leq t.$$

This condition is symmetric, since $\widehat{\widehat{s}} = s$.

For *up-compatible* $s, t \in pF_\omega(A,2)$, it follows from 1.2 that their join, $(s \vee t)$, exists in $pF_\omega(A,2)$, being given by

$$\begin{cases} dom\,(s \vee t) = dom\, s \cup dom\, t; \\ (s \vee t)_{|dom s} = s \text{ and } (s \vee t)_{|dom t} = t. \end{cases}$$

Since sequences with disjoint domains are compatible, their join exist in $pF_\omega(A,2)$. In particular, if $a \notin dom\, s$, $s \vee \langle\, a, 1\,\rangle$ and $s \vee \langle\, a, 0\,\rangle$ are in $pF_\omega(A,2)$.

If $s \in pF_\omega(A,2)$, define the **length** of s, $l(s)$, by

[length] $$l(s) = card(\text{dom } s).$$

The basic properties of the collection $\mathcal{B} = \{V_s : s \in pF_\omega(A,2)\}$ are in

LEMMA 18.5. *Let A be a set and $s, t \in pF_\omega(A,2)$. Then,*

a) $V_\emptyset = 2^A$. Moreover, $V_t \subseteq V_s$ iff $s \leq t$.

b) $V_s \cap V_t \neq \emptyset$ iff s and t are up-compatible and $V_s \cap V_t = V_{(s \vee t)}$.

c) $$2^A - V_s = \bigcup \{V_{\widehat{s_{|B}}} : B \subseteq dom\ s,\ B \neq \emptyset\} = \bigcup \{V_{(a, 1-s(a))} : a \in dom\ s\}.$$

d) For all $a \notin dom\ s$,

$$V_s = V_{(s \vee (a,1))} \cup V_{(s \vee (a,0))} \text{ and } V_{(s \vee (a,1))} \cap V_{(s \vee (a,0))} = \emptyset.$$

In particular, for all $a \in A$, $V_{(a,1)} \cap V_{(a,0)} = \emptyset$ and $V_{(a,1)} \cup V_{(a,0)} = 2^A$.

e) A subset $C \subseteq 2^A$ is clopen iff it is a finite union of elements of $\mathcal{B}$.

PROOF. Straightforward. Only (e) needs compactness. $\square$

It follows from 18.5.(e) that the cardinality of $B(2^A)$ is equal to that of A. Moreover, 18.5.(d) implies that, whenever A is infinite, $B(2^A)$ is *atomless*, that is, every non-empty clopen has a proper non-empty clopen subset.

Deeper topological properties of 2^A come from combinatorial results in Set Theory, like the Erdös-Rado Theorem 2.39, sometimes referred to as the Δ-system

Lemma. Recall (2.35), that a topological space is ccc if every family of non-empty pairwise disjoint opens is at most countable. From 2.42 we get

COROLLARY 18.6. *All dyadic spaces are ccc.*

PROOF. Clearly, the continuous image of a ccc space is ccc. Thus, it suffices to verify that 2^A is ccc, for all sets A. Let $\{U_i : i \in I\}$ be a family of non-empty pairwise disjoint opens in 2^A. For $i \in I$, choose a basic clopen $V_{s_i} \subseteq U_i$; since the U_i are pairwise disjoint, Lemma 18.5.(b) implies that $s_i \perp^* s_j$, if $i \neq j$. By Corollary 2.42, the cardinal of I must be countable. $\square$

REMARK 18.7. Note that Theorem 1.29 implies that 2^S is separable whenever $card(S) \leq 2^{\mathbb{N}}$. $\square$

The space 2^A is also a topological group, because pointwise multiplication of its elements is a continuous map. This compact topological group carries a measure λ, called **Haar measure**, the unique regular Borel measure whose value in each basic clopen V_s is given by

$$\lambda(V_s) = 2^{-l(s)},$$

where $l(s)$ is the length of s (defined in [length], page 172). λ is translation invariant, that is, for all Borel sets B in 2^A and $f \in 2^A$,

$$\lambda(B) = \lambda(f \cdot B),$$

where $f \cdot B = \{fg : g \in B\}$. Furthermore, 2^A is an example of a **profinite group** that is, a compact totally disconnected topological group. One can find information about such groups in [**46**] and [**41**], as well as in the references therein.

If $K \subseteq I$, there is a map, $\pi_K : 2^I \longrightarrow 2^K$, given by $f \mapsto f_{|K}$. Clearly, π_K is surjective; it also has the following properties :

LEMMA 18.8. *Let $K \subseteq I$ be sets, $s \in pF_\omega(I, 2)$ and $t \in pF_\omega(K, 2)$.*
a) π_K is continuous.
b) $\pi_K(V_s) = V_{s_{|K}}$ and $\pi_K^{-1}(V_t) = V_t$.
c) If $U \subseteq 2^K$, U is clopen in 2^K iff $\pi_K^{-1}(U)$ is clopen in 2^I.
d) If $U \in Reg(2^K)$, then $\pi_K^{-1}(U) = int\, \pi_K^{-1}(\overline{U}) \in Reg(2^I)$.
e) There is a homeomorphism

$$h_K : 2^I \longrightarrow 2^K \times 2^{(I-K)}, \text{ defined by } h(x) = \langle\, \pi_K(x), \pi_{(I-K)}(x)\,\rangle,$$

such that for all $w \in pF_\omega((I - K), 2)$,

$$h_K(V_s) = V_{s_{|K}} \times V_{s_{|(I-K)}} \quad \text{and} \quad h_K^{-1}(V_t \times V_w) = V_{t \vee w}.$$

PROOF. Item (a) is clear and (e) is straightforward; (c) and (d) are immediate consequences of 18.5.(e) and the fact that for $A \subseteq 2^K$ and $C, D \subseteq 2^I$ we have

$$\pi_K(\pi_K^{-1}(A)) = A \quad \text{and} \quad \pi_K(C \cup D) = \pi_K(C) \cup \pi_K(D),$$

The second assertion in (b), as well as that $\pi_K(V_s) \subseteq V_{s_{|K}}$, are clear from the definition of π_K. If $g \in V_{s|K}$, then g and s are compatible; hence, any extension to I of $(g \vee s)$ will be in V_s and its restriction to K is equal to g. $\square$

Remark 18.9. Let X be a topological space. A bijective function $h : I \longrightarrow J$, induces a homeomorphism, $H : X^J \longrightarrow X^I$, by composition : $f \mapsto h \circ f$. Note that for $s \in pF_\omega(J, 2)$, $H(V_s) = V_{h \circ s}$. In particular, if I is infinite, then $X^{I \times I}$ is homeomorphic to X^I. $\square$

The next result gives a topological characterization of completeness of $B(X)$, X a Boolean space.

Lemma 18.10. *For a Boolean space X, the following are equivalent:*
(1) *$B(X)$ is a cBa;* (2) *$B(X) = Reg(X)$;*
(3) *The closure of every open set in X is clopen.*

Proof. Since the interior of the closure of any open set is regular, it is clear that (2) and (3) are equivalent. Moreover, $Reg(X)$ is always a cBa (10.5) and so (2) implies (1). It remains to check that (1) implies (2). Let U be a regular open in X and write

$$U = \bigcup_{i \in I} V_i,$$

with $\{V_i : i \in I\} \subseteq B(X)$. Let $C = \bigvee_{i \in I} V_i$, this sup being taken in $B(X)$. Note that for all $i \in I$, $V_i \subseteq C$, since the partial order in $B(X)$ is set inclusion. Consequently, $U \subseteq C$. Now, if the open set $(C - \overline{U})$ is not empty, then there is $V \in B(X)$ such that $V \subseteq (C - \overline{U})$. But then, $(C - V)$ is a clopen set containing all the V_i, but properly contained in C, which is impossible. Hence, $U \subseteq C \subseteq \overline{U}$; since C is clopen and U is regular, we conclude that $C = U$, ending the proof. $\square$

Condition (3) in 18.10 is the definition of **extremally disconnected space**. Well, no infinite dyadic space is extremally disconnected. Since we have not yet presented Stone duality, we shall prove this just for 2^A, in a way that has some geometric content. The first step is

Corollary 18.11. *$B(2^{\mathbb{N}})$ is not a complete Boolean algebra.*

Proof. Let $\mathbb{N} \xrightarrow{f} 2$ be the constant function with value 1. For an integer $n \geq 0$, define

$$s_n = f_{|[0,n]} \quad \text{and} \quad V_n = V_{s_n};$$

$\{V_n : n \geq 0\}$ is a strictly decreasing sequence of clopens in $2^{\mathbb{N}}$. For $n \geq 0$, set

$$U_0 = V_0 \quad \text{and} \quad U_{n+1} = V_{n+1} - V_n.$$

The sequence $\{U_n\}$ consists of non-empty pairwise disjoint clopens in $2^{\mathbb{N}}$. It is straightforward to verify that the map

$$S \subseteq \mathbb{N} \longmapsto \text{int}\,\overline{\bigcup_{n \in S} U_n} \in Reg(2^{\mathbb{N}})$$

is injective. Hence, $Reg(2^{\mathbb{N}})$ has cardinality equal to $2^{\mathbb{N}}$, and so must be distinct from $B(2^{\mathbb{N}})$, which is countable by 18.5.(e). $\square$

The method of proof of 18.11 is useful in many contexts. It is based in finding a properly decreasing sequence, which in turn produces a sequence of non-empty, pairwise disjoint elements.

By Stone duality, Exercise 18.20 entails that if X is extremally disconnected, there is a continuous surjection, $X \longrightarrow \beta\mathbb{N}$, where $\beta\mathbb{N}$ is the Stone-Čech compactification of the discrete space $\mathbb{N}$. But a result due to Engelking and Pelczyński ([**13**]), implies that $\beta\mathbb{N}$ is not a continuous image of a dyadic compact; hence, Corollary 18.6 applies to show that no dyadic space is extremally disconnected. We shall return to this question later in the text. For the moment, we prove

PROPOSITION 18.12. *If A is an infinite set, $B(2^A)$ is not a complete Boolean algebra.*

PROOF. Select an infinite countable subset $K \subseteq A$. Since $2^{\mathbb{N}}$ is homeomorphic to 2^K (18.9), there is a regular open U in 2^K, which is not clopen. With notation as in 18.8, if $V = \pi_K^{-1}(U)$, then 18.8.(c) and (d) guarantee that V is a regular open in 2^A, which is not clopen. By 18.10, $B(2^A)$ is not a cBa. $\square$

From Corollary 14.6 we obtain

PROPOSITION 18.13. *$Reg(2^A)$ is the completion of $B(2^A)$.*

PROOF. Write $\bigvee^*$ for the sup operation in $Reg(2^A)$ and B for $B(2^A)$. By 10.5, $Reg(2^A)$ is a complete Boolean algebra. It must be shown that the canonical injection, $B \xrightarrow{*} Reg(2^A)$, satisfies conditions (1) and (2) of Corollary 14.6. Since set intersection is the meet operation in both BAs in consideration, we conclude that $*$ preserves meets. Let U_i, $i \in I$, be a family of clopens and let $V = \sup_B U_i$. Since B is a basis for the topology on 2^A, we have
$$\bigvee{}^* U_i \;=\; \mathrm{int}\,\overline{\bigcup U_i} \;=\; V.$$
Hence, $*$ is a regular embedding; its image is dense in $Reg(2^A)$ because if $V \in Reg(2^A)$, then
$$\bigvee{}^* \{U \in B : U \subseteq V\} \;=\; \mathrm{int}\,\overline{\bigcup\{U \in B \;:\; U \subseteq V\}} \;=\; \mathrm{int}\,\overline{V} \;=\; V,$$
completing the proof. $\square$

REMARK 18.14. In Theorem 14.4, if the map $f : H \longrightarrow D$ is simply a HA-morphism, then f_* might not even be a lattice morphism. For instance, consider the HA-embedding $B(X) \xrightarrow{\iota} \Omega(X)$, with $X = 2^A$. The only extension of ι to $Reg(X)$ that preserves finite meets, is the canonical map into $\Omega(X)$. To see this, let f be a $\wedge$-preserving extension of ι to $Reg(X)$. To show that $f(U) = U$, it suffices to check that $U \subseteq f(U) \subseteq \overline{U}$. Since each $V \in B(X)$ is kept fixed by f, it is clear that $U \subseteq f(U)$. Suppose, to get a contradiction, that $f(U)$ is not contained in $\overline{U}$. Then, there is $V \in B(X)$ such that $V \cap f(U) \neq \emptyset$, while $V \cap U = \emptyset$. But then,
$$\emptyset \;=\; f(V \cap U) \;=\; f(V) \cap f(U) \;=\; V \cap f(U),$$
the desired contradiction. Now, observe that the canonical injection from $Reg(X)$ into $\Omega(X)$ is not a lattice morphism, since it does not preserve finite joins. $\square$

DEFINITION 18.15. *Let $\{A_i : i \in I\}$ be a family of non-empty sets and let $A = \prod_{i \in I} A_i$.*
a) For $a = \langle a_i \rangle$ and $b = \langle b_i \rangle$ in A, set $[a = b] \;=\; \{i \in I : a_i = b_i\}$.

b) Let $A \xrightarrow{f} T$ be map and let J be a subset of I. We say that **f depends only on J** *if for all $a = \langle a_i \rangle$ and $b = \langle b_i \rangle$ in A*

$$J \subseteq [a = b] \quad \textit{implies} \quad f(a) = f(b).$$

c) Let S be a subset of A and let J be a subset of I. **S depends only on J** *if its characteristic map* [1], *$\chi_S : A \longrightarrow \{0, 1\}$, depends only on J.*

REMARK 18.16. Note that if $f : A \longrightarrow T$,

(*) $$f \text{ depends only on } \emptyset \quad \text{iff} \quad f \text{ is constant on } A.$$

Moreover, for $S \subseteq A$, S depends only on J iff for all a, $b \in A$

$$a \in S \text{ and } J \subseteq [a = b] \quad \text{implies} \quad b \in S. \qquad \square$$

The following result yields another characterization of dependence. Its proof is left to the reader.

LEMMA 18.17. *For non-empty sets $\{A_i : i \in I\}$, set*

$$A = \prod_{i \in I} A_i \quad \textit{and} \quad A_{|J} = \prod_{j \in J} A_j.$$

Let $\pi_J : A \longrightarrow A_{|J}$ be the projection that forgets the coordinates outside J. Then, for all sets T, the map

$$f \in [A_{|J}, T] \longmapsto f \circ \pi_J \in [A, T],$$

is a natural bijective correspondence between the set of maps from $A_{|J}$ to T, and the set of maps from A to T that depend only on J. $\square$

Recall (1.17) that $\mathbb{C}(X)$ is the algebra of continuous real valued maps on the topological space X.

PROPOSITION 18.18. *Let $\{X_i : i \in I\}$ be compact Hausdorff spaces and let $X = \prod_{i \in I} X_i$. Then, any map in $\mathbb{C}(X)$ depends only on a countable subset of I.*

PROOF. Let

$$\mathcal{A} \; = \; \{h \in \mathbb{C}(X) : h \text{ depends only on a finite subset of } I\}.$$

Then, $\mathcal{A}$ is closed under addition, multiplication and, because of (*) in Remark 18.16, contains the constant functions. Since X is compact and Hausdorff, the following celebrated result applies :

Theorem (Stone-Weierstrass) *If X is a compact Hausdorff space, any subalgebra of $\mathbb{C}(X)$, containing the constant functions, is dense in $\mathbb{C}(X)$ with the sup norm.*

Hence, $\mathcal{A}$ is dense in $\mathbb{C}(X)$, i.e., if $f \in \mathbb{C}(X)$, there is a sequence $h_n \in \mathcal{A}$, such that h_n converges *uniformly* to f :

$$\lim_{n \to \infty} \; ||h_n \; - \; f||_\infty = 0,$$

where $||g||_\infty = \sup_{x \in X} |g(x)|$. For $n \geq 1$, let $J_n \subseteq_f I$ be the subset on which h_n depends. Since the convergence is uniform, it is clear that f depends only on the countable subset of coordinates $\bigcup_{n \geq 1} J_n$. $\square$

[1] $\chi_S(a) = 1 \;$ iff $\; a \in S$.

Exercises

18.19. A BA B is a $\boldsymbol{\sigma}$**-algebra** if it has all countable joins and meets. Prove that an infinite σ-algebra is uncountable. Conclude that any infinite cBa must be uncountable, giving another proof of Corollary 18.11. $\square$

18.20. If X be an infinite extremally disconnected space, then there is an injective BA-morphism $f : 2^{\mathbb{N}} \longrightarrow B(X)$. $\square$

18.21. Let X be a topological space that contains at least two distinct points. Let I be a set. Write $S(I)$ for the group of permutations of I, that is, bijections from I onto I. For each $\omega \in S(I)$, define

$$\omega^* : X^I \longrightarrow X^I, \text{ given by } \langle\, x_i \,\rangle \mapsto \langle\, x_{\omega(i)} \,\rangle.$$

Then

a) ω^* is a homeomorphism.

b) $\omega \in S(I) \mapsto \omega^* \in Homeo(X)$, the group of homeomorphisms of X, is an injective group homomorphism.

c) When is the homomorphism in (b) is an isomorphism ?

d) For x, $y \in X^I$, define

$$x \; E \; y \quad \text{iff} \quad \text{there is } \omega \in S(I), \text{ such that } \omega^* x = y.$$

Then, E is an equivalence relation on X^I. The class x/E is called the **orbit** of x by $S(I)$. What can be said about the space of orbits of this action, $X^I/S(I)$? $\square$

CHAPTER 19

Spectra of Rings and Lattices

In this Chapter we study the duality that between lattices and commutative rings and topological spaces. Both instances of the duality, Stone spaces and the spectrum of a ring, will be presented at the same time in order to underline similarities and differences. In the next Chapter, we shall describe a general setting of which both cases are particulars. References for parts of this Chapter are [**2**], [**3**], [**29**], [**33**], [**67**], [**25**] and [**34**].

In this chapter, all lattices are distributive with $\perp$ and $\top$, and all rings are commutative with 1.

We restate Theorems 4.24 and 9.5 in a way that underlines the fundamental analogy between them.

THEOREM 19.1. *a) Let L be a lattice. Let F be a filter in L and S be a non-empty ud subset of L, such that $F \cap S = \emptyset$. Then, there is a prime filter P in L, such that $F \subseteq P$ and $P \cap S = \emptyset$.*
b) Let R be a ring. Let I be an ideal in R and S be a multiplicative subset of R, such that $I \cap S = \emptyset$. Then, there is a prime ideal P in R, such that $I \subseteq P$ and $P \cap S = \emptyset$.

DEFINITION 19.2. *Let L be a lattice and R be a ring. We define*

$$S(L) \quad = \quad \{P : P \text{ is a } \textbf{proper} \text{ prime filter in } L\}$$

and

$$Spec(R) \quad = \quad \{P : P \text{ is a } \textbf{proper} \text{ prime ideal in } R\}$$

called the **Stone space** *of L and the* (Zariski) **prime spectrum** *of R, respectively.*

For $a \in L$ and $r \in R$, set

$$S_a = \{P \in S(L) : a \in P\} \quad \text{and} \quad Z_r = \{P \in Spec(R) : r \notin P\}.$$

If I is an ideal in R, set

$$F_I \quad = \quad \{P \in Spec(R) : I \subseteq P\}.$$

Let $S_a^c = S(L) - S_a$ and $Z_r^c = Spec(R) - Z_r$.
Notice that $S_\perp = \emptyset = Z_0$, $S_\top = S(L)$ and $Z_1 = Spec(R)$.

REMARK 19.3. a) By 9.12.(d), if I is a proper ideal in a ring R, then

$$(\star) \qquad \sqrt{I} \quad = \quad \bigcap \{P \in Spec(R) : I \subseteq P\} \quad = \quad \bigcap_{r \in I} Z_r^c.$$

Recall that $\eta = \sqrt{0}$, the intersection of all prime ideals in R, is the ideal of nilpotent elements in R (Corollary 9.13). For $r \in R$, (r) is the (principal) ideal generated by r, $(r) = \{\alpha r : \alpha \in R\}$.
b) To deal with quotients by ideals, we set down some basic terminology. If I is an ideal in the ring R, R/I denotes the quotient ring, whose elements are equivalence classes r/I, $r \in R$. Recall that

$$r/I = s/I \quad \text{iff} \quad (r - s) \in I.$$

Let $\pi_I : R \longrightarrow R/I$ be the quotient homomorphism, $\pi_I(r) = r/I$. If $r, s \in R$, we may write $r \equiv s \bmod I$ [1] is synonymous with $r/I = s/I$. It is well known that all ring congruences on R are of the form $*/I$, for some ideal $I \subseteq R$.

If $S \subseteq R$ is a subset of R, set

$$S/I = \{s/I \in R/I : s \in S\} = \pi_I(S).$$

If (S) is the ideal generated by S in R, then $(S)/I$ is an ideal, exactly the ideal generated by S/I in R/I, that is :

$$(S)/I = (S/I).$$

In particular, $(r)/I = (r/I)$, for all $r \in R$. □

LEMMA 19.4. *For ideals I and J in a ring R,*
a) $\sqrt{I} \subseteq \sqrt{J}$ *iff* $F_J \subseteq F_I$ [2].
b) $I/\eta \subseteq J/\eta \;\Rightarrow\; \sqrt{I} \subseteq \sqrt{J}$.

PROOF. Item (a) is clear from $(\star)$ in 19.3; for (b), let $P \in Spec(R)$ be such that $J \subseteq P$; the hypothesis in (b) means that for every $r \in I$, there is $s \in J$ such that $(r - s) \in \eta$. Since every prime contains η, we get $I \subseteq P$ and the conclusion follows from (a). □

PROPOSITION 19.5. *Let L, L', L'' be lattices, and let R, R', R'' be rings. Let a, b be elements of L and r, s be elements of R.*
a) $S_a \cap S_b = S_{a \wedge b}$, $S_a \cup S_b = S_{a \vee b}$ *and* $Z_r \cap Z_s = Z_{rs}$.
b) $a \leq b$ *iff* $S_a \subseteq S_b$; $a = b$ *iff* $S_a = S_b$.
c) $Z_r \subseteq Z_s$ *iff* $\sqrt{r} \subseteq \sqrt{s}$; $\sqrt{r} = \sqrt{s}$ *iff* $Z_r = Z_s$.
d) $(r/\eta) = (s/\eta) \;\Rightarrow\; Z_r = Z_s$.
e) A lattice morphism, $L \xrightarrow{f} L'$, induces a map

$$f_S : S(L') \longrightarrow S(L), \text{ given by } P \mapsto f^{-1}(P),$$

such that for all $a \in L$, ${f_S}^{-1}(S_a) = S_{f(a)}$. Further, if $L' \xrightarrow{g} L''$ is a lattice morphism, then $(f \circ g)_S = f_S \circ g_S$.
f) A ring homomorphism, $R \xrightarrow{f} R'$, induces a map

$$f_Z : Spec(R') \longrightarrow Spec(R), \text{ given by } P \mapsto f^{-1}(P),$$

such that for all $r \in R$, ${f_Z}^{-1}(Z_r) = Z_{f(r)}$. Further, if $R' \xrightarrow{g} R''$ is a ring homomorphism, then $(f \circ g)_Z = f_Z \circ g_Z$.

[1] Read r is congruent to s modulo I.
[2] Notation as in 19.2.

PROOF. a) is immediate from the definitions of prime filter and prime ideal. In the lattice case, primeness is needed only to prove that $S_{(*)}$ preserves joins.
b) By (a)

$$a \leq b \;\text{ iff }\; a = a \wedge b \;\Rightarrow\; S_a = S_{a\wedge b} = S_a \cap S_b \;\Rightarrow\; S_a \subseteq S_b.$$

For the converse, if $a \leq b$ is false, then the filter $a^{\rightarrow}$ and the ideal $b^{\leftarrow}$ are disjoint; by 19.1.(a), $P \in Spec(R)$, such that $a \in P$ and $b \notin P$, that is, $P \in (S_a - S_b)$. (c) and (d) follow from 19.4.
e) By Lemma 4.22.(c) f_S is indeed a map from $S(L')$ to $S(L)$. The assertion about composition is immediate. For $a \in L$, we have
$Q \in {f_S}^{-1}(S_a)$ iff $f_S(Q) \in S_a$ iff $f^{-1}(Q) \in S_a$ iff $fa \in Q$ iff $Q \in S_{fa}$.
f) It is clear that the inverse image of a proper prime ideal by a ring homomorphism is a proper prime ideal, and that composition behaves as asserted. The argument to prove that ${f_Z}^{-1}(Z_r) = Z_{f(r)}$ is exactly as in (e), with $\notin$ substituted for $\in$ in the appropriate places. □

By Proposition 19.5.(a),

$$\{S_a\}_{a\in L} \quad \text{and} \quad \{Z_r\}_{r\in R}$$

are basis for topologies on $S(L)$ and $Spec(R)$, respectively, called the **Stone topology** on $S(L)$ and the **Zariski topology** on $Spec(R)$.

By items (e) and (f) in Proposition 19.5, the rules [3]

$$\text{(Stone)} \qquad \begin{cases} S : \boldsymbol{\mathcal{D}} \longrightarrow \mathbf{Top} \\ L \longmapsto S(L) \\ f \in [L,\, L'] \;\mapsto\; f_S \in [S(L'),\, S(L)] \end{cases}$$

$$\text{(Spec)} \qquad \begin{cases} Spec : \mathbf{CR} \longrightarrow \mathbf{Top} \\ R \longmapsto Spec(R) \\ f \in [R,\, R'] \;\mapsto\; f_Z \in [Spec(R'),\, Spec(R)] \end{cases}$$

are contravariant functors, where **Top** is the category of topological spaces and **CR** is the category of commutative rings with identity; S is the *Stone space functor* and *Spec* is the *Zariski spectrum functor*. As will be seen later, these functors take their values in a subcategory of **Top**, that of spectral spaces and spectral maps.

Since η is the intersection of all prime ideals in a ring R, we get

COROLLARY 19.6. *Let η be the ideal of nilpotent elements in a ring R and let $\pi_\eta : R \longrightarrow R/\eta$ be the quotient homomorphism. Then, $(\pi_\eta)_Z =_{def} \eta_*$ is a homeomorphism of $Spec(R/\eta)$ onto $Spec(R)$.*

PROOF. Since η_* is continuous, it suffices to check that it is open and bijective (1.18).
η_* is a bijection : By the fundamental theorem of homomorphisms for rings (see [**2**], Chapter 1), there is a bijective correspondence between ideals in R/η and ideals in R containing η, given by

[3] $[A,\, B]$ is the set of morphisms from A to B; see 16.1.

$$J \subseteq R/\eta \;\mapsto\; {\pi_\eta}^{-1}(J) \subseteq R, \quad \text{with inverse} \quad K \subseteq R \;\mapsto\; K/\eta.$$

Hence, it is sufficient to verify that the above correspondence preserves primeness. Given $P \in Spec(R)$, P/η is a proper ideal in R/η. Suppose $r/\eta \cdot s/\eta \in P/\eta$; hence, there is $t \in P$, such that $(rs - t) \in \eta \subseteq P$ and so $rs \in P$. Since P is prime, $r \in P$ or $s \in P$, and P/η is a prime ideal. Since inverse image preserves primeness, η_* is indeed a bijection.

$\underline{\eta_* \text{ is open}}$: It is enough to show that for $r \in R$, $\eta_*(Z_{r/\eta})$ is open in $Spec(R)$. We prove, in fact, that it is equal to Z_r. For $(r/\eta) \in R/\eta$, we have,

$$\begin{aligned}\eta_*(Z_{r/\eta}) &= \{\eta_*(Q) : Q \in Spec(R/\eta) \text{ and } (r/\eta) \notin Q\} \\ &= \{P \in Spec(R) : (r/\eta) \notin P/\eta\} \\ &= \{P \in Spec(R) : r \notin P\} = Z_r,\end{aligned}$$

ending the proof. □

Whenever a lattice has additional structure, this is reflected in its Stone space. Recall that $\overline{A}$ is the *closure* of the subset A in a topological space.

PROPOSITION 19.7. *Let L be a distributive lattice, $a, b \in L$ and $A \subseteq L$.*

a) If $a \to b = max\,\{x \in L : x \wedge a \leq b\}$ exists in L, then

$$S_{a\to b} = S_a \to S_b,$$

where the last $\to$ is that in $\Omega(S(L))$. If L is a Heyting algebra, the map $a \longmapsto S_a$ embeds L as a sub-HA of $\Omega(S(L))$.

b) If $a \in B(L)$, then S_a is clopen in $S(L)$ and $S_a^c = S_{\neg a}$.

c) If L is a BA, $S(L)$ is a totally disconnected Hausdorff space.

d) If $\bigvee A$ exists in L, then $\bigcup_{a\in A} S_a \subseteq S_{\bigvee A}$. Moreover,

(1) *If L is a $[\wedge, \bigvee]$-lattice, then $S_{\bigvee A} \subseteq \overline{\bigcup_{a\in A} S_a}$.*

(2) *If L is a Boolean algebra, then $S_{\bigvee A} = \overline{\bigcup_{a\in A} S_a}$.*

e) If $\bigwedge A$ exists in L, then $S_{\bigwedge A} = \mathrm{int}\,(\bigcap_{a\in A} S_a)$.

PROOF. a) Since $a \wedge (a \to b) \leq b$, 19.5.(a) entails

$$S_a \cap S_{a\to b} \subseteq S_b.$$

Hence, $S_{a\to b} \subseteq S_a^c \cup S_b$, and so $S_{a\to b} \subseteq \mathrm{int}\,(S_a^c \cup S_b) = S_a \to S_b$ in $S(L)$. To prove the reverse containment, suppose $S_c \subseteq S_a^c \cup S_b$; then

$$S_a \cap S_c = S_{a\wedge c} \subseteq S_b,$$

and 19.5.(b) yields $a \wedge c \leq b$. Then, $c \leq (a \to b)$, and so $S_c \subseteq S_{a\to b}$. Since the set of S_c, $c \in L$, is a basis of opens in $S(L)$, it follows that $\mathrm{int}\,(S_a^c \cup S_b) = S_{a\to b}$, as needed. By item (a) and (b) in 19.5, the map $a \in L \mapsto S_a \in \Omega(S(L))$ is a lattice embedding. By what has just been proven, if L is a HA, it will be a HA-embedding.

b) An immediate consequence of 19.5.(a).

c) If L is a BA, given distinct prime filters, F, G in L, 5.12 furnishes $a \in L$, such that $a \in F$ and $\neg a \in G$. Thus, $F \in S_a$, $G \in S_{\neg a}$ with $S_a \cap S_{\neg a} = \emptyset$, and $S(L)$ is Hausdorff. It follows immediately from (b) that $\{S_a : a \in L\}$ is a basis of clopens for the Stone topology in $S(L)$, showing that it is totally disconnected (Definition 18.1).

d) The first assertion follows from 19.5.(b). If L is a $[\wedge, \bigvee]$-lattice, to show that $S_{\bigvee A} \subseteq \overline{\bigcup_{a\in A} S_a}$, it suffices to prove that, for $P \in S_{\bigvee A}$ and $b \in P$, $S_b \cap \bigcup_{a\in\ A} S_a \neq \emptyset$. Since P is a proper filter and both b and $\bigvee A$ are in P, we get $b \wedge \bigvee A = \bigvee_{a\in\ A} (b \wedge a) \neq \bot$. Thus, there is $a \in A$, such that $b \wedge a \neq \bot$. Hence, $S_b \cap S_a \neq \emptyset$, as needed to verify (1). For (2), recall that a BA is a $[\wedge, \bigvee]$-lattice and every prime filter is maximal (8.7, 5.14). If $P \in \overline{\bigcup_{a\in\ A} S_a}$, then the $[\wedge, \bigvee]$-law and the fact that P is proper, imply that $P \cup \{\bigvee A\}$ has the fip. Since P is maximal, this forces $\bigvee A \in P$. Therefore, $\overline{\bigcup_{a\in\ A} S_a} \subseteq S_{\bigvee A}$.
e) Is an immediate consequence of the fact the collection S_a, $a \in L$, is a basis for $S(L)$ and Proposition 19.5.(b). $\square$

As observed in the proof of 19.7.(a), 19.5 entails that any distributive lattice is isomorphic to a sublattice of parts of a set : the map $a \mapsto S_a$ is a lattice isomorphism from L into $2^{S(L)}$; by 19.7.(e), this mapping is in fact an injective $[\vee, \bigwedge]$-morphism from L into $\Omega(S(L))$. Moreover, when L is a HA, this $[\vee, \bigwedge]$-morphism is also a HA-morphism. Another consequence of these results is a famous Theorem of M. Stone : every Boolean algebra is isomorphic to a subalgebra of parts of a set. We shall return to this theme, registering the full statement Stone's Theorem with due pomp and circumstance (Theorem 20.5).

Proposition 19.8. *Let L be a lattice and let R be a ring. The Stone space of L and the Zariski spectrum of R are* T0 *compact spaces and for all $a \in L$ and $r \in R$, S_a and Z_r are compact opens. Moreover, in both cases, the set of compact opens is a* **sublattice** *of the frames $\Omega(S(L))$ and $\Omega(Spec(R))$, respectively.*

Proof. If $P \neq Q$ are prime filters in $S(L)$, then $P \vartriangle Q \neq \emptyset$; hence, there is $a \in L$, such that either $P \in S_a$ and $Q \notin S_a$, or $Q \in S_a$ and $P \notin S_a$. Thus, $S(L)$ is T0. The argument for $Spec(R)$ is analogous.

Since $S(L) = S_\top$ and $Spec(R) = Z_1$, we shall show that each S_a and each Z_r, $a \in L$ and $r \in R$, is compact. In both cases it is enough to prove that coverings by basic opens have finite subcoverings.

For $a \in L$, suppose that b_i, $i \in I$, are such that $S_a \subseteq \bigcup_{i\in I} S_{b_i}$. Let K be the ideal generated by the b_i, $i \in I$, that is (Lemma 3.13.(b))

$$K = \{c \in L : c \leq \bigvee_{j\in\ J} b_j,\ J \subseteq_f I\}.$$

If the filter $a^{\rightarrow}$ had empty intersection with I, Theorem 19.1.(a) would yield a prime filter P, such that $a \in P$ and $b_i \notin P$, $i \in I$, an impossibility since the union of the S_{b_i} covers S_a. Therefore, $a \in K$, and there is a finite $J \subseteq I$, such that $a \leq \bigvee_{j\in J} b_j$. But then, $S_a \subseteq \bigcup_{j\in\ J} S_{b_j}$, verifying that S_a is compact.

For the ring case, suppose $\{r\} \cup \{s_i : i \in I\} \subseteq R$ are such that $Z_r \subseteq \bigcup_{i\in I} Z_{s_i}$. Let K be the ideal generated by the s_i, that is,

$$K = \{t \in R : t = \textstyle\sum_{j\in J} \alpha_j s_j, \text{ with } J \subseteq_f I \text{ and } \alpha_j \in R\}.$$

If $\{r^n : n \in \mathbb{N}\}$ had empty intersection with K, 19.1.(b) yields a prime ideal P, such that $s_i \in P$, while $r \notin P$, which is impossible because the Z_{s_i} cover Z_r. Thus, for some $n \in \mathbb{N}$ and some $J \subseteq_f I$, we have

$$r^n = \textstyle\sum_{j\in J} \alpha_j s_j,$$

with $\alpha_j \in R$. Now, if P is a prime ideal such that $r \notin P$, then for some $j \in J$, s_j cannot be in P. Otherwise, $r^n \in P$, forcing $r \in P$. Hence, $Z_r \subseteq \bigcup_{j\in J} Z_{s_j}$, as needed. To finish the proof, note that the finite union of compacts is always compact. The crucial point is that, both in the lattice and in the ring case, what has just been proven and 19.5.(a) assure that the finite intersection of compact opens is compact. □

The *proof* of 19.8 in the ring case yields

COROLLARY 19.9. *Let R be a ring and $\{b\} \cup \{c_j : j \in J\} \subseteq R$. The following are equivalent :*

(1) $Z_b \subseteq \bigcup_{j\in J} Z_{c_j}$;

(2) *b is in the radical of the ideal generated by $\{c_j : j \in J\}$.* □

A natural question is if S_a and Z_r are a Stone space and a Zariski spectrum, respectively. Although an affirmative answer follows from general results [4], we shall give explicit descriptions in each case.

PROPOSITION 19.10. *If L is lattice, S_a is homeomorphic to $S(a^{\leftarrow})$, $\forall\, a \in L$.*

PROOF. Note that the principal ideal $a^{\leftarrow}$ is a distributive lattice with the same bottom as L and $\top = a$. We may of course assume that $a \neq \top_L$, otherwise there is nothing to prove. Hence, $a^{\leftarrow}$ *is not* a sublattice of L. Nevertheless, the maps

$$\begin{cases} P \in S_a \longmapsto \alpha(P) = P \cap a^{\leftarrow}; \\ f : L \longrightarrow a^{\leftarrow},\ f(x) = x \wedge a, \end{cases}$$

are respectively, a function from S_a into $S(a^{\leftarrow})$ and a lattice morphism from L onto $a^{\leftarrow}$. If $f_S : S(a^{\leftarrow}) \longrightarrow (L)$ is the continuous Stone dual of f, note that

$$Im\, f_S \subseteq S_a$$

because any prime filter in $a^{\leftarrow}$ contains a (it is $\top$ in $a^{\leftarrow}$). Consequently, f_S is a continuous map from $S(a^{\leftarrow})$ into $S_a \subseteq S(L)$. We shall show that

$$\alpha \circ f_S = Id_{S(a^{\leftarrow})} \quad \text{and} \quad f_S \circ \alpha = Id_{S_a}. \tag{*}$$

If $R \in S(a^{\leftarrow})$, then

$$f_S(R) = f^{-1}(R) = \{x \in L : x \wedge a \in R\} \tag{**}$$

and so, it is easily established that

$$\alpha(f_S(R)) = f_S(R) \cap a^{\leftarrow} = R,$$

verifying the first equation in (*). For the second, let P be a prime filter in S_a. Then, since $a \in P$, (**) yields

$$\begin{aligned} f_S(P \cap a^{\leftarrow}) &= \{x \in L : x \wedge a \in (P \cap a^{\leftarrow})\} = \{x \in L : x \wedge a \in P\} \\ &= \{x \in L : x \in P\} = P, \end{aligned}$$

completing the proof of (*). Hence, α is a bijection, with a continuous inverse f_S. To end the proof it is now enough to check that α is continuous. If $b \leq a$, to make matters clearer, write (momentarily)

[4] Every spectral space is the Stone space of a distributive lattice and Zariski spectrum of a commutative ring with identity; see 20.5 for the lattice case.

$$T_b = \{R \in S(a^{\leftarrow}) : b \in R\}$$

for the basic compact open determined by b in $S(a^{\leftarrow})$. Then,

$$\alpha^{-1}(T_b) = S_b.$$

Indeed, by 19.5.(e), we have $f_S^{-1}(S_b) = T_{fb} = T_b$, and so the equations in (*) entail $\alpha^{-1}(T_b) = \alpha^{-1}(f_S^{-1}(S_b)) = (f_S \circ \alpha)^{-1}(S_b) = S_b$, as desired. $\square$

For the ring case, Proposition 9.39 and Corollary 9.13 yield

Corollary 19.11. *Let R be a ring.*
a) If S is a multiplicative subset of R and $\iota_S : R \longrightarrow RS^{-1}$ is the canonical ring homomorphism of 9.36.(*b*)*, then*

$$\iota_{SZ} : Spec(RS^{-1}) \longrightarrow Z_S = \bigcap_{s \in S} Z_s$$

is a homeomorphism.
b) If $Z_r \neq \emptyset$ in $Spec(R)$ and $S = \{r^n : n \geq 0\}$, then Z_r is homeomorphic to $Spec(RS^{-1})$.

Proof. Item (a) is immediate from 9.39.(b). For (b), note that if $Z_r \neq \emptyset$, then r cannot be nilpotent (9.13). Hence, $S = \{r^n : n \geq 0\}$ is a proper multiplicative set and item (a) guarantees that $Spec(RS^{-1})$ is homeomorphic to

$$Z_S = \bigcap_{s \in S} Z_s = Z_r,$$

as claimed. $\square$

Definition 19.12. *If X is a topological space, write $\Lambda(X) \subseteq \Omega(X)$, for the join semilattice of compact opens in X.*

Notice that if X is Hausdorff and compact, $\Lambda(X)$ is precisely the BA of clopens in X, $B(X)$. The following result will be useful in the next Chapter.

Corollary 19.13. *If L is a distributive lattice, then the map*

$$\sigma_L : L \longrightarrow \Lambda(S(L)), \textit{ given by } a \mapsto S_a,$$

is a lattice isomorphism.

Proof. By items (a) and (b) in 19.5, σ_L is an injective lattice morphism. If $U \in \Lambda(S(L))$, then, since $\{S_b : b \in L\}$ is a basis for the Stone topology, there is $B \subseteq_f L$ such that $U = \bigcup_{b \in B} S_b = S_{\bigvee B}$, showing that σ_L is a surjection. $\square$

In general, $\Lambda(X)$ is not a sublattice of $\Omega(X)$. The following example might be useful in understanding what might go wrong.

Example 19.14. Let $C = 2^{\mathbb{N}}$ be the Cantor space, as in Chapter 18. C is a Boolean space and so $\Lambda(C) = B(C)$ is a BA and a basis for its topology. Let $*$ be a point distinct from all those in C and write O for the sequence in C whose coordinates are all equal to zero. We define a topology on $X = C \cup \{*\}$ by the following rule :

$$U \subseteq X \text{ is open} \quad \text{iff} \quad \begin{cases} (i)\ U \cap C \text{ is open in } C \\ \qquad\qquad \text{and} \\ (ii)\ \text{If } * \in U, \text{ then } \exists\, V \in \nu_O \cap B(C), \\ \quad \text{such that } (V - \{O\}) \subseteq U. \end{cases}$$

It is readily checked that this defines a topology on X. The filter of neighborhoods of $*$, ν_*, is generated by sets of the form $V_* = (V - \{\mathrm{O}\}) \cup \{*\}$, where V is a clopen neighborhood of O in C. On the other hand, the filter of neighborhoods of all points in C remains unchanged. Hence, O and $*$ are **inseparable** in X, that is, if $V \in \nu_*$ and $U \in \nu_O$, then $V \cap U \neq \emptyset$. However, X is T1, because all points are closed.

If V is any clopen neighborhood of O in C, then both V and V_* are compact in X. For V this is clear, since the topology induced in C by X is the original topology. For V_*, let U_i, $i \in I$, be an open covering of V_* in X and select $k \in I$, such that $* \in U_k$. Note that the complement of U_k is *a compact set in* C; thus, its intersection with V_* is a compact set in C and so can be covered by a finite subfamily, $\mathcal{F}$, of the U_i, such that $U_k^c \cap V_* \subseteq \bigcup \mathcal{F}$. But then, $U_k \cup \mathcal{F}$ is a finite subcovering of V_*. In particular, X is a compact T1 space and C is a compact open set in X. In fact, the above reasoning shows that X is a compact T1 space, with a basis of compact opens. But the set of compact opens in X **is not** a sublattice of 2^X : X_* is compact open in X, C is compact open in X, but their intersection, $(C - \{\mathrm{O}\})$, is not compact : the family $\{(C - V) : V \in \nu_O\}$ is a collection of closed sets with the finite intersection property, whose intersection is empty.

Hence, C cannot be the spectrum of a ring, nor the Stone space of a lattice. $\square$

Definition 19.15. *A continuous map, $X \xrightarrow{f} Y$, of topological spaces is* **spectral** *iff the inverse image of every compact open set in Y is compact in X.*

Note that if X is compact and Y is Hausdorff, every continuous map from X to Y is spectral (1.24). Propositions 19.5 and 19.8 yield

Corollary 19.16. *If $g : R \longrightarrow T$ is a ring homomorphism and $f : L \longrightarrow P$ is a lattice morphism, then g_Z and f_S are spectral maps.* $\square$

Recalling Definition 2.43, Proposition 19.8 also yields

Corollary 19.17. *If L is a lattice and R is a ring, then $\Omega(S(L))$ and $\Omega(Spec(R))$ are compact algebraic frames.* $\square$

We may ask when $\Omega(S(L))$ and $\Omega(Spec(R))$ are zero-dimensional frames. The answers appear as Corollary 19.18 and Proposition 19.22.

Corollary 19.18. *Let L be a lattice and $S(L)$ its Stone space.*

a) For $a \in L$, the following are equivalent :

(1) *a is complemented in L.* (2) *S_a is clopen in $S(L)$.*

b) The following are equivalent :

(1) *L is a Boolean algebra.* (2) *$S(L)$ is Hausdorff.*

Proof. a) We already know that (1) $\Rightarrow$ (2) (Proposition 19.7.(c)). To prove (2) $\Rightarrow$ (1), suppose S_a is clopen in $S(L)$. Then, the same is true of its complement S_a^c; closed sets are quasi compact in any compact space and so S_a^c is a compact open in $S(L)$. Hence, there is $B \subseteq_f L$ such that $S_a^c = \bigcup_{b \in B} S_b$. But then, 19.5.(a) yields $S_a^c = S_{\bigvee B}$. From this relation we obtain

$$S_a \cup S_{\bigvee B} = S_\top \quad \text{and} \quad S_a \cap S_{\bigvee B} = S_\bot.$$

Another application of 19.5.(a) shows that $\bigvee B$ is the complement of a in L.

b) By Proposition 19.7.(c), we have (1) $\Rightarrow$ (2). For (2) $\Rightarrow$ (1), first observe that if $S(L)$ is Hausdorff, then compact sets are closed and so S_a is clopen in $S(L)$, for all $a \in L$. By (a), $B(L) = L$ and L is a BA. ☐

To answer the question of when $\Omega(Spec(R))$ is zero-dimensional, we introduce the BA of idempotents in a ring and prove a result concerning the lifting of idempotents.

Definition 19.19. *Let R be a ring. An element e in R is* **idempotent** *if $e^2 = e$. Write $B(R)$ for the set of idempotents in R.*
A commutative unitary ring R is **regular** [5] *iff for all $r \in R$, there is $e \in B(R)$ such that $(r) = (e)$.*

Remark 19.20. Notation as in 19.19, define operations $\wedge$ and $\vee$ on $B(R)$ as follows :

$$e \wedge f = ef \quad \text{and} \quad e \vee f = e + f - ef.$$

Then, $B(R)$ is a **Boolean algebra**, wherein $\neg e = (1 - e)$.

A ring homomorphism, $f : R \longrightarrow R'$, gives rise to a BA-morphism, $B(f) : B(R) \longrightarrow B(R')$, given by $e \mapsto f(e)$. In fact, the associations

$$R \mapsto B(R) \quad \text{and} \quad f \mapsto B(f),$$

define a functor from the category **CR**, of commutative rings with identity, to the category **BA** of Boolean algebras. ☐

Proposition 19.21. *Let R be a ring and I an ideal contained in η, that is every element of I is nilpotent. Let π_I be the quotient map from R to R/I. Then, $B(\pi_I)$ is a BA-isomorphism from $B(R)$ onto $B(R/I)$.*

Proof. To simplify notation, write q for $B(\pi_I)$. Recall that for each $e \in B(R)$, $q(e) = \pi_I e$. We must show that $q : B(R) \longrightarrow B(R/I)$ is a bijection.

If $e, f \in B(R)$, then for odd integers $n \geq 1$, $(e - f)^n = e - f$. If $(e - f)$ is nilpotent, then $e = f$, and q is injective. Now let x/I be an idempotent in R/I. We must show that there is $e \in B(R)$, such that $e/I = x/I$. Note that if $z, t \in R$, satisfy $(z + t) = 1$ and $zt = 0$, then both are idempotents, with t the complement of z in $B(R)$. We shall use this simple observation below.

Since $x/I \cdot x/I = x^2/I = x/I$ and $I \subseteq \eta$, we conclude that

$$x^2 - x = x(1 - x) \in I;$$

hence, this element is nilpotent, say $x^n(1 - x)^n = 0$. Then,

$$1 = 1^{2n-1} = (x + (1-x))^{2n-1} = \textstyle\sum_{j=0}^{2n-1} c_j x^{2n-j-1}(1-x)^j,$$

where the c_j are the usual binomial coefficients. Set

$$e = \textstyle\sum_{j=0}^{n-1} c_j x^{2n-j-1}(1-x)^j \quad \text{and} \quad f = \textstyle\sum_{j=n}^{2n-1} c_j x^{2n-j-1}(1-x)^j.$$

We have $e + f = 1$, while $ef = 0$, since all monomials of ef contain $x^n(1-x)^n$ as a factor. It remains to show that $e \equiv x$ mod I. Observe that $e \equiv x^{2n-1}$ mod

[5] Sometimes called *von Neumann regular.*

I, since $e - x^{2n-1}$ is a multiple of $x(1 - x)$. From $x \equiv x^2 \bmod I$, we get, by successively multiplying by x on both sides of this congruence, $x \equiv x^{2n-1} \bmod I$, which implies $e \equiv x \bmod I$, as desired. □

PROPOSITION 19.22. *Let R be a ring.*
a) For $U \subseteq Spec(R)$, the following are equivalent :
(1) *U is clopen in $Spec(R)$;*
(2) *There is $e \in B(R)$, such that $U = Z_e$.*
In particular, for $r \in R$, Z_r is clopen in $Spec(R)$ iff there is $e \in B(R)$ such that $(r/\eta) = (e/\eta)$ [6]*. Moreover, the map $e \in B(R) \mapsto Z_e$ is an isomorphism from $B(R)$ onto the BA of clopens in $Spec(R)$.*
b) The following are equivalent :
(1) *$Spec(R)$ is a T1 space;* (2) *$Spec(R)$ is Hausdorff;*
(3) *R/η is a regular ring.*

PROOF. a) If $e \in R$ satisfies $e(1 - e) \in \eta$, then Z_e is clopen in $Spec(R)$: from $e(1 - e) \in \eta$ and $e + (1 - e) = 1$, we get

$$Z_e \cap Z_{(1-e)} = \emptyset \quad \text{and} \quad Z_e \cup Z_{(1-e)} = Spec(R),$$

and so $Z_{(1-e)}$ is precisely Z_e^c. Since $(r/\eta) = (e/\eta)$ implies $\sqrt{r} = \sqrt{e}$ (19.4.(b)), Proposition 19.5 assures that (2) $\Rightarrow$ (1).
<u>(1) $\Rightarrow$ (2) :</u> Note that both U and U^c are open and compact. Thus, there are $S, T \subseteq_f R$ such that

$$U = \bigcup_{s\in S} Z_s \quad \text{and} \quad U^c = \bigcup_{t\in T} Z_t,$$

and so

$$(*) \qquad \begin{cases} \bigcup_{s\in S} Z_s \cup \bigcup_{t\in T} Z_t &= \quad Spec(R) \\ & \text{and} \\ \bigcup_{s\in S} Z_s \cap \bigcup_{t\in T} Z_t &= \quad \emptyset. \end{cases}$$

Let I and J be the ideals generated by S and T in R, respectively, i.e.,

$$I = \{\textstyle\sum_{s\in S} \alpha_s s : \alpha_s \in R\} \quad \text{and} \quad J = \{\textstyle\sum_{t\in T} \beta_t T : \beta_t \in R\}.$$

Fact. *$I \cap J \subseteq \eta$ and $I + J = R$.*
Proof. For the first relation, let $x \in I \cap J$; then,

$$x = \textstyle\sum_{t\in T} \beta_t t = \sum_{s\in S} \alpha_s s,$$

for some β_t, $\alpha_s \in R$. If $P \in Spec(R)$, there are two alternatives : if $T \subseteq P$, it is clear that $x \in P$; if T is not contained in P then, for some $t \in T$, $P \in Z_t$. The second equality in (*) entails $P \notin \bigcup_{s\in S} Z_s$, and so $S \subseteq P$. Consequently, $x = \sum_{s\in S} \alpha_s s \in P$, that is, x is in all prime ideals in R and so must be nilpotent.

To verify the equality in the statement, assume that $I + J$ is a proper ideal in R. Then, by 9.4.(b), there is a maximal ideal P in R such that $I + J \subseteq P$. Since every maximal ideal is prime (9.4.(c)), this contradicts the first equation in (*), ending the proof of the Fact.

By the Fact, there are $u \in I$ and $v \in J$ such that

$$(**) \qquad u + v = 1.$$

[6]Because by Proposition 19.5.(d), this condition is equivalent to $Z_r = Z_e$.

Therefore, $u^2 + uv = u$, with $uv \in I \cap J \subseteq \eta$. Hence, u is congruent to u^2 modulo η, that is, u/η is an idempotent in R/η. Similarly, v/η is an idempotent in R/η, in fact, $v/\eta = (1-u)/\eta$, the complement of u/η in $B(R/\eta)$. By 19.21, there is $e \in B(R)$ such that $e/\eta = u/\eta$; note that (**) implies that $(1-e)/\eta = v/\eta$. By Proposition 19.5.(d) we know that

$$Z_e \;=\; Z_u \quad \text{and} \quad Z_{1-e} \;=\; Z_v,$$

and so, to complete the proof it suffices to show that $U = Z_u$. Since $u \in I$, say $u = \sum_{s\in S} \alpha_s s$, it follows that if $P \in \mathit{Spec}(R)$, then

$$\begin{aligned} P \in Z_u \;&\Leftrightarrow\; u \notin P \;\Rightarrow\; \exists\, s \in S \text{ such that } s \notin P \\ &\Rightarrow\; P \in \textstyle\bigcup_{s\in S} Z_s = U, \end{aligned}$$

showing that $Z_u \subseteq U$. Analogously, $Z_v \subseteq U^c$. Now observe that

$$\begin{cases} uv \in \eta & \Rightarrow \quad Z_u \cap Z_v \;=\; \emptyset; \\ u + v = 1 & \Rightarrow \quad Z_u \cup Z_v \;=\; \mathit{Spec}(R). \end{cases}$$

Since $Z_u \subseteq U$ and $Z_v \subseteq U^c$, it is immediate that $Z_u = U$ and $Z_v = U^c$, as desired.

We omit the straightforward verification that the map $e \mapsto Z_e$ is a BA-isomorphism.

b) It is enough to verify that (2) $\Leftrightarrow$ (3) and that (1) $\Rightarrow$ (2). By Corollary 19.6, $\mathit{Spec}(R)$ is homeomorphic to $\mathit{Spec}(R/\eta)$. Since R/η has no nilpotent elements, we may assume, without loss of generality, that R is nilpotent free that is, $\eta = (0)$.
<u>(2) $\Rightarrow$ (3) :</u> If $\mathit{Spec}(R)$ is Hausdorff, then Z_r is clopen for all $r \in R$. Therefore, by (a), for each $r \in R$, there is an idempotent $e \in R$ such that $(r) = (e)$. But this is exactly the definition of R being regular.
<u>(3) $\Rightarrow$ (2) :</u> Suppose R is a regular ring. Given $P \neq Q$ in $\mathit{Spec}(R)$, assume, without loss of generality, that $Q - P \neq \emptyset$. Choose $r \in Q - P$ and $e \in B(R)$ such that (r) $= (e)$. Then, $e \in Q - P$; since $e(1-e) = 0$ and both primes in consideration are proper, it must be true that $e \notin P$ and $(1-e) \notin Q$. Thus, $P \in Z_e$, $Q \in Z_{(1-e)}$, and $\mathit{Spec}(R)$ is Hausdorff.
<u>(1) $\Rightarrow$ (2) :</u> Write Λ for the lattice of compact opens in $\mathit{Spec}(R)$. It follows from what we have just proven that $\mathit{Spec}(R)$ is Hausdorff iff Λ is a Boolean algebra, in fact a BA isomorphic to $B(R)$. Our tactic will be to show that Λ is a BA, by proving that all prime filters in Λ are maximal (Proposition 5.14). For a **prime filter** $F \subseteq \Lambda$, define

$$M_F = \{r \in R : Z_r \in F\}.$$

Fact. *Let F and G be prime filters in Λ.*
i) $F \subseteq G$ iff $M_F \subseteq M_G$.
ii) M_F is a multiplicative set in R, in fact the complement of a prime ideal in $\mathit{Spec}(R)$.
Proof. Clearly $F \subseteq G$ implies $M_F \subseteq M_G$. For the converse, let U be a compact open in F; then $U = \bigcup_{i=1}^{n} Z_{a_i} \in F$, and so, $Z_{a_i} \in F$, for some $i \leq n$. Thus, $a_i \in M_F \subseteq M_G$ and since U contains Z_{a_i}, we conclude that $U \in G$.
ii) First observe that for all $r, s \in R$, we have $Z_{r+s} \subseteq Z_r \cup Z_s$; for if $(r+s)$ does not belong to a prime ideal, then either r or s are not in it. Hence, M_F^c is closed under addition. Let $r \in M_F^c$ and $\alpha \in R$; since $Z_{\alpha r} = Z_\alpha \cap Z_r$ and $Z_r \notin F$, it

follows that $Z_{\alpha r} \notin F$, and so $\alpha r \in M_F^c$. We have shown that M_F^c is an ideal in R. It follows immediately from the definition of filter and 19.5.(a), that M_F is a multiplicative set containing 1. But then, M_F^c must be a proper prime ideal in R, ending the proof of the Fact.

To finish the proof, note that by 19.26.(d), $Spec(R)$ is T1 iff every prime ideal in R is maximal. If $F \subseteq \Lambda$ is not maximal, let G be a proper prime extension of F. Then, M_G^c is properly contained in M_F^c, contradicting the maximality of the latter. Hence, all prime filters in Λ are maximal and Λ is a BA, as desired. □

For future reference, we register

Corollary 19.23. *The Stone space of a Boolean algebra, and the prime spectrum of regular rings, are Boolean spaces.* □

We shall later see that a space is Boolean *iff* it is the Stone space of a BA.

Recall that a space is **connected** if it cannot be written as a disjoint union of non-empty opens. Hence,

Corollary 19.24. *$Spec(R)$ is connected iff $B(R) = \{0, 1\}$; $S(L)$ is connected iff $B(L) = \{\bot, \top\}$.*

Exercises

19.25. Let R be a ring, I, J ideals in R and $r \in R$. Notation is as in 19.2.
a) $F_{(r)} = Z_r^c$.
b) F_I is a closed set in $Spec(R)$ with the Zariski topology.
c) $I \subseteq J$ iff $F_J \subseteq F_I$.
d) $F_I = F_J$ iff $\sqrt{I} = \sqrt{J}$.
e) There is natural and order reversing bijective correspondence between the closed sets in $Spec(R)$ and radical ideals in R.
f) Describe the relations between closed sets, open sets and ideals in $S(L)$. □

19.26. Let L be a lattice and R a ring.
a) If P, $Q \in S(L)$, then $P \in \overline{\{Q\}}$ iff $P \subseteq Q$.
b) If P, $Q \in Spec(R)$, then $P \in \overline{\{Q\}}$ iff $Q \subseteq P$.
c) The following are equivalent for a distributive lattice L :
(1) $S(L)$ is a T1 space. (2) L is a Boolean algebra.
(3) Every prime ideal in L is maximal. (4) $S(L)$ is a Hausdorff space.
d) Let $MSpec(R)$ be the set of maximal ideals in $Spec(R)$ and $mS(L)$ be the set of minimal prime filters in $S(L)$.
(1) A point in $Spec(R)$ ($S(L)$) is closed iff it is in $MSpec(R)$ (resp., $mS(L)$).
(2) $MSpec(R)$ and $mS(L)$ are dense compact sets in $Spec(R)$ and $S(L)$, respectively, which are $T1$ in the induced topology.
(3) What can be said about maximal filters in $S(L)$ and minimal prime ideals in $Spec(R)$? □

CHAPTER 20

Spectral Spaces and Stone Duality

In order to unify the topological properties that were disclosed in the preceding chapter we introduce

Definition 20.1. *A topological space T is* **spectral** *iff*
[*spec* 1] *: T is* T0 *and quasi-compact.*
[*spec* 2] *: The set of quasi-compact opens in T, $\Lambda(T)$, is a sublattice and a basis of $\Omega(T)$.*
[*spec* 3] *: Every non-empty irreducible closed set has a generic point.*

Definition 12.8 describes the concepts of irreducibility and generic point. In fact, many of the results of Chapter 12 will be used here.

Recall (19.15) that a continuous map is *spectral* if the inverse image of a quasi-compact open is quasi-compact. Spectral spaces and spectral maps constitute a category, written **Spec**. Since any continuous function between Boolean spaces is spectral, **BTop**, the category of Boolean spaces is a full subcategory of **Spec**.

By 12.12, every spectral space is sober. In fact, spectral is the same as sober and the set of quasi-compact opens is both a sublattice and a basis of its topology.

Theorem 20.2. *The Stone space of a distributive lattice and the Zariski spectrum of a commutative ring are spectral spaces. If L is a distributive lattice and R is a commutative ring*
a) $S(L)$ is Hausdorff iff $S(L)$ is T1 *iff L is a BA.*
iff $S(L)$ is a Boolean space.
b) $Spec(R)$ is Hausdorff iff $Spec(R)$ is T1 *iff R is a regular ring*
iff every prime ideal in R is maximal
iff $Spec(R)$ is a Boolean space.

Proof. By 19.8, 19.18 and 19.22, the only condition still to verify is that irreducible closed sets have generic points. For the lattice case, the method of proof is the same as that of Proposition 12.11. Therefore, we leave the details of this case to the reader and turn to the ring case.

Let K be a non-empty irreducible closed set in $Spec(R)$ and define $P = \bigcap K$. Suppose $rs \in P$; then for all $Q \in K$, $Q \in Z_r^c \cup Z_s^c$. The irreducibility of K requires that $K \subseteq Z_r^c$ or $K \subseteq Z_s^c$. Hence, either r or s belongs to all primes in K, i.e., is in P. This proves that P is prime, as well as that $K \subseteq \overline{\{P\}}$ (19.26.(b)). Now, note that a basic open Z_r is disjoint from K iff $r \in P = \bigcap K$. Thus, every basic open containing P has non-empty intersection with K. Hence, $P \in \overline{K} = K$. $\square$

The reader will find in [**67**] many interesting examples of spectra of rings.

REMARK 20.3. In Chapter 19 we defined the Stone space functor,

$$S : \boldsymbol{\mathcal{D}} \longrightarrow \mathbf{Spec}.$$

By Corollary 19.13, if L is a distributive lattice, we have a natural isomorphism $\sigma_L : L \longrightarrow \Lambda(S(L))$, given by $a \;\mapsto\; S_a$. We now construct a contravariant functor

$$\Lambda : \mathbf{Spec} \longrightarrow \boldsymbol{\mathcal{D}},$$

as follows : if X is a spectral space and $X \xrightarrow{f} Y$ is a morphism of spectral spaces, define

$*$ $\Lambda(X)$ is the distributive lattice of quasi-compact opens in X;
$*$ $\Lambda(f) : \Lambda(Y) \longrightarrow \Lambda(X)$ is given by $U \;\mapsto\; f^{-1}(U)$.

The fact that f is spectral guarantees that this is well defined, while the preservation of all set theoretic operations by inverse image assures that $\Lambda(f)$ is a lattice morphism. Clearly,

$$\Lambda(Id_X) = Id_{\Lambda(X)} \quad \text{and} \quad \Lambda(f \circ g) = \Lambda(g) \circ \Lambda(f). \qquad \square$$

PROPOSITION 20.4. *For a spectral space X, define*

$$\lambda_X : X \longrightarrow S(\Lambda(X)), \textit{ by } \;\lambda_X(x) = \nu_x \cap \Lambda(X),$$

that is, $\lambda_X(x)$ is the prime filter of quasi-compact open neighborhoods of x in X. Then, λ_X is a homeomorphism of X onto $S(\Lambda(X))$, such that for all compact open U in X, $\lambda_X(U) = S_U$.

PROOF. Since X is T0 and $\Lambda(X)$ is a basis for the topology on X, it is clear that λ_X is injective; it remains to show that it is surjective, continuous and open. For surjectivity, let F be a prime filter in $\Lambda(X)$. Let G be the filter generated by F in $\Omega(X)$, that is,

$$G = \{U \in \Omega(X) : \exists\, V \in F \text{ such that } V \subseteq U\}.$$

Fact. *G is a* **point** *in* $\Omega(X)$ (12.1).

Proof. Suppose $S \subseteq \Omega(X)$, satisfies $\bigcup S \in G$. Fix a compact $V \in F$, such that $V \subseteq \bigcup S$. For each $U \in S$, we may write $U = \bigcup_{k \in I_U} W_k$, with $W_k \in \Lambda(X)$, for all $k \in I_U$. Therefore,

$$\bigcup S = \bigcup_{k \in \mathcal{I}} W_k,$$

with $\mathcal{I} = \bigcup_{U \in S} I_U$. By compactness, there are $k_1, \ldots, k_n \in \mathcal{I}$ with $V \subseteq \bigcup_{i=1}^{n} W_{k_i}$. Thus, $\bigcup_{i \leq n} W_{k_i} \in F$; since F is prime, there is $i \leq n$ such that $W_{ki} \in F$. If we select $U \in S$, such that $W_{k_i} \subseteq U$, then $U \in G$, as needed.

Since X is sober (12.6), there is $x \in X$, such that $\nu_x = G$; then, $\nu_x \cap \Lambda(X) = F$, showing that λ_X is onto $S(\Lambda(X))$.

For $U \in \Lambda(X)$, we have $S_U = \{F \in S(\Lambda(X)) : U \in F\}$. If $x \in U$, then it is clear that $U \in \nu_x \cap \Lambda(X)$, that is $\lambda_X(x) \in S_U$. Hence, $\lambda_X(U) \subseteq S_U$. The reverse containment follows from the fact that λ_X is surjective. For if $F \in S_U$, there is $x \in X$ such that $\lambda_X(x) = F = \nu_x \cap \Lambda(X)$, and so $x \in U$. Since λ_X is bijective, the equality $\lambda_X(U) = S_U$, U compact open in X, implies that it is continuous and open, ending the proof. $\square$

Our discussion has led to the fundamental

THEOREM 20.5. (M. Stone) *The functors S and Λ establish a* duality [1] *between the categories $\boldsymbol{\mathcal{D}}$ and* **Spec**, *which restricts to a duality between the categories* **BA** *and* **BTop**. *There are natural equivalences*

$$\sigma : Id_{\boldsymbol{\mathcal{D}}} \longrightarrow \Lambda \circ S \quad \text{and} \quad \lambda : Id_{\boldsymbol{Spec}} \longrightarrow S \circ \Lambda,$$

that is

[lat] *: For all $L \in Ob(\boldsymbol{\mathcal{D}})$, $\sigma_L : L \longrightarrow \Lambda(S(L))$ is an isomorphism. For all lattice morphisms $L \xrightarrow{f} P$ the diagram below left is commutative.*

$$\begin{tikzcd} L \arrow[r, "\sigma_L"] \arrow[d, "f"'] & \Lambda(S(L)) \arrow[d, "\Lambda(f_S)"] \\ P \arrow[r, "\sigma_P"'] & \Lambda(S(P)) \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\lambda_X"] \arrow[d, "f"'] & S(\Lambda(X)) \arrow[d, "(\Lambda(f))_S"] \\ Y \arrow[r, "\lambda_Y"'] & S(\Lambda(Y)) \end{tikzcd}$$

[spec] *: For all $X \in Ob($**Spec**$)$, $\lambda_X : X \longrightarrow S(\Lambda(X))$ is a homeomorphism. For all spectral maps, $f : X \longrightarrow Y$, the diagram above right is commutative.*

PROOF. The only statements that remain to be verified are the commutativity of the diagrams in [*lat*] and [*Spec*], which follow straightforwardly from the definitions of the concepts involved. $\square$

By Theorem 20.5, every BA is isomorphic to the BA of clopens of its Stone space and every Boolean space X is homeomorphic to the Stone space of a BA, namely $S(B(X))$.

We shall now develop some applications of Stone duality for Boolean algebras. We start with the construction of the **Booleanization** of a topological space. Since we are also interested in how closed sets behave in this process, we introduce

DEFINITION 20.6. *Let $F \subseteq K$ be non-empty closed sets in a topological space X. K is an* **essential extension** *of F, if there is a clopen U in X, such that $U \cap F = \emptyset$ and $U \cap K \neq \emptyset$.*

If X has a basis consisting of clopen sets, every extension is essential. On the other hand, in the real unit interval, every extension is inessential.

THEOREM 20.7. *Let X a topological space. Then, there is a Boolean space γX, together with a continuous function, $\gamma : X \longrightarrow \gamma X$, such that*
a) For all $u \in B(X)$, $\gamma^{-1}(S_u) = u$.
b) The image of γ is dense in γX.

[1] As in 16.25. The functor Λ is defined in 20.3; the functor S by condition (Stone) in page 180.

c) For all Boolean spaces Y and continuous maps $X \xrightarrow{f} Y$, there is a **unique** *continuous map, $\gamma f : \gamma X \longrightarrow Y$, such that the diagram below is commutative :*

$$\begin{tikzcd}
X \arrow[rr, "\gamma"] \arrow[dr, "f"'] & & \gamma X \arrow[dl, "\gamma f"] \\
& Y &
\end{tikzcd}$$

d) For F closed in X, define

$$\gamma F = \bigcap \{u : u \text{ is clopen in } \gamma X \text{ and } F \subseteq \gamma^{-1}(u)\}.$$

Then, $F \mapsto \gamma F$ is an increasing map from closed sets in X to closed sets in γX. Moreover, for closed sets $F \subseteq K$ in X, K is an essential extension of F iff $\gamma F \neq \gamma K$.

e) For all $u \in B(X)$, γu is (naturally homeomorphic to) $S_u \subseteq \gamma X$ [2].

Proof. Let $B(X)$ be the BA of clopens in X and let γX be the Stone space of $B(X)$. For $x \in X$, consider the set $\gamma(x) = \{u \in B(X) : x \in u\}$; clearly, $\gamma(x)$ is closed under finite intersections and contains all supersets of any of its elements. It is also clear that for all $v \in B(X)$, either v or its complement is in $\gamma(x)$. Thus, $\gamma(x)$ is an ultrafilter on $B(X)$, that is, a point of γX. Define $\gamma : X \longrightarrow \gamma X$ by $x \longmapsto \gamma(x)$. Recalling that the basic clopens of γX are given by

$$S_u = \{\mathcal{F} \in \gamma X : u \in \mathcal{F}\}, \quad u \in B(X),$$

then $\gamma^{-1}(S_u) = \{x \in X : u \in \gamma(x)\} = u$. Hence, γ is continuous and its image is dense in γX. Now let $f : X \longrightarrow Y$ be a continuous map from X to the Boolean space Y. Inverse image by f induces a BA-morphism from $B(Y)$ to $B(X)$; by Stone duality, this BA-morphism induces a continuous map, $\gamma f : \gamma X \longrightarrow Y$, such that

1. For all $\mathcal{F} \in \gamma X$, $\gamma f(\mathcal{F}) = \bigcap \{w \in B(Y) : f^{-1}(w) \in \mathcal{F}\}$.
2. For all $w \in B(Y)$, $(\gamma f)^{-1}(w) = f^{-1}(w)$.

Because Y is compact and has a basis of clopens, the intersection of clopens in the right-hand side of (1) has exactly one point. If $x \in X$, note that $f(x)$ belongs to all clopen w such that $x \in f^{-1}(w)$, or equivalently, $f^{-1}(w) \in \gamma(x)$. Thus, $\gamma f(\gamma(x)) = f(x)$. The fact that continuous functions with dense image in a Hausdorff space are epimorphisms in **HTop** (16.38.(c)) guarantees the uniqueness of γf, proving of items (a), (b) and (c). Item (d) is straightforward and left to the reader. For (e), if $u \in B(X)$, (a) and items (a) and (b) in 19.5 guarantee that γ^{-1} establishes an isomorphism between $B(u)$ and $B(S_u)$. Since S_u is a Boolean space, Stone duality (20.5) yields $\gamma u = S(B(u)) = S(B(S_u)) = S_u$, completing the proof. $\square$

Definition 20.8. *The space γX constructed in Theorem* 20.7 *is called the* **Booleanization** *of X.*

[2] Here u is considered as a topological space, with the induced topology.

Booleanization is a functor from **Top** to **Btop**. If $f : X \longrightarrow Y$ is a continuous function, then $f^*{}_{|B(Y)}$ is a BA morphism from $B(Y)$ to $B(X)$. Hence, the Stone dual of $f^*{}_{|B(X)}$ yields a continuous map

$$\gamma f : \gamma X \longrightarrow \gamma Y$$

that describes the operation of the functor γ on morphisms in **Top**.

Recall (1.17) that $\mathbb{C}(X, Y)$ is the set of continuous maps from X to Y. Let

[Cb] $\qquad \mathbb{C}_b(X, Y) = \{f \in \mathbb{C}(X, Y) : \overline{Im\, f} \text{ is compact in } Y\}.$

For $u \in \Omega(X)$, recall that $f_{|u} : u \longrightarrow Y$ is the restriction of f to u. Since $\overline{Im\, f_{|u}} \subseteq \overline{Im\, f}$ and closed subsets of a compact set are compact, $\cdot_{|u}$ is also map from $\mathbb{C}_b(X, Y)$ to $C_b(u, Y)$. The results in 1.24 entail that $\mathbb{C}_b(X, Y)$ is $\mathbb{C}(X, Y)$ whenever

$*$ Y is compact or $*$ X is compact and Y is Hausdorff.

As an application of the Booleanization functor, we prove

Proposition 20.9. *Let X be a topological space and Y a totally disconnected space* (18.1). *For each $v \in \Omega(\gamma X)$, the map*

$$\alpha_v : \mathbb{C}_b(v, Y) \longrightarrow \mathbb{C}_b(\gamma^{-1}(v), Y), \text{ given by } \alpha_v(f) = f \circ \gamma,$$

is an isomorphism, such that if $v' \subseteq v$ are opens in γX, the following diagram is commutative :

$$\begin{tikzcd}
\mathbb{C}_b(v, Y) \arrow[r, "\alpha_v"] \arrow[d, "\cdot_{|v'}"'] & \mathbb{C}_b(\gamma^{-1}(v), Y) \arrow[d, "\cdot_{|v'}"] \\
\mathbb{C}_b(v', Y) \arrow[r, "\alpha_{v'}"'] & \mathbb{C}_b(\gamma^{-1}(v'), Y)
\end{tikzcd}$$

In particular, for all $u \in B(X)$, $\mathbb{C}(S_u, Y)$ is isomorphic to $\mathbb{C}_b(u, Y)$.

Proof. Since Y is Hausdorff and the image of X by γ is dense in γX, it follows from 16.38.(c) that α_v is injective, for all $v \in \Omega(\gamma X)$.

$$\gamma^{-1}(v) \xrightarrow{\gamma} v \xrightarrow{f} Y.$$

It is also clear that the displayed diagram is commutative. Note that the arguments up to now work for **any** Hausdorff topological space in place of Y. It is to prove that α_v is *onto* that we use the remaining hypotheses.

Let $v \in \Omega(\gamma X)$; since γX is Boolean with a basis indexed by $B(X)$, there is $A \subseteq B(X)$ such that

$$v = \bigcup_{u \in A} S_u.$$

Hence, by 20.7.(a), $\gamma^{-1}(v) = \bigcup_{u \in A} u$. Now let $f \in C_b(\gamma^{-1}(v), Y)$; since a compact subset of a totally disconnected space is a Boolean space, f is a continuous map from $\gamma^{-1}(v)$ to the Boolean space $K = \overline{Im\, f}$. By items (c) and (e) in 20.7, for each $u \in A$, there is a *unique* continuous $\gamma(f_{|u}) : S_u \longrightarrow K$ such that the following diagram is commutative

$$\begin{tikzcd} u \arrow[rr, "\gamma"] \arrow[dr, "f_{|u}"'] & & S_u \arrow[dl, "\gamma(f_{|u})"] \\ & K & \end{tikzcd}$$

Consider the family

$$\mathcal{F} = \{\gamma(f_{|u}) : u \in A\}.$$

Since $S_u \cap S_{u'} = S_{u \cap u'}$ (19.5.(a)) and both $\gamma(f_{|u})$ and $\gamma(f_{|u'})$ are extensions of $f_{|u\cap u'}$ to $S_{u\cap u'}$, uniqueness entails $\gamma(f_{|u})_{|u\cap\, u'} = \gamma(f_{|u'})_{|u\cap u'}$, that is, the family $\mathcal{F}$ is a collection of compatible partial continuous maps, defined on opens, from v to K. Since continuity is a local property and v is the union of the S_u, it follows from 1.2 that there is a unique *continuous* function $\gamma f : v \longrightarrow K$, such that $\gamma f_{|u} = \gamma(f_{|u})$, for all $u \in A$. Since over each $u \in A$ the diagram above is commutative, the same property must hold in $\gamma^{-1}(v)$, and so $\gamma \circ \gamma f = f$, completing the verification that α_v is a bijection. The final assertion is a consequence of 20.7.(a) and the fact that $\mathbb{C}_b(S_u, Y) = \mathbb{C}(X, Y)$ because S_u is compact and Y is Hausdorff. $\square$

Proposition 20.9 applies, in particular, to the collection of all continuous maps from a space X to a set A with the discrete topology, whose image is finite in A.

Next, we prove a beautiful result, due to M. Hochster ([**28**]), that ties spectral spaces to Boolean spaces. This construction corresponds to the **constructible topology** on the spectra of commutative rings.

THEOREM 20.10. (Hochster) *For a spectral space X, define*

$$\mathcal{B} = \{U \cap V^c : U, V \in \Lambda(X)\},$$

where V^c is the complement of V in X. Then,
a) $\mathcal{B}$ is a basis for a **compact Hausdorff** *topology on X, with which X becomes a Boolean space, written X_c.*
b) The identity map $\iota : X_c \longrightarrow X$ is spectral; ι is a homeomorphism iff X is Boolean.
c) If $Y \xrightarrow{f} X$ is a spectral map and Y is Boolean, then $f : Y \longrightarrow X_c$ is continuous.

PROOF. (a) Since $\Lambda(X)$ is a sublattice of and a basis for $\Omega(X)$, $\mathcal{B}$ is closed under finite intersection and hence, a basis topology on X. Note that for $W \in \Lambda(X)$, $\{W, W^c\} \subseteq \mathcal{B}$, since $X \in \Lambda(X)$. Thus, the topology generated on X by $\mathcal{B}$ is Hausdorff and has a basis of clopen sets. Hence, the substantive part of (a) is showing that X is *compact* in newly defined topology.

In any topological space, compact sets are closed under finite unions. Furthermore, the intersection of a compact set and a closed set is compact (1.24.(b)). Hence, all elements of $\mathcal{B}$ as well as their complements, are compact in the original topology on X. Further, since finite intersections of complements of elements in $\mathcal{B}$ can be expressed as finite unions of members of $\mathcal{B}$, we get the following simple, but crucial

Fact 1. *All finite intersections of sets of the form $U \cup V^c$, U, V in $\Lambda(X)$, are compact in* (the original topology of) X.

The reader should remark the importance in all this of $\Lambda(X)$ being closed under finite intersections. Now, since $\mathcal{B}$ is a basis for the topology on X_c, every closed set in X_c is the intersection of a collection of closed sets (in X_c) of the form $U \cup V^c$, with $U, V \in \Lambda(X)$. Thus, to show that X_c is compact, it enough to verify that if A is a collection of sets of the form $U \cup V^c$, with U and V compact opens in X, possessing the finite intersection property, then $\bigcap A \neq \emptyset$. Such a set A will remain fixed throughout the remaining part of the argument. By Fact 1, the intersection of all finite subsets of A is compact in the original topology of X.

Define

$$T = \{F \subseteq X : F \text{ is } \textit{closed in } X \text{ and } F \cap \bigcap B \neq \emptyset, \text{ for all } B \subseteq_f A\},$$

partially ordered by inclusion. Clearly, T is not empty since $X \in T$. Let C be a chain in T. We show that the closed set $\bigcap C$ is in T. To see this, fix $W = \bigcap B$, $B \subseteq_f A$. For $F \in C$, $F \cap W$ is a non-empty closed set in W, and so, since C is a chain, the family

$$\{F \cap W : F \in C\}$$

has the fip. The compactness of W implies that $\bigcap C \cap \mathrm{W} \neq \emptyset$. By Zorn's Lemma (2.20), there is K minimal in T; K is closed in X and $K \cap W \neq \emptyset$, for all $W \in A$.

Fact 2. *K is irreducible in X.*

Proof. Suppose $K = F \cup G$, with F and G closed in X. If for some $B \subseteq_f A$, we had $G \cap \bigcap B = \emptyset$, then, it follows that *for all* finite subsets $B' \subseteq A$, we must have $F \cap \bigcap B' \neq \emptyset$. Since K is minimal with this property, it follows that $K = F$. The argument just presented is symmetric in F and G, proving that K is irreducible.

Since X is spectral, K has a generic point p, $K = \overline{\{p\}}$. The proof of (a) will be finished as soon as we show that $p \in W$, for all $W \in A$. Observe that for open U in X,

$$K \cap U \neq \emptyset \quad \text{iff} \quad p \in U.$$

Let $U \cup V^c \in A$; if $p \notin U$, then $K \cap U = \emptyset$, because U is open. Consider $F = K \cap V^c$; F is closed and non-empty, because $K \in T$. If B is a finite subset of A, set $W = \bigcap B$; then $W \cap (U \cup V^c)$ is the intersection of a finite subset of A and so, since K is in T and $K \cap U = \emptyset$, we must have

$$K \cap W \cap V^c = F \cap W \neq \emptyset.$$

Since B is an arbitrary finite subset of A, we conclude that $F \in T$. From $F \subseteq K$ and the minimality of K in T, it follows that $F = K$, and $p \in V^c$, verifying (a).

Part (b) is immediate from the construction. As for (c), if the map f from Y to X is spectral, then for all $U \in \Lambda(X)$, $f^{-1}(U)$ is compact open in Y, i.e., *clopen*. But this is just what is required for f to be continuous, when X is given the topology in (a). $\square$

The topology of Theorem 20.10 on X_c is called the **constructible topology** associated to the spectral space X. A consequence of the above is

Corollary 20.11. *If L is a distributive lattice, there is a BA B and lattice embedding, $\beta : L \longrightarrow B$, such that for all lattice morphisms, $f : L \longrightarrow B'$, with B' a BA, there is a* **unique** *lattice morphism, $g : B \longrightarrow B'$, such that $g \circ \beta = f$.*

$$\begin{tikzcd} L \arrow[r, "\beta"] \arrow[d, "g"'] & B \arrow[dl, "f"] \\ B' & \end{tikzcd}$$

Proof. Let B be the BA of clopens of $S(L)_c$; there is a natural lattice morphism, $\beta : L \longrightarrow B$, given by $a \in L \mapsto S_a$. Now Stone duality and Theorem 20.10 will complete the proof. $\square$

Recall that a topological space is a **Baire space** iff the intersection of any countable collection of dense open sets is dense. It is well known that locally compact Hausdorff spaces and complete metric spaces are Baire spaces. In general, T0 compact spaces are not Baire spaces. However, for spectral spaces we have

Corollary 20.12. *Every spectral space is a Baire space.*

Proof. By Theorem 20.10.(a), in a spectral space, any collection of compact opens with the finite intersection property has non-empty intersection. Let $\{O_n\}$, $n \in \mathbb{N}$, be a countable collection of dense opens in the spectral space X. We must show that $O = \bigcap O_n$ is dense in X. Let U be a non-empty open set in X. Since O_1 is open and dense, we can choose $W_1 \in \Lambda(X)$, such that $W_1 \subseteq O_1 \cap U$. Now, O_2 is a dense open set, and so has non-empty open intersection with W_1; select $W_2 \in \Lambda(X)$ such that $W_2 \subseteq W_1 \cap O_2$. By induction, we construct a decreasing sequence of non-empty compact opens in X, W_n, such that $W_n \subseteq \bigcap_{i=1}^{n} U \cap O_i$. But then, $\emptyset \neq \bigcap W_n \subseteq U \cap O$, as needed. $\square$

The interplay between the original topology on a spectral space and the constructible topology has interesting consequences. To explore some of these we introduce the following

Definition 20.13. *A subset E of a spectral X is* **pro-constructible** *if E is closed* **in the constructible topology of X**.

Recall from 20.10.(a) that for a spectral space X, X_c denotes X endowed with the constructible topology. As always, $\Omega(X)$ indicates the frame of opens in X.

REMARK 20.14. Let X be a spectral space.
a) Since the constructible topology in X is finer than its spectral topology, every closed set in the original spectral topology is pro-constructible.
b) Because X_c is a Boolean space, with a basis of clopens given by the set $\mathcal{B}$ in 20.10.(a), if $E \subseteq X$, the following are equivalent :

(1) E is pro-constructible;
(2) E is an intersection of clopens in X_c;
(3) E is a compact subset of X_c [3];
(3) There are compact opens **in $\boldsymbol{X}$**, $\{U_i : i \in I\}$, $\{V_i : i \in I\}$, such that $E \;=\; \bigcap_{i \in I} U_i \cup V_i^c$.

c) It is clear that the finite intersection of pro-constructible subsets of X is pro-constructible. Moreover, if E is pro-constructible and $F \subseteq E$ is closed in X_c, then F is pro-constructible. In particular, it follows from (a) that subsets of pro-constructible sets, that are closed in the spectral topology, are also pro-constructible. $\square$

PROPOSITION 20.15. *Let X be a spectral space.*

a) If $E \neq \emptyset$ is a pro-constructible subset of X, which is irreducible in the spectral topology [4], then E has a (unique) generic point.

b) If K is a pro-constructible subset of X, then, with the topology induced by X, K is a spectral space. Moreover, the constructible topology associated to the spectral space K is that induced by X_c on K.

PROOF. Let $\Lambda(X)$ be a basis of compact opens in X, that is a sublattice of $\Omega(X)$.
a) Let $\overline{E}$ be the closure of E *in the spectral topology* of X. By Lemma 12.10.(b), $\overline{E}$ is irreducible in X and so there is $p \in \overline{E}$ such that p is generic for $\overline{E}$. We contend that $p \in E$, which will complete the proof of (a). Assume otherwise; since E is closed in X_c and

$$\mathcal{B} \;=\; \{U \cap V^c : U,\, V \in \Lambda(X)\}$$

is a basis of clopens for the topology in X_c, there are $U,\, V \in \Lambda(X)$ such that

$$p \in U \cap V^c \quad \text{and} \quad U \cap V^c \cap E \;=\; \emptyset. \tag{I}$$

Because $p \in V^c$, a closed set in X, we get that $E \subseteq \overline{\{p\}} = \overline{E} \subseteq V^c$. Hence, the equality in (I) yields

$$\emptyset \;=\; U \cap V^c \cap E \;=\; U \cap E,$$

that is impossible because U is an open neighborhood of $p \in \overline{E}$. The uniqueness of p follows immediately from Lemma 12.10.(c).
b) It is clear that, with the topology induced by X, K is a $T0$ space, as well as that

$$\Lambda(X)_{|K} \;=_{def}\; \{U \cap K : U \in \Lambda(X)\}$$

[3] And consequently also in X.
[4] This is a subtle, but important point; since X_c is Boolean, its only non-empty irreducible subsets are its points.

is a basis and a sublattice of $\Omega(K) = \{W \cap K : W \in \Omega(X)\}$. We must show that for all $U \in \Lambda(X)$, $U \cap K$ is compact in K. Since K is closed in X_c, U is clopen in X_c and this last space is compact, it follows that $U \cap K$ is compact in X_c. Now, the fact that the constructible topology is finer than the spectral topology guarantees that $U \cap K$, with the topology induced by X, is also compact. In particular, K itself is a compact space.

It remains to check that if E is a non-empty irreducible closed set in K, then E has a generic point. By Remark 20.14.(c), we know that E is pro-constructible and so the desired conclusion follows immediately from (a). It is straightforward that the constructible topology associated to the topology induced by X on K is that induced by X_c on K, ending the proof. $\square$

As an application, we prove that certain limits of spectral spaces $-$ in particular, products and projective limits $-$ are spectral, and also determine the constructible topology on these limits. We start with

THEOREM 20.16. *If X_i, $i \in I$, is a family of spectral spaces, then $X = \prod_{i \in I} X_i$, with the product topology, is a spectral space. Moreover,*

$$X_c = \prod_{i \in I} X_{ic},$$

that is, the constructible topology on X is the product of the constructible topologies on each coordinate.

PROOF. Since each X_i is a compact space, it follows from Tychonoff's Theorem 1.28 that $X = \prod_{i \in I} X_i$, with the product topology, is a compact space; it is straightforward that X is $T0$; if $\Lambda(X_i)$ is the family of compact opens in X_i, $i \in I$, Exercise 1.32 guarantees that, with $\mathcal{L}(\alpha) = \prod_{i \in \alpha} \Lambda(X_i)$,

$$\mathfrak{L} = \{\mathfrak{p}(U) : U \in \mathcal{L}(\alpha) \text{ and } \alpha \in Fin(I)\}, \tag{*}$$

is a basis of compact opens for the product topology on X and a sublattice of $\Omega(X)$.

Let $\mathfrak{X} = \prod_{i \in I} X_{ic}$ be the product of the Boolean spaces that arise when the X_i are endowed with the constructible topology. Again by Tychonoff's Theorem, $\mathfrak{X}$ is compact and, in fact, a Boolean space, because by Exercise 1.32, the set

$$\mathfrak{B} = \{\mathfrak{p}(V) : v \in \mathcal{B}(\alpha) \text{ and } \alpha \in Fin(I)\}, \tag{**}$$

where $\mathcal{B}(\alpha) = \prod_{i \in \alpha} \mathcal{B}_i$ [5], is a basis for the product topology on $\mathfrak{X}$, consisting of clopen sets [6]. Set

$$\gamma : \mathfrak{X} \longrightarrow X, \text{ given by } \gamma(\xi) = \xi; \tag{***}$$

since the topology on $\mathfrak{X}$ is finer than that in X (because this is true for each coordinate), γ is a *continuous* map.

For $i \in I$, we shall write π_i both for the projection of X onto X_i and for the projection of $\mathfrak{X}$ onto X_{ic}.

[5] $\mathcal{B}_i = \{U \cap V^c : U, V \in \Lambda(X_i)\}$, as in Theorem 20.10.

[6] **As sets**, we have $X = \mathfrak{X}$; but it seemed advisable to give distinct names to distinct *topological spaces.*

Since X is compact, $T0$ and has a basis of compact opens that is a sublattice of $\Omega(X)$, to show that it is spectral we must prove that every non-empty irreducible closed set in X has a generic point. This is a consequence of the following:

FACT 20.17. *Let E be a non-empty irreducible subset of X and let $u \in X$. For each $i \in I$, let $u_i = \pi_i(u)$ and $E_i = \pi_i(E)$*
a) $u \in \overline{E} \quad \Leftrightarrow \quad \forall\, i \in I,\, u_i \in \overline{E_i}$.
b) If E is closed in X, then $\forall\, i \in I$, E_i is an irreducible pro-constructible subset of X_i.

Proof. a) The implication ($\Rightarrow$) is immediate from the continuity of each π_i. For the converse, assume, to obtain a contradiction, that $u \notin \overline{E}$; then, there is a finite subset α of I and for each $j \in \alpha$ an open set $U_j \subseteq X_J$, so that the open $U = \prod_{j\in\alpha} U_j \times \prod_{i\in I\setminus\alpha} X_i$ is a neighborhood of u, disjoint from E. For $j \in \alpha$, let $V_j = U_j \times \prod_{i\neq j} X_i$; then, $U = \bigcap_{j\in\alpha} V_j$ and so $E \subseteq \bigcup_{j\in\alpha} V_j^c$, a union of closed sets in X. Since E is irreducible, for some $k \in \alpha$ we have $E \subseteq V_k^c$, i.e., $E \cap V_k = \emptyset$; since $u \in V_k$, this contradicts the assumption $u \in \overline{E}$, establishing (a).

b) Since the topology on $\mathfrak{X}$ is finer than that of X, E is closed, and so compact in $\mathfrak{X}$. The continuity of $\pi_i : \mathfrak{X} \longrightarrow X_{ic}$ entails that $\pi_i(E)$ is compact in X_{ic}. Since X_{ic} is Hausdorff, $\pi_i(E)$ is closed in X_{ic} and thus pro-constructible in X_i. With respect to irreducibility, suppose that F_1, F_2 are closed sets in X_i such that

$$\pi_i(E) \;\subseteq\; F_1 \cup F_2.$$

Then, $E \;\subseteq\; \pi_i^{-1}(F_1) \cup \pi_i^{-1}(F_2)$; since E is irreducible in X, we conclude that

$$E \;\subseteq\; \pi_i^{-1}(F_1) \quad \text{or} \quad E \;\subseteq\; \pi_i^{-1}(F_2). \tag{****}$$

Hence, (****) and the surjectivity of π_i entails

$$\pi_i(E) \;\subseteq\; \pi_i(\pi_i^{-1}(F_1)) \;=\; F_1 \quad \text{or} \quad \pi_i(E) \;\subseteq\; \pi_i(\pi_i^{-1}(F_2)) \;=\; F_2,$$

and $\pi_i(E)$ is irreducible in X, completing the proof of the Fact.

Now let F be a non-empty irreducible closed in X. Fact 20.17.(b) and Proposition 20.15.(a) yield, for each $i \in I$, a generic point, $x_i \in \pi_i(F)$. Let $x \in X$ be the point whose coordinates are the $x_i \in \pi_i(F)$, $i \in I$. Recalling that $\overline{\pi_i(F)} = \overline{\{x_i\}}$, it follows readily from item (a) in Fact 20.17 that x belongs to F and is a generic point for F, establishing the spectrality of X.

To finish the proof we need to verify that $\mathfrak{X}$ is homeomorphic to X_c. In view of item (c) of Theorem 20.10 and the fact that a continuous bijection between compact Hausdorff spaces is a homeomorphism, it suffices to check that the continuous map γ in (***) above *is spectral*, which in turn reduces to verifying that if $\mathfrak{p}(U) \in \mathcal{L}$, then $\gamma^{-1}(\mathfrak{p}(U))$ is compact in $\mathfrak{X}$. But this is clear, since $\gamma^{-1}(\mathfrak{p}(U)) = \mathfrak{p}(U) \in \mathfrak{B}$ (as in (**)), in fact a clopen set in $\mathfrak{X}$, ending the proof. $\square$

Theorem 20.16 and Proposition 20.15 yield

THEOREM 20.18. *Let $\langle I, \leq\rangle$ be a poset and let $D : I \longrightarrow$ **Spec** be an I-diagram of spectral spaces, that is, a family $\{X_i : i \in I\}$ of spectral spaces and spectral maps, $f_{ij} : X_i \longrightarrow X_j$ ($i \leq j$ in I), satisfying for all $i \leq j \leq k$ in I*

$$\begin{array}{lcl} (1)\ f_{ii} & = & Id_{X_i}; \\ (2)\ f_{ik} & = & f_{jk} \circ f_{ij}. \end{array} \qquad \begin{tikzcd} X_i \arrow[rr, "f_{ij}"] \arrow[dr, "f_{ik}"'] & & X_j \arrow[dl, "f_{jk}"] \\ & X_k & \end{tikzcd}$$

Then, $T = \varprojlim D$ exists in **Spec** *and the constructible topology on T is the limit of the I-diagram whose vertices are the X_{ic} and whose arrows are the f_{ij} $(i \leq j)$.*

Proof. The proof will show that in fact the limit of D, *in the category of topological spaces*, is spectral and hence the limit of D in **Spec**.

Let $X = \prod_{i \in I} X_i$; by 20.16, X is spectral and that $X_c = \prod_{i \in I} X_{ic}$. Recalling Corollary 17.12, let

$$T \;=\; \{z \in X : \forall\, i \leq j \text{ in } I,\ f_{ij}(\pi_i(\mathrm{z})) \;=\; \pi_j(z)\}, \tag{I}$$

endowed with the topology induced by $X = \prod_{i \in I} X_i$. We now prove

Fact 20.19. *T is pro-constructible in X.*

Proof. It must be verified that T is closed in X_c. Let $x = (x_i)_{i \in I}$ be a point in X_c, outside T. Hence, for some $i \leq j$ in I, we have that

$$x_j \;=\; \pi_j(x_j) \;\neq\; f_{ij}(x_i) \;=\; f_{ij}(\pi_i(x)).$$

Since X_{jc} is a Hausdorff space, there are disjoint clopens in X_{jc}, V_j, V'_j, such that $x_j \in V_j$ and $f_{ij}(x_i) \in V'_j$. The continuity of f_{ij} guarantees that $V_i = f_{ij}^{-1}(V'_j)$ is a clopen set in X_{ic}. Let $\alpha = \{i,\, j\}$ and consider, with notation as in (**) in the proof of 20.16, the clopen $\mathfrak{p}(U) \in \mathcal{B}$, where $U = \langle V_i, V_j \rangle$. Then, $\mathfrak{p}(U)$ is a clopen neighborhood of x in X_c, that is disjoint from T. Indeed, clearly $x \in \mathfrak{p}(U)$; if $y = (y_i)_{i \in I} \in \mathfrak{p}(U)$, then $y_i \in V_i$, while $y_j \in V_j$. Hence, $f_{ij}(y_i) \in V'_j$, which is disjoint from V_j; in particular, $f_{ij}(y_i) \neq y_j$, showing that $y \notin T$. The above reasoning has shown that T^c is open in X_c, as needed to establish Fact 20.19.

It follows immediately from Proposition 20.15 that T, with the topology induced by X, is a spectral space. Moreover, the same result guarantees that T_c is the space resulting from endowing T with the topology of X_c.

Consider the systems

$$\begin{cases} \mathcal{T} & = \quad \langle T;\ \{\pi_{i_{|T}} : T \longrightarrow X_i\};\rangle \\ \mathcal{T}_c & = \quad \langle T_c;\ \{\pi_{i_{|T_c}} : T_c \longrightarrow X_{ic}\}.\rangle \end{cases}$$

The proof will be completed as soon as it is verified that

$$\mathcal{T} \;=\; \varprojlim D \quad \text{and} \quad \mathcal{T}_c \;=\; \varprojlim D_c, \tag{II}$$

where D_c is the I-diagram having as vertices the X_{ic} and as arrows the f_{ij}. We shall prove the first equality in (II); the proof of the second is analogous and left to the reader. The first step is to show that $\mathcal{T}$ is a cone over D, that is, for $i \leq j$ in I, the diagram below left is commutative :

$$\begin{tikzcd} T \arrow[r, "\pi_i|_T"] \arrow[dr, "\pi_j|_T"'] & X_i \arrow[d, "f_{ij}"] \\ & X_j \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r, "g_i"] \arrow[dr, "g_j"'] & X_i \arrow[d, "f_{ij}"] \arrow[dl, phantom, "\text{(Y)}"] \\ & X_j \end{tikzcd}$$

For $x = (x_i)_{i \in I} \in T$, we have, by the definition of T in (I),

$$f_{ij}(\pi_i(x)) \;=\; \pi_j(x) \;=\; x_j,$$

and the diagram above left is indeed commutative. Next, let

$$\mathcal{Y} \;=\; \langle\, Y;\ \{g_i : Y \longrightarrow X_i\}\,\rangle$$

be a cone over D in **Spec**, i.e., Y is a spectral space, the g_i are spectral maps and for all $i \leq j$ in I, diagram (Y) above right is commutative. For $y \in Y$, define $f : Y \longrightarrow T$ by

$$f(y)(i) \;=\; g_i(y),$$

that is, $f(y)$ is the I-sequence whose i^{th}-coordinate is $g_i(y)$. To check that $f(y)$ is indeed in T, assume that $i \leq j$ in I. Then, $\mathcal{Y}$ being a cone over D, the commutativity of diagram (Y) above yields

$$f_{ij}(f(y)(i)) \;=\; f_{ij}(g_i(y)) \;=\; g_j(y) \;=\; f(y)(j),$$

as needed. The uniqueness of f is straightforward. It remains to show that f is spectral, which reduces to verifying that if $\mathfrak{p}(U) \in \mathcal{L}$ (as in (*) in the proof of 20.16), then $f^{-1}(\mathfrak{p}(U))$ is a compact open in Y. Let $\alpha \in Fin(I)$ be such that $U = \langle\, U_i \,\rangle_{i \in \alpha}$; then, it is easily established that

$$f^{-1}(\mathfrak{p}(U)) \;=\; \textstyle\bigcap_{i \in \alpha} g_i^{-1}(U_i),$$

a compact open in Y because it is spectral and each g_i is a spectral map. Hence, the cone $\mathcal{T}$ is $\varprojlim D$, ending the proof. $\square$

Remark 20.20. a) If fact, Theorem 20.16 is a consequence of Theorem 20.18, for the former corresponds to the case in which the poset I is discrete, i.e., $i \leq j$ iff $i = j$. However, it seemed that the exposition would become clearer if 20.16, that is interesting in its own right, was proven first.

For a rather comprehensive information on the category of spectral spaces, the reader is referred to Theorem 7 and 8, as well as Propositions 9 and 10 in [**28**].

b) As a special case of Theorem 20.18, we obtain that a projective limit of spectral spaces is spectral, as well as that the constructible topology on the projective limit is the projective limit of the corresponding constructible topologies.

c) If $I = \emptyset$, then Theorems 20.16 and 20.18 correspond to the *empty product*, yielding

$$\varprojlim D \;=\; \{*\},$$

i.e., a singleton, endowed with the only possible topology it can have. Clearly, such a space is spectral and its constructible and spectral topologies coincide since $\{*\}$ is a *Boolean space*. $\square$

Exercises

20.21. Prove that a topological space T is Boolean iff it is spectral and Hausdorff. $\square$

20.22. Show that the construction in Theorem 20.10 originates a functor from **Spec** to **BTop**, and study its properties. $\square$

20.23. Let $L = [0, 1]$ be the unit real interval considered as a complete chain. Determine $S(L)$ and the constructible topology on $S(L)$.
Hint : $S(L)$ is the disjoint union $((0, 1] \times \{0\}) \cup ([0, 1) \times \{1\})$ $\square$

20.24. Using Stone duality, show that the category **Spec**, of spectral spaces and maps, is a complete and cocomplete category [7]. $\square$

[7] Freyd's Theorem 16.31 may be useful. A nice proof would generalize Theorem 20.18 and the analogous result for colimits of I-diagrams in **Spec**.

CHAPTER 21

Projective Compact Hausdorff Spaces

The first section of this Chapter discusses extremally disconnected spaces, while in the second it will be shown that the category of compact Hausdorff spaces has enough projectives.

1. Extremally Disconnected Spaces

We now turn to the discussion of the effect of completeness of a lattice or of a BA on the topological properties of their Stone spaces.

DEFINITION 21.1. *A topological space X is*
a) **completely spectral (cs)** *iff it is spectral and the interior of the intersection of any family of compact open sets is compact open.*
b) **extremally disconnected (ed)** *iff it is Hausdorff and the closure of every open set is clopen.*

From 19.7.(e) and Stone duality (20.5) we obtain

PROPOSITION 21.2. *Let L be a distributive lattice.*
a) L is complete iff $S(L)$ is a completely spectral space. Stone duality establishes a natural bijective correspondence between complete distributive lattices and completely spectral spaces.
b) L is a complete Boolean algebra iff $S(L)$ is a compact extremally disconnected space. Stone duality establishes a natural and bijective correspondence between complete Boolean algebras and compact extremally disconnected spaces. □

If B is a Boolean algebra, there is a natural lattice morphism from B to the cBa of regular opens in $S(B)$:

$$a \in B \;\; \mapsto \;\; S_a \in B(S(B)) \subseteq Reg(S(B)).$$

Note that the operation of join in $Reg(S(B))$ involves taking the closure of a union, but for a **finite** number of clopens this is just their set theoretic union. Thus, the above map is in fact a lattice morphism. Further, it is injective and by Proposition 19.7.(d) it is a **regular** embedding. Since the family $\{S_a : a \in B\}$ is a basis for the topology on $S(B)$, we can generalize 18.13, with the help of 14.6 :

PROPOSITION 21.3. *If B is a BA, then $Reg(S(B))$ is the completion of B.*

If X is a set then 2^X is a cBa and so its Stone space, $S(2^X)$, is a compact extremally disconnected space, written βX.

REMARK 21.4. By Exercise 21.21, it is possible to map X into βX in a natural way :

(β) $$x \in X \longmapsto x^{\rightarrow} = \{A \subseteq X : x \in A\} \in \beta X.$$

If X is given the discrete topology (all points are open), then the above map is a homeomorphism of X onto its image, for all principal ultrafilters in βX are isolated points. Further, if S_A is a non-empty clopen set in βX, then A is a non-empty subset of X and so, any of its points will determine a principal ultrafilter in S_A. This means that the image of map defined in (β) is a dense open set in βX. Thus, the Stone space construction has given us a **compactification** of the discrete space X. To develop the topological and functional analytic properties of this construction, we need to recall the relation between filters and convergence in a topological space. $\square$

Recall (§ after 2.26) that a subset S of a lattice L is a **(proper) filter basis** if $S \neq \emptyset$, $\perp \notin S$ and

$*$ S is *down directed*, that is, for $a, b \in S$, $\exists\ c \in S$ such that $c \leq a \wedge b$.

DEFINITION 21.5. *Let X, Y be topological spaces and let $F \subseteq 2^X$ be a filter basis. Let $x \in X$, $y \in Y$ and let $X \xrightarrow{f} Y$ be a map.*

a) x is an **accumulation point** *of F iff $x \in \bigcap_{A \in F} \overline{A}$.* [1]

b) F **converges to** *x ($F \longrightarrow x$ or $\lim F = x$) iff ν_x is contained in the filter generated by F.*

c) f **converges to y along F** *($\lim_F f = y$) iff $f(F) = \{f(A) : A \in F\}$ is a filter basis in Y, which converges to y.*

Exercise 21.22 registers some of the fundamental properties of these concepts. Now we prove

PROPOSITION 21.6. *Let X be a set, βX be the Stone space of 2^X and let $\beta : X \longrightarrow \beta X$ be the map in 21.4.(β). If Y is a compact space and $f : X \longrightarrow Y$ is a continuous map, then f has a* **unique continuous** *extension to βX, that is, there is $g : \beta X \longrightarrow Y$, such that $g \circ \beta = f$.*

PROOF. We begin with

Fact. *Let F be an ultrafilter on 2^X. Then, $f(F) = \{f(A) : A \in F\}$ is a convergent filter base, to a* **unique** *point in the closure of $f(X)$. Further, if $F = x^{\rightarrow}$ is principal, then $f(F)$ converges to fx.*

Proof. Since F is a proper filter and $f(A \cap B) \subseteq f(A) \cap f(B)$, $f(F)$ is a filter base in Y. By compactness (21.22.(d)), this filter base has an accumulation point, $y \in \bigcap_{A \in F} \overline{f(A)}$. Clearly, $y \in \overline{f(X)}$.

To prove that $f(F) \longrightarrow y$, let V be an open neighborhood of y in Y; since $V \cap \bigcap_{A \in F} \overline{f(A)} \neq \emptyset$, it follows that for $A \in F$, $f^{-1}(V) \cap A$ is non-empty. Thus, $\{f^{-1}(V)\} \cup F$ has the fip, and the maximality of F yields $f^{-1}(V) \in F$. Since

[1] $\overline{A}$ is the closure of A.

$f(f^{-1}(V)) \subseteq V$, we conclude that V is in the filter generated by $f(F)$, as needed. Because Y is Hausdorff, a filter base can have only one limit (21.22.(c)).

If F is principal, $F = x^{\rightarrow}$, then fx is an accumulation point of $f(F)$. The proof given above shows that $f(F)$ converges to any its accumulation points. Hence, $f(x^{\rightarrow}) \longrightarrow fx$, completing the proof of the Fact.

Define $g : \beta X \longrightarrow Y$ by $g(F) = \lim_F f$; by the Fact, g is a function and an extension of f. It remains to prove that it is continuous. Let V be an open set in Y; to show that $g^{-1}(V)$ is open in βX, let F be such that $g(F) \in V$. Since compact spaces are regular, choose $W \in \nu_{g(F)}$, such that $\overline{W} \subseteq V$ and let $A = g^{-1}(W)$. Exactly as above, we can show that $\{A\} \cup F$ has the fip, and so $A \in F$, that is, $F \in S_A$. If $G \in S_A$ then the construction of g yields

$$g(G) \in \overline{f(A)} \subseteq \overline{W} \subseteq V.$$

Thus, $F \in S_A \subseteq g^{-1}(V)$, as needed. Uniqueness is immediate, because X is dense in βX and Y is Hausdorff. $\square$

Proposition 21.6 shows that βX is **the Stone-Čech compactification** of the discrete space X, since the defining properties of the Stone-Čech compactification of X are : X is densely embedded in a compact space Y such that every continuous function from X to any compact space can be extended to Y. Clearly, this property determines the space Y, whenever it exists, up to homeomorphism. By Exercise 16.44, every completely regular space has a Stone-Čech compactification. However, the method described here to obtain it is quite distinct from the one in 16.44.

Recall (1.16(a)) that the **density**, $d(X)$, of a space X is the least cardinal such that X it has a dense subset of that cardinality. Clearly, X always has a dense subset of cardinality $d(X)$. Note that if I is a set, then, $d(\beta I) = \text{card}(I)$. Hence, $\beta\mathbb{N}$ is separable, i.e., $d(\beta\mathbb{N}) = \omega$.

COROLLARY 21.7. *Every compact topological space is the continuous image of a compact extremally disconnected space of the same density. In particular, all separable compact spaces are a continuous image of* $\beta\mathbb{N}$.

PROOF. Let Y be a dense subset of cardinality $d(X)$, of the compact space X. By 21.6, the injection of Y into X has a continuous extension, $f : \beta Y \longrightarrow X$. Since Im f is dense and compact in a Hausdorff space, we conclude that f is onto X. To finish the proof, just observe that $d(\beta Y) = \text{card}(Y) = d(X)$. $\square$

By 21.7, extremally disconnected compacts are "projective generators" in the category of compact spaces. We will discuss this theme in depth in the next section.

We now wish to show that dyadic spaces cannot be extremally disconnected, generalizing Proposition 18.13. Our method will be distinct from that in [**13**]. To this end, we prove (compare 18.20)

LEMMA 21.8. *Every infinite cBa contains a copy of* $2^{\mathbb{N}}$.

PROOF. Let B be an infinite cBa. We start by constructing a strictly decreasing sequence, a_n, $n \geq 0$, of elements of B, such that $a_n^{\leftarrow}$ is infinite, for all $n \geq 0$. Set $a_0 = \top$; by induction, assume that we have constructed a_k, with $a_{k+1} < a_k$,

$0 \leq k \leq (n-1)$. Since $a_n^{\leftarrow}$ is infinite, select $b < a_n$; because $b \vee (\neg\, b \wedge a_n) = a_n$, all $c \in a_n^{\leftarrow}$ can be written as a join of elements in $b^{\leftarrow}$ and $(\neg\, b \wedge a_n)^{\leftarrow}$. It follows that either $b^{\leftarrow}$ or $(\neg\, b \wedge a_n)^{\leftarrow}$ must be infinite. Set

$$a_{n+1} = \begin{cases} b & \text{if } b^{\leftarrow} \text{ is infinite} \\ \neg\, b \wedge a_n & \text{otherwise} \end{cases}$$

For $n \geq 0$, let $b_n = (a_n \wedge \neg\, a_{n+1}) = a_n \bigtriangleup a_{n+1}$. Note that the b_n are all pairwise disjoint and distinct from $\bot$. Moreover,

$$\bigvee_{n \geq 0} a_n \;=\; \bigvee_{n \geq 0} b_n \;=\; \top.$$

Now define $f : 2^{\mathbb{N}} \longrightarrow B$, by

$$f(S) \;=\; \bigvee_{n \in S} b_n.$$

Clearly, $f(\emptyset) = \bot$ $(= \bigvee \emptyset)$ and $f(\mathbb{N}) = \top$. If S, $T \in 2^{\mathbb{N}}$ and $n \in (S - T)$, then $b_n \leq f(S)$, but $b_n \wedge f(T) = \bot$, showing that f is injective. Next, the distributive law in 8.4 yields, for all $S \in 2^{\mathbb{N}}$,

$$f(S) \wedge f(\mathbb{N} - S) \;=\; \bigvee_{k \in S} b_k \wedge \bigvee_{j \notin S} b_j \;=\; \bigvee_{(k,j) \in (S \times S^c)} b_k \wedge b_j \;=\; \bot,$$

because the b_n are pairwise disjoint. Clearly, $f(S) \vee f(\mathbb{N} - S) = \top$, and hence, $f(\mathbb{N} - S) = \neg\, f(S)$, that is, f preserves complements. To complete the proof, it is enough to show that f preserves meets. Another application of 8.4 yields

$$\begin{aligned} f(S) \wedge f(T) \;&=\; \bigvee_{k \in S} b_k \wedge \bigvee_{j \in T} b_j \;=\; \bigvee_{(k,j) \in S \times T} b_k \wedge b_j \\ &=\; \bigvee_{n \in S \cap T} b_n \;=\; f(S \cap T), \end{aligned}$$

completing the proof. $\square$

REMARK 21.9. The proof of 21.8 applies to σ-algebras, that is BAs with all countable joins and meets. Hence, any σ-algebra (or cBa) has cardinal larger than or equal to that of the continuum $(2^{\mathbb{N}})$. $\square$

A categorical duality establishes a correspondence between dual concepts : monic and epic, injectives and projectives, etc. In the case of Stone duality, an embedding of BAs originates an epic in the category of Boolean spaces, which by Exercise 16.40 and compactness, are onto continuous maps in **BTop**. Similarly, an epic in **BA**, which is an onto BA-morphism by 5.17, will give rise to a monic in **BTop**, that is an injective continuous map. We shall make use of these correspondences in what follows, without further notice.

COROLLARY 21.10. *$\beta\mathbb{N}$ is a continuous image of every extremally disconnected space.*

PROOF. If X is extremally disconnected, then $B(X)$ is a cBa (21.2.(b)). The injective BA-morphism, $f : 2^{\mathbb{N}} \longrightarrow B(X)$, of 21.8, yields, by Stone duality, a continuous surjection, $f_S : X \longrightarrow \beta\mathbb{N}$, as desired. $\square$

PROPOSITION 21.11. *$\beta\mathbb{N}$ is not a dyadic compact.*

PROOF. Assume, to get a contradiction, that there is a continuous surjection $\pi : 2^A \longrightarrow \beta\mathbb{N}$. For each $n \geq 0$, let $V_n = \pi^{-1}(\{n\})$ be the clopen inverse image of the principal ultrafilters $n \in \beta\mathbb{N}$. Clearly,

$$(*) \qquad \begin{cases} (i)\ k \neq j \ \ \Rightarrow \ \ V_k \cap V_j = \emptyset; \\ (ii)\ \bigcup_{n \geq 0} V_n \text{ is a dense open in } 2^A, \end{cases}$$

where the second property follows from the fact that $\mathbb{N}$ is dense in $\beta\mathbb{N}$. By 18.5.(e), for $n \geq 0$, there is $s_n \in pF_\omega(A, 2)$, such that $V_n = V_{s_n}$. Let $K = \bigcup_{n \geq 0} dom\, s_n$. Then K is countable. To keep notation straight, write σ_n for s_n considered as an element of $pF_\omega(K, 2)$; by Lemma 18.8, we have, for all $n \geq 0$,

$$V_n = V_{\sigma_n} \times 2^{(A-K)}.$$

Note that the collection $\{V_{\sigma_n} \ ; \ n \geq 0\}$ satisfies the properties in (*), with K in place of A. For $x \in 2^K$, write $\widehat{x}$ for the element of 2^A given, for $i \in A$, by

$$\widehat{x}_i = \begin{cases} x_i & \text{if } i \in K \\ 1 & \text{if } i \notin K. \end{cases}$$

Note that if $x \in V_{\sigma_n}$, then $\widehat{x} \in V_n$, $n \geq 0$.

Fact 1. *Let $(x_n)_{n \geq 0} \subseteq 2^K$ be a sequence such that $x_n \in V_{\sigma_n}$, $n \geq 0$. Then, there is an infinite $T \subseteq \mathbb{N}$, such that the sequence $\widehat{x}_t$, $t \in T$, is convergent in 2^A.*

Proof. Since K is countable, 2^K is metric compact. Thus, x_n has a convergent subsequence, $x_{n_k} \longrightarrow x$ in 2^K. Clearly, $\widehat{x}_{n_k}$ converges to $\widehat{x}$. Now, just take $T = \{n_k : k \geq 0\}$; since the x_n are all distinct (the V_{σ_n} are disjoint), T is an infinite subset of $\mathbb{N}$.

But we also have

Fact 2. *Let T be an infinite subset of $\mathbb{N}$ and let $\mathcal{T} = \{x_t : t \geq T\}$ be a sequence of elements in 2^A, such that $x_t \in V_t$, for all $t \in T$. Let Y be a compact space. Then, any map from $\mathcal{T}$ to Y can be extended to a continuous map from 2^A into Y.*

Proof. Let $f : \mathcal{T} \longrightarrow Y$ be a map; define $g : \mathbb{N} \longrightarrow Y$ by

$$g(n) = \begin{cases} f(x_n) & \text{if } n \in T \\ p & \text{if } n \notin T, \end{cases}$$

where p is any point in Y. By 21.6, g has a (unique) continuous extension, $\widehat{g} : \beta\mathbb{N} \longrightarrow Y$. But then $\widehat{g} \circ \pi$ is an extension of f to 2^A.

$$\begin{tikzcd}
2^A \arrow[r, "\pi"] \arrow[d, "\widehat{g} \circ \pi"'] & \beta\mathbb{N} \arrow[dl, "\widehat{g}"] \\
Y &
\end{tikzcd}$$

Let $\mathcal{T} = \{\widehat{x}_t : t \in T\}$ be the convergent sequence in Fact 1, with limit $\widehat{x}$. Since T is infinite, write $T = P \cup Q$, with P, Q infinite and disjoint. Let $f : \mathcal{T} \longrightarrow \{0, 1\}$ given by

$$f(x_t) = \begin{cases} 1 & \text{if } t \in P \\ 0 & \text{if } t \in Q. \end{cases}$$

By Fact 2, f has a continuous extension to 2^A. But this is impossible, because the subsequences of $\mathcal{T}$ indexed by P and Q both converge to $\widehat{x}$. $\square$

We can now prove a strong generalization of 18.13, namely

THEOREM 21.12. *If X is a dyadic space, then*
a) No continuous image of X is extremally disconnected. In particular, X is not extremally disconnected.
b) No infinite Boolean algebra of clopens in X is complete.

PROOF. a) Since the class of dyadic spaces is closed under continuous images, it follows from Proposition 21.11 and Corollary 21.10, that no continuous image of a dyadic space can be extremally disconnected.
b) Let X be a dyadic compact and let $B \subseteq B(X)$ be a complete subalgebra of clopens in X. Assume, to get a contradiction, that B is infinite. Let $X \xrightarrow{\gamma} \gamma X$ be the Booleanization of X, as in 20.7. Since X is compact and the image of γ is dense in γX, it follows that γ is surjective. Moreover, since γX is the Stone space of $B(X)$, the embedding $B \longrightarrow B(X)$ yields, by Stone duality, a continuous surjection, from γX to $S(B)$. Hence, $S(B)$ is a continuous image of X. But, by 21.2, $S(B)$ is extremally disconnected, contradicting (a). $\square$

2. Projective Compacts

In this section we discuss projectives in the category of compact Hausdorff spaces and present the construction of the **Gleason projective cover** of these spaces.

As already mentioned in section 1, a duality establishes a natural correspondence between dual notions, such as epics and monics, injectives and projectives, injective hulls and projective covers. Consequently, Stone duality provides a way to transfer results about the category **BA** to the category **BTop**. For instance, Theorem 20.5, Corollary 15.3, Exercise 15.5 and Corollaries 15.4 and 21.3 yield

COROLLARY 21.13. *A Boolean space is projective in* **BTop** *iff it is extremally disconnected. Every Boolean space X is the continuous image of $S(Reg(X))$, and this compact extremally disconnected space is the projective cover of X in* **BTop**.

We go a step further, investigating what happens in the category of compact Hausdorff spaces. The properties in Lemma 1.10 will be of constant use.

PROPOSITION 21.14. *Let X be a compact (Hausdorff) space.*
a) The following are equivalent:
(1) *X is extremally disconnected.*
(2) *For all open U, V in X, $\overline{U \cap V} = \overline{U} \cap \overline{V}$.*
(3) *For all open U, V in X, $U \cap V = \emptyset$ implies $\overline{U} \cap \overline{V} = \emptyset$.*
b) If X is a projective compact space, then X is extremally disconnected.
c) X is a projective compact space iff it satisfies :

[*split*] *If Y is a compact space and $Y \xrightarrow{f} X$ is a onto continuous map, there is a continuous $X \xrightarrow{g} Y$, such that $f \circ g = Id_X$.*

Proof. a) (3) $\Rightarrow$ (1) : If $U \in \Omega(X)$, since $(U \cup \text{int}\ U^c)$ is dense in X, and their intersection is empty, (3) yields

$$X = \overline{U \cup int\ U^c} = \overline{U} \cap \overline{\text{int}\ U^c}, \quad \text{with } \overline{U} \cap \overline{\text{int}\ U^c} = \emptyset,$$

proving that $\overline{U}$ is clopen in X.
(1) $\Rightarrow$ (3) : Let U, V be opens in X. Note that $U \cap V = \emptyset$, forces $U \cap \overline{V} = \emptyset$. Since $\overline{V}$ is clopen, we get $\overline{U} \cap \overline{V} = \emptyset$, verifying (3).
(3) $\Rightarrow$ (2) : For U, $V \in \Omega(X)$, note that $(U \cap V)$ and int $(U - V)$ are disjoint opens, whose union is dense in U. Moreover, int $(U - V)$ is also disjoint from V. Thus, taking closures we get

$$\text{i) } \overline{U} = \overline{U \cap V} \cup \overline{\text{int}\ (U - V)} \qquad \text{and} \qquad \text{ii) } \overline{V} \cap \overline{\text{int}\ (U - V)} = \emptyset.$$

The intersection of the equation in (i) with $\overline{V}$ proves (2). Clearly, (2) implies (3).
b) Let βX be the Stone-Čech compactification of X with the discrete topology. By 21.6, there is a continuous surjection, $\beta X \xrightarrow{f} X$. Since X is projective, there must be a continuous g from X to βX, such that $f \circ g = Id_X$.

$$\begin{tikzcd}
& X \arrow[dl, "g"'] \arrow[d, "Id_X"] \\
\beta X \arrow[r, "f"'] & X
\end{tikzcd}$$

Clearly, g is injective. Let U and V be disjoint opens in X; if it is shown that their closures also disjoint, then (a) implies that X is extremally disconnected. Consider $A = f^{-1}(U)$ and $B = f^{-1}(\text{V})$; these are disjoint open sets in βX, and so $\overline{A}$ and $\overline{B}$ are disjoint clopens in βX. Since $f \circ g = Id_X$, we have $U \subseteq g^{-1}(\overline{A})$ and $V \subseteq g^{-1}(\overline{B})$, and the inverse images of $\overline{A}$ and $\overline{B}$ are disjoint and clopen in X. It is now immediate that $\overline{U}$ must be disjoint from $\overline{V}$.
c) If X is projective in **CTop**, the definition of projectiveness (16.36) implies that X has [*split*]. For the converse, suppose X satisfies [*split*] and we are given a continuous surjection, $Y \xrightarrow{f} Z$ and a continuous map $X \xrightarrow{g} Z$, with Y, Z compact spaces. Let

$$T = \{\langle x, y \rangle \in X \times Y : gx = fy\};$$

T is a subspace of $X \times Y$, with the product topology. Since T is precisely the inverse image of the diagonal of $Z \times Z$ by the continuous map

$$(g \times f) : X \times Y \longrightarrow Z \times Z, \quad \langle x, y \rangle \mapsto \langle gx, fy \rangle,$$

T is closed in $X \times Y$ and so a compact Hausdorff space. Furthermore, the restrictions of the projections onto the first and second coordinates to T, yield continuous maps, α from T to X and β from T to Y, respectively. Clearly, $g \circ \alpha = f \circ \beta$.

$$\begin{tikzcd} T \arrow[r, "\alpha"] \arrow[d, "\beta"'] & X \arrow[d, "g"] \\ Y \arrow[r, "f"'] & Z \end{tikzcd}$$

Since f is onto, the same is true of α. By [*split*], there is a continuous $X \xrightarrow{h} T$, such that $\alpha \circ h = Id_X$. It is straightforward to show that $\beta \circ h : X \longrightarrow Y$ satisfies $f \circ \beta \circ h = g$, completing the proof that X is projective. $\square$

DEFINITION 21.15. *A continuous surjection, $X \xrightarrow{f} Y$, is* **essential** *iff for all closed sets $F \subseteq X$, if $f(F) = Y$, then $F = X$* [2].

In the next result we use the adjoint pair associated to a continuous map in Example 4.6, and discussed in full generality in Theorem 7.8, to give a characterization of closed essential surjections. This will, of course, apply directly to compact spaces, since all continuous maps from a compact space to a Hausdorff space are closed. We also take the opportunity to register how this type of adjoint pair behaves in relation to complemented elements.

PROPOSITION 21.16. *a) Let L and P be complete distributive lattices. Consider the adjoint pair*

$$L \xrightarrow{f} P \quad \textit{and} \quad P \xrightarrow{g} L,$$

with g right adjoint to f. If f is a $[\wedge, \bigvee]$-morphism and $g(\bot) = \bot$, then, for all complemented elements x in L, $g(f(x)) = x$.

b) Let $X \xrightarrow{f} Y$ be a continuous map and let (f^*, f_*) be the adjoint pair associated to f as in 4.6. Then,

(1) *If Im f is dense in Y, then for all $V \in B(Y)$, $f_*(f^*(V)) = V$.*

(2) *If f is a closed continuous surjection, then*

$$f \textit{ is essential} \quad \textit{iff} \quad \textit{ker } f_* = \{\emptyset\}.$$

PROOF. a) The adjointness relation in Theorem 7.8 reads

[ad] For $u \in L$ and $v \in P$, $fu \leq v$ iff $u \leq gv$.

It follows that for all $u \in L$, $g(f(u)) \geq u$. Now suppose $x \in B(L)$; since f is a lattice morphism and g a $\bigwedge$-morphism with $g(\bot) = \bot$, we have

$$g(f(x)) \wedge g(f(\neg x)) = g(fx \wedge f(\neg x)) = g(f(x \wedge \neg x)) = \bot.$$

Hence, $u = g(fx) \geq x$, $w = g(f(\neg x)) \geq \neg x$ and $u \wedge w = 0$. But this entails $u = x$ and $w = \neg x$.

b) If $X \xrightarrow{f} Y$ is a continuous map, recall that the right adjoint to the inverse image morphism from $\Omega(Y)$ to $\Omega(X)$ is (4.6)

$$f_*V = \bigcup \{W \in \Omega(Y) : f^*W = f^{-1}(W) \subseteq V\}.$$

[2] Some authors use the term *minimal* for essential surjections.

Consequently, if the image of f is dense in Y, the only open set in Y whose inverse image is empty is $\emptyset$. The conclusion in (1) now follows from (a). To verify (2), let U be a non-empty open set in X. Because f is essential and closed, there is a non-empty open V in Y, such that $V \cap f(X - U) = \emptyset$. Then,

$$f^*V \cap (X - U) \subseteq f^*V \cap f^{-1}(f(X - U)) = \emptyset,$$

showing that $f^*V \subseteq U$. By adjointness, $V \subseteq f_*U$, and so $f_*U \neq \emptyset$, proving that the kernel of $f_* = \{\emptyset\}$. Conversely, if ker $f_* = \{\emptyset\}$, let F be a proper closed subset of X and $U = (X - F)$. Since $U \neq \emptyset$, we have $f_*U \neq \emptyset$. From the adjointness relation and $f_*U = f_*U$, we obtain $f^*(f_*U) \subseteq U$. But this means that f restricted to F is not surjective, for there is a non-empty open in Y, whose inverse image is disjoint from F. Hence, f is essential. $\square$

Proposition 21.17. *Let $X \xrightarrow{f} Y$ be a continuous surjection of compact spaces.*
a) There is a compact subspace T of X, such that $f_{|T}$ is an essential surjection onto Y.
b) If f is essential and Y is extremally disconnected, then f is a homeomorphism.

Proof. a) Define

$$\mathcal{V} = \{F \subseteq X : F \text{ is closed in } X \text{ and } f(F) = Y\},$$

partially ordered by inclusion. If C is a chain in $\mathcal{V}$, then, for each $y \in Y$, the family $\{f^{-1}(y) \cap F : F \in C\}$ has the fip. Since $f^{-1}(y)$ is compact, this implies that $\bigcap C \cap f^{-1}(y) \neq \emptyset$, that is, $\bigcap C \in$ V. By Zorn's Lemma (2.20), $\mathcal{V}$ has a minimal element, T. Clearly, f restricted to T is an essential surjection onto Y.
b) Since continuous maps between compact spaces are closed and f is onto Y, to prove that f is a homeomorphism, it is enough to show that it is injective. Let z, t be distinct points in X and assume, to get a contradiction, that $fz = ft = y$. Fix disjoint opens, $U \in \nu_z$ and $V \in \nu_t$ and let $f_* : \Omega(X) \longrightarrow \Omega(Y)$ be the right adjoint to the inverse image morphism f^*, as in 21.16.(b). Since $U \cap V = \emptyset$, we have $f_*U \cap f_*V = \emptyset$. Y being extremally disconnected, it follows that $\overline{f_*U}$ is disjoint from $\overline{f_*V}$. Let S and T be the clopen complements of $\overline{f_*U}$ and $\overline{f_*V}$, respectively, in Y. Clearly, $S \cup T = Y$. Consequently, $y = fz = ft$ must be in either S or T and we can suppose, without loss of generality, that $y \in T$. Consider the open set $V \cap f^*T$; it is non-empty, since it contains t. On the other hand, recalling that T is complemented in $\Omega(Y)$ and item (b) in 21.16, we get

$$f_*(V \cap f^*T) = f_*V \cap f_*(f^*T) = f_*V \cap T = \emptyset,$$

that is impossible, because f is essential. $\square$

If X is a compact Hausdorff space, let $G(X) =_{def} S(Reg(X))$ be the Stone space of the cBa $Reg(X)$. $G(X)$ is the **Gleason cover** of X; it is extremally disconnected and its points are the ultrafilters in $Reg(X)$.

If $F \in G(X)$, $\{\overline{V} : V \in F\}$ is a collection of closed sets in X with the fip. By compactness, $A = \bigcap_{V \in F} \overline{V} \neq \emptyset$. Let $x \in A$; since X is regular, for each $U \in \nu_x$, we can select a regular open $W \in \nu_x$, such that $W \subseteq U$. Notice that $\{W\} \cup F$ has the fip in $Reg(X)$; the maximality of F implies that $W \in F$. We have shown (21.5) that the filter generated by F in $\Omega(X)$ (or 2^X) contains ν_x, that is, $F \longrightarrow x$. Since X is Hausdorff, A must consist of a single point. Therefore, we have constructed a map

$$\mathfrak{g} : G(X) \longrightarrow X,$$

whose basic properties are described in

PROPOSITION 21.18. *With $\mathfrak{g} : G(X) \longrightarrow X$ as above,*
a) $\mathfrak{g}$ is onto X.
b) For all open $U \in \Omega(X)$,

$$\mathfrak{g}^*U \;=\; \mathfrak{g}^{-1}(U) \;=\; \bigcup \{S_V : V \in Reg(X) \text{ and } \overline{V} \subseteq U\}.$$

In particular, $\mathfrak{g}$ is continuous and $\overline{\mathfrak{g}^*U} = S_U$, for all $U \in Reg(X)$.
c) $\mathfrak{g}$ is an essential continuous surjection from $G(X)$ onto X.
d) If $G(X) \xrightarrow{f} G(X)$ is a continuous map such that $\mathfrak{g} \circ f = \mathfrak{g}$, then $f = Id_{G(X)}$.

PROOF. a) Let x be a point in X; the collection of regular open neighborhoods of x is a proper filter in $Reg(X)$ and so can be extended to an ultrafilter F on $Reg(X)$. Clearly, the filter generated by F in $\Omega(X)$ contains ν_x, that is $\mathfrak{g}F = x$.
b) Let U be open in X and let F be an ultrafilter in $Reg(X)$, such that $\mathfrak{g}F \in U$. Let V be a regular open neighborhood of $\mathfrak{g}F$, such that $\overline{V} \subseteq U$. The definition of $\mathfrak{g}$ implies that for all $G \in S_V$, $\mathfrak{g}G \in \overline{V} \subseteq U$. Hence, $S_V \subseteq \mathfrak{g}^*U$, proving that $\mathfrak{g}^*U$ is open and contained in the right side of the equality stated in (b). The reverse containment is an immediate consequence of the construction of $\mathfrak{g}$. The continuity of $\mathfrak{g}$ is clear. Observe that U is the sup, in the cBa $Reg(X)$, of the collection of regular $V \subseteq U$, such that $\overline{V} \subseteq U$. By Proposition 19.7.(d),

$$S_U \;=\; \overline{\bigcup \{S_V \;:\; V \in Reg(X) \text{ and } \overline{V} \subseteq U\}} \;=\; \overline{\mathfrak{g}^*\, U}.$$

(c) For $U \in Reg(X)$, we have $\mathfrak{g}^*U \subseteq S_U$; the adjointness relation $[ad]$ (see proof of 21.16 or 7.8) implies that $U \subseteq \mathfrak{g}_*(S_U)$. Recalling that S_U is empty iff U is empty, that the clopens in $G(X)$ are of the form S_U, $U \in Reg(X)$, and that they form a basis for $G(X)$, we conclude that ker $\mathfrak{g}_* = \{\emptyset\}$. By Proposition 21.16.(b), $\mathfrak{g}$ is an essential surjection.
(d) Let $f : G(X) \longrightarrow G(X)$ be a continuous map, such that $\mathfrak{g} \circ f = \mathfrak{g}$. Since $\mathfrak{g}(\text{Im } f) = X$ and $\mathfrak{g}$ is an essential surjection, it follows that f is onto $G(X)$. Moreover, f must be essential, for if for some proper closed subset, $A \subseteq G(X)$, we had $f(A) = G(X)$, then $\mathfrak{g}(f(A)) = \mathfrak{g}(A) = X$, contradicting the essential nature of $\mathfrak{g}$. But then, the fact that $G(X)$ is extremally disconnected implies that f is a homeomorphism. Now, for U in $Reg(X)$, (a) entails that $\mathfrak{g}^*U$ dense in S_U. Since f is a homeomorphism, $f^*(\mathfrak{g}^*U)$ is dense in the clopen set $f^*(S_U)$. On the other hand, $\mathfrak{g} \circ f = \mathfrak{g}$ yields $f^*(\mathfrak{g}^*U) = \mathfrak{g}^*U$. Thus, $f^*(S_U) = S_U$, for all $U \in Reg(X)$. But the only continuous map from $G(X)$ to $G(X)$ that can satisfy this property is $Id_{G(X)}$, ending the proof. □

We summarize our discussion in

THEOREM 21.19. *a) A compact Hausdorff space is projective iff it is extremally disconnected.*

b) For all compact spaces X, the Gleason cover of X, $G(X) \xrightarrow{\mathfrak{g}} X$, is the projective cover of X in the category of compact spaces.

Proof. (a) By items (b) and (c) in Proposition 21.14, it is enough to show that if X is extremally disconnected, every continuous surjection $Y \xrightarrow{f} X$ has a section that is, a continuous $g : X \longrightarrow Y$, such that $f \circ g$ is the identity on X. By 21.17.(a), there is a closed subset F of Y, such that $h = f_{|F} : F \longrightarrow X$ is an essential surjection. Since X is extremally disconnected, part (b) of 21.17 implies that h is a homeomorphism. The inverse of h provides the desired continuous section of f.
(b) Let $Y \xrightarrow{f} X$ be a continuous surjection, with Y compact and extremally disconnected. Since Y is projective, there is a continuous map $g : Y \longrightarrow G(X)$, such that $\mathfrak{g} \circ g = f$. It remains to show that g is onto. Since $G(X)$ is also projective, there is $h : G(X) \longrightarrow Y$, such that $f \circ h = \mathfrak{g}$. From $\mathfrak{g} \circ g = f$, we get $\mathfrak{g} \circ (g \circ h)$ $= \mathfrak{g}$, with $g \circ h$ a continuous map from $G(X)$ to $G(X)$. By Proposition 21.14.(d), $g \circ h = Id_{G(X)}$. Hence, g is onto $G(X)$ and the proof is complete. □

It is quite clear that the construction of taking the Gleason cover of a compact space is idempotent, that is, $G(G(X)$ is naturally homeomorphic to $G(X)$.

Exercises

21.20. A compact Hausdorff ed space is Boolean [3]. □

21.21. Recall that an atom in a lattice L is a minimal element $\neq \perp$ (2.6).
a) A principal filter is maximal iff it is generated by an atom.
b) Let $Atm(L)$ be (possibly empty) set of atoms in L. The map $a \mapsto a^{\rightarrow}$ defines a bijective correspondence between the atoms of L and the open set of isolated points [4] of $S(L)$. □

21.22. Let $X \xrightarrow{f} Y$ be a map, where X and Y are topological spaces. Let F be a filter basis in X. Then,
a) If $F \longrightarrow x$, then x is an accumulation point of F. If F is a maximal filter in $\Omega(X)$ and x is an accumulation point of F, then $F \longrightarrow x$. In particular, if an ultrafilter in 2^X has an accumulation point, then it converges to that point.
b) f is continuous iff for all $x \in X$ and filters $F \subseteq 2^X$, $F \longrightarrow x \Rightarrow \lim_F f = f(x)$.
c) If X is a topological space and $x, y \in X$, the following are equivalent :
(1) X is a Hausdorff space.
(2) For all filter bases $F \subseteq 2^X$, $x = lim\ F = y \Rightarrow x = y$.
(3) For all ultrafilters $F \subseteq 2^X$, $x = lim\ F = y \Rightarrow x = y$.
(4) For all filter bases G **on $\boldsymbol{\Omega(X)}$**, $x = lim\ G = x \Rightarrow x = y$.

[3] Compact spaces are normal; but even regularity suffices!
[4] A point p is isolated if $\{p\}$ is open.

d) Consider the following conditions :

(1) X is a compact space.

(2) Every filter basis on 2^X has an accumulation point.

(3) Every ultrafilter on 2^X converges.

(4) Every ultrafilter in $\Omega(X)$ converges.

Then, (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Rightarrow$ (4). If X is *regular* (1.20), all four conditions are equivalent.

e) Give an example of a *Hausdorff* non-compact, non-regular space in which all ultrafilters of opens are convergent [5]. $\square$

21.23. Let I be a set and $\{F_i : i \in I\}$ be a family of fields indexed by I. Let $R = \prod_{i \in I} F_i$. Then, $Spec(R)$ is homeomorphic to the Stone-Čech compactification of the discrete space I. $\square$

21.24. Let **CTop** be the category of compact Hausdorff spaces and let $\mathcal{F}$: **CTop** $\longrightarrow$ **Set** be the forgetful functor. What is the relation between βX and $\mathcal{F}$? $\square$

[5] T0, non-Hausdorff examples are easy to find; T2 examples are harder.

Part 4

Presheaves over Topological Spaces

CHAPTER 22

Geometric Sheaves

In this chapter many proofs are left to the reader. Its contents describe the basic geometrical object associated to sheaves over topological spaces. General references for this chapter are [**75**], [**19**], [**22**] and [**66**]. In [**21**] the reader will find an interesting survey of the history of Sheaf Theory and its mathematical significance.

Notation and results of section 1.2 may be used without explicit reference. Recall that $\Omega(X)$ is the frame of opens of the topological space X.

Definition 22.1. *Let $pF(X, E)$ be the set of partial maps from X to E* (2.12). *For $s, t \in pF(X, E)$, define*

$$[\![s = t]\!] \;=\; \{x \in dom\, s \cap dom\, t : s(x) = t(x)\}$$

Let $E \xrightarrow{p} X$ be a map. A **section** *for p is a map $s \in pF(X, E)$ such that*

$$dom\, s \subseteq Im\, p \quad and \quad p \circ s = Id_{doms}.$$

If $dom\, s = A \subseteq X$, s is a **section over** *A.*

If E and X are topological spaces, p is a **local homeomorphism** *iff for all $e \in E$, there is an open $U \in \nu_e$ such that $p(U) \in \Omega(X)$ and $p_{|U} : U \longrightarrow p(U)$ is a homeomorphism* [1].

The basic properties of local homeomorphisms and their sections are described in

Proposition 22.2. *If $E \xrightarrow{p} X$ is a local homeomorphism, then*
a) p is continuous and open.
b) A section of p over an open subset of X is an open map. If s is a **continuous** *section of p over $u \in \Omega(X)$, then s and $p_{|s(u)}$ are inverse homeomorphisms between $u \in \Omega(X)$ and $s(u) \in \Omega(E)$.*
c) $p(E) \in \Omega(X)$ and $E \xrightarrow{p} p(E)$ is a local homeomorphism.
d) Let $q : F \longrightarrow X$ be a local homeomorphism and $f : E \longrightarrow F$ be a map such that $q \circ f = p$. Then, f is continuous iff f is a local homeomorphism.

[1] ν_e is the filter of open neighborhoods of e; see the paragraph right after 1.9. Homeomorphisms are defined in 1.17.(d).

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p"'] & & F \arrow[dl, "q"] \\ & X & \end{tikzcd} \qquad \begin{tikzcd} U \arrow[rr, "f"] \arrow[dr, "p"'] & & f(U) \arrow[dl, "q"] \\ & p(U) & \end{tikzcd}$$

Proof. (a) Let $u \in \Omega(X)$; if $e \in p^{-1}(u)$, there is $U \in \nu_e$, such that $p_{|U}$ is a homeomorphism from U onto $p(U) \in \Omega(X)$. Consider $w = u \cap p(U)$; w is open in X, with $p(e) \in w$. Since $p_{|U}$ is a homeomorphism, $W = p^{-1}(w) \cap U$ is open in U and consequently, in E. Further, $e \in W \subseteq p^{-1}(u)$, and p is continuous. The argument to show that p is open is similar. Clearly, (c) follows from (a).
(b) Let $u \in \Omega(X)$ and s be a section of p over u. If $e = s(x)$, there is $W \in \nu_e$, with $p_{|W}$ a homeomorphism from W onto $p(W) \in \nu_x$. Set $v = p(W) \cap u \in \nu_x$; since p is continuous and $p \circ s = Id_u$, we have $p^{-1}_{|W}(v) = s(v)$, open in E, with $e \in s(v) \subseteq s(u)$. Since e is arbitrary, $s(u)$ must be open in E.

If s is a **continuous** section of p over $u \in \Omega(X)$, then, s is a continuous open map from u onto the open set $s(u) \subseteq E$ which is injective, because $p \circ s = Id_u$. Thus, s is a homeomorphism from u onto $s(u)$. Now, the fact that s is a right inverse to p forces $p_{|s(u)}$ and s to be inverse homeomorphisms between u and $s(u)$.
(d) Clearly, it is enough to prove that if f is continuous, then it is a local homeomorphism. Let $e \in E$; select $W \in \nu_{f(e)}$ such that $q_{|W}$ is a homeomorphism from W onto the open set $q(W)$. Since f is continuous and p is a local homeomorphism, there is $U \in \nu_e$, such that $f(U) \subseteq W$ and $p_{|U}$ is a homeomorphism from U onto the open set $p(U)$. We have $p(U) = q(f(U))$ and $W \cap q^{-1}(p(U)) = f(U) \in \Omega(E)$. Further, $p_{|U}$ and $q_{|f(U)}$ are homeomorphisms, with $q_{|f(U)} \circ f_{|U} = p_{|U}$. Hence, $f_{|U}$ is a homeomorphism from U onto $f(U) \in \Omega(F)$, as needed. $\square$

Definition 22.3. *Let X be a topological space. A* **geometrical sheaf of sets over X** *consists of a triple,*

$$\mathcal{E} = \langle E, p, X \rangle,$$

where E is a topological space and p a local homeomorphism. E is called the **sheaf space**, *X the* **base space** *and p the* **projection**, *although we are* **not** *requiring that p be onto X.*

For $x \in X$, $p^{-1}(x)$ is called the **stalk** *of $\mathcal{E}$ at x.*

If $\mathcal{F} = \langle F, q, X \rangle$ is a sheaf over X, a **morphism** $\mathcal{E} \xrightarrow{f} \mathcal{F}$, *is a continuous map, $E \xrightarrow{f} F$, such that $q \circ f = p$.*

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p"'] & & F \arrow[dl, "q"] \\ & X & \end{tikzcd}$$

Since the (set-theoretical) composition of morphisms is a morphism and the identity is a morphism, we have a category, $\boldsymbol{Shg(X)}$*, of geometric sheaves over* X*.*

EXAMPLE 22.4. If $\mathcal{E} = \langle E, p, X \rangle$ is a sheaf over X, the projection p is a continuous open map. If U is any open set in E, $\langle U, p_{|U}, X \rangle$ is a sheaf over X. In fact, if $S \subseteq X$ is given the induced topology from X, then $\langle p^{-1}(S),\ p_{|p^{-1}(S)},\ S \rangle$ is a sheaf over S. $\square$

EXAMPLE 22.5. Proposition 22.2.(d) implies that if $\mathcal{E} \xrightarrow{f} \mathcal{F}$ is a morphism of sheaves, then the triple $\langle E, f, F \rangle$ is a sheaf over F, where E and F are the sheaf spaces of $\mathcal{E}$ and $\mathcal{F}$, respectively. $\square$

DEFINITION 22.6. *Let* $\mathcal{E} = \langle E, p, X \rangle$ *be a sheaf and let* S, T *be subsets of* X*.*
a) Write $\mathcal{E}(S)$ *for the set of* **continuous** *sections of* p *with domain* S *:*

$$\mathcal{E}(S) \quad = \quad \{S \xrightarrow{s} E : s \text{ is a continuous section of } p\}$$

Unless explicit mention to the contrary, it is assumed that S *has the topology induced by* X*. Note that* $\mathcal{E}(\emptyset)$ *has precisely one element, written* $*$*, corresponding to the unique section of* $\mathcal{E}$ *over the empty set.*

For $x \in X$, $\mathcal{E}_x =_{def} \mathcal{E}(\{x\}) = p^{-1}(x)$ *is the stalk of* $\mathcal{E}$ *at* x*.*

b) If $S \subseteq T$*, define the* **restriction map**

$$\rho_{TS} : \mathcal{E}(T) \longrightarrow \mathcal{E}(S), \text{ given by } t \longmapsto t_{|S}.$$

When $S = \{x\}$*, write* ρ_{Tx} *for* $\rho_{T\{x\}}$*. Note that if* $S \subseteq T \subseteq K$*, then*

$$\rho_{KS} = \rho_{TS} \circ \rho_{KT} \quad \text{and} \quad \rho_{SS} = Id_{\mathcal{E}(S)}.$$

We shall frequently use our standard notation for restriction to indicate the maps ρ_{TS}*.*

c) Let $|\mathcal{E}| = \coprod_{u \in \Omega(X)} \mathcal{E}(u)$*, called the* **domain** *of the sheaf* $\mathcal{E}$ [2]*. The elements of* $|\mathcal{E}|$ *will be identified with the corresponding sections of* $\mathcal{E}$*, and called the* **sections** *of* $\mathcal{E}$*.*

$*$ *If* s *is a section of* $\mathcal{E}$*, write* Es *for* $dom\, s$ (the extent of s).

$*$ $\mathcal{E}(p(E))$ *is the set of* **global sections** *of the sheaf* $\mathcal{E}$*.*

COROLLARY 22.7. *Let* $\mathcal{E} = \langle E, p, X \rangle$ *be a sheaf over* X*.*
a) If $u \in \Omega(X)$ *and* $s \in \mathcal{E}(u)$*, then* s *is a homeomorphism of* u *onto the open set* $s(u)$ *in* E*, with inverse* $p_{|s(u)}$*. Moreover, the set*

[2] Notice that the disjoint union is taken over the *open* sets in X.

$$\{s(u) : s \in |\mathcal{E}|,\, u \in \Omega(X)\}$$

is a basis for the topology of E.
b) If τ_1 and τ_2 are topologies on E, such that p is a local homeomorphism with respect to both, then $\tau_1 = \tau_2$.

PROOF. The first assertion in (a) is a restatement of 22.2.(b). By the definition of local homeomorphism, given $U \in \Omega(E)$, we may write $U = \bigcup_{i \in I} W_i$, where $p_{|W_i}$ is a homeomorphism onto the open subset $p(W_i)$, $i \in I$. Let s_i be the sections of p given by the inverses of $p_{|W_i}$; clearly, $U = \bigcup_{i \in I} s_i(p(W_i))$, showing that the of images of opens in X, by the continuous sections of p, constitute a basis for the topology on E. Item (b) is left to the reader. □

The next result describes the main properties of sections in a sheaf.

PROPOSITION 22.8. *Let a, b, $c \in |\mathcal{E}|$, where $\mathcal{E} = \langle\, E, p, X\,\rangle$ is a sheaf over X.*
a) $[\![a = b]\!] = \bigcup\, \{u \in \Omega(Ea \cap Eb) : a_{|u} = b_{|u}\}$. In particular, $[\![a = b]\!]$ is open in $Ea \cap Eb$ and in X.
b) The map $[\![* = *]\!]$ has the following properties :

(1) $[\![a = b]\!] = [\![b = a]\!]$; (2) $[\![a = b]\!] \cap [\![b = c]\!] \subseteq [\![a = c]\!]$;

(3) $Ea = [\![a = a]\!]$; (4) $Ea = Ea = [\![a = b]\!] \;\Leftrightarrow\; a = b$.

c) Suppose $\{s_i : i \in I\} \subseteq |\mathcal{E}|$ satisfies
[compatible] For all i, $j \in I$, $s_{i|Es_i \cap Es_j} = s_{j|Es_i \cap Es_j}$;
then, there is a **unique** *$t \in |\mathcal{E}|$ such that*
(1) *$Et = \bigcup_{i \in I} Es_i$;* (2) *For all $i \in I$, $t_{|Es_i} = s_i$.*

PROOF. a) The definition of $[\![a = b]\!]$ is in 22.1. To keep notation straight, we shall, momentarily, write

$$\|a = b\| = \bigcup\, \{u \in \Omega(Ea \cap Eb) : a_{|u} = b_{|u}\}.$$

It is clear that $\|a = b\| \subseteq [\![a = b]\!]$; for the reverse inclusion, let $x \in Ea \cap Eb$, $e = a(x) = b(x)$ and $W = a(Ea) \cap b(Eb) \in \nu_e$. By 22.2.(b), $p_{|W}$ is a homeomorphism between W and $p(W) \in \nu_x$. Moreover, $a_{|p(W)}$ and $b_{|p(W)}$ are inverses to $p_{|W}$. Thus, $a_{|p(W)} = b_{|p(W)}$ and $x \in \|a = b\|$, establishing the desired equality.
b) Items (1), (3) and (4) are clear. For (2), note that if $u \in \Omega(Ea \cap Eb)$ and $v \in \Omega(Eb \cap Ec)$, then $(u \cap v) \in \Omega(Ea \cap Ec)$, and

$$a_{|u} = b_{|u} \text{ and } b_{|v} = c_{|v} \;\Rightarrow\; a_{|u \cap v} = c_{|u \cap v}. \tag{I}$$

Since $\Omega(X)$ is a frame, 8.4 and (I) yield

$$\begin{aligned}
[\![a = b]\!] \cap [\![b = c]\!] &= \bigcup \left\{u \cap v : \begin{array}{c} u \in \Omega(Ea \cap Eb),\, v \in \Omega(Eb \cap Ec), \\ a_{|u} = b_{|u} \text{ and } b_{|v} = c_{|v} \end{array}\right\} \\
&\subseteq \bigcup\, \{w \in \Omega(Ea \cap Ec) : a_{|w} = c_{|w}\} \\
&= [\![a = c]\!].
\end{aligned}$$

c) Observe that the hypothesis entails that $\{s_i : i \in I\} \subseteq pF(X, E)$ is compatible, according to 1.2. Hence, by this same result, there is a *unique* $t \in pF(X, E)$ satisfying conditions (1) and (2) (keep in mind that extent is the same as domain, in this context). It remains to see that t is a section for p over Et. But this follows easily from the fact that each s_i is a section for p. $\square$

EXAMPLE 22.9. Let Y be a set, considered as a topological space with the discrete topology (all points are open). Let $E = X \times Y$, with the product topology; then, the natural projection onto the first coordinate, π_X, is a local homeomorphism, and $\mathcal{E} = \langle E, \pi_X, X \rangle$ is a sheaf over X. This sheaf is called the **constant sheaf**, of stalk Y, over X. Note that for each $u \in \Omega(X)$, $\mathcal{E}(u) = Y^u$. $\square$

EXAMPLE 22.10. Let $Y \xrightarrow{p} X$ be a continuous surjection. The triple $\mathcal{C} = \langle Y, p, X \rangle$ is a **covering space**, if for all $x \in X$, there is $u \in \nu_x$, such that $p^{-1}(u)$ can be written as a disjoint union $\bigcup_{i \in I} v_i$, such that $p_{|v_i}$ is a homeomorphism of v_i onto u. Clearly, all covering spaces $-$ an important topological construction $-$, are geometric sheaves. References for this topic are [**47**], [**30**] and [**23**]. We give two examples of this construction, which will be useful latter.

Let S^1 be the unit circle in the complex plane, i.e., the set of $z \in \mathbb{C}$, such that $|z| = 1$, with the topology induced by $\mathbb{C}$. Define $exp: \mathbb{R} \longrightarrow S^1$ by $t \mapsto e^{it}$; then $\mathcal{U}(S^1) = \langle \mathbb{R}, exp, S^1 \rangle$ is a covering space, the *universal covering space* of S^1.

Let S^n be the n-sphere in $\mathbb{R}^{n+1}$, that is,

$$S^n = \{\langle a_1, \ldots, a_n, a_{n+1} \rangle \in \mathbb{R}^{n+1} : \textstyle\sum_{i=1}^{n=1} a_i^2 = 1\}.$$

The identification of antipodal points in S^n, produces a covering space,

$$\mathcal{U}(\mathbb{RP}^n) = (S^n, \pi, \mathbb{RP}^n),$$

called the *universal covering space of the real projective space of dimension* n, $\mathbb{RP}^n$. For these examples, we have

$*$ The stalks of $\mathcal{U}(S^1)$ are naturally isomorphic to $\mathbb{Z}$, while the stalks of $\mathcal{U}(\mathbb{RP}^n)$ are naturally isomorphic to $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$.

$*$ In both cases, the set of global sections **is empty**.

(a) To see this for $\mathcal{U}(S^1)$, suppose that $s : S^1 \longrightarrow \mathbb{R}$ is a continuous section; since S^1 is connected, the image of s in $\mathbb{R}$ must be an interval. Now, let $A = S^1 - \{x\}$, where x is any point in S^1. The restriction of s to A is continuous and A is still connected, but its image is the union of two disjoint intervals in $\mathbb{R}$, a contradiction.

(b) For $\mathcal{U}(\mathbb{RP}^n)$ the argument is more sophisticated (and also applies to $\mathcal{U}(S^1)$). Let $\pi_1(X)$ be the first homotopy group of the space X; if $f : X \longrightarrow Y$ is a continuous map, let $f_* : \pi_1(X) \longrightarrow \pi_1(Y)$ be the homomorphism induced by f on homotopy.

Since the sphere is simply connected, we have $\pi_1(S^n) = \{0\}$. On the other hand, since $\mathcal{U}(\mathbb{RP}^n)$ is a universal covering space with structure group $\mathbb{Z}_2$, $\pi_1(\mathbb{RP}^n) = \mathbb{Z}_2$. Now assume that $s : \mathbb{RP}^n \longrightarrow S^n$ is a global section of $\mathcal{U}(\mathbb{RP}^n)$. Then, since $p \circ s$ is the identity on $\mathbb{RP}^n$ and homotopy is a functor, we obtain a commutative diagram

$$\begin{tikzcd} \pi_1(\mathbb{RP}^n) \arrow[rr, "s_*"] \arrow[dr, "Id"'] & & \pi_1(S^n) \arrow[dl, "p_*"] \\ & \pi_1(\mathbb{RP}^n) & \end{tikzcd}$$

Hence, s_* would have to be injective, which is clearly impossible. $\square$

Example 22.11. Let $\{A_i : i \in I\}$ be a family of non-empty sets. Endow I with the discrete topology, and define $E = \coprod_{i\in I} A_i$ and a map $E \xrightarrow{p} I$, such that $p(A_i) = \{i\}$. Since all points in I are open, $\mathcal{E} = (E, p, I)$ is a geometric sheaf, and indeed, a covering space. Note that for $u \subseteq I$, $\mathcal{E}(u) = \prod_{i\in u} A_i$; in particular, the set of global sections of $\mathcal{E}$ is $\prod_{i\in I} A_i$. $\square$

In what follows, $\mathcal{E} = \langle E, p, X \rangle$ is a sheaf over X.

Remark 22.12. Let $x \in X$ and $A = \coprod_{u\in\nu_x} \mathcal{E}(u)$; for $s, t \in A$, define

$$s \equiv t \quad \text{iff} \quad \exists\, w \in \nu_x, \text{ such that } w \subseteq Es \cap Et \text{ and } s_{|W} = t_{|W}.$$

By 22.8.(a), for all $s, t \in A$,

$$s \equiv t \quad \text{iff} \quad [\![s = t]\!] \in \nu_x.$$

It is straightforward to verify that $\equiv$ is an equivalence relation on A. Corollary 22.7.(b) implies that there is a natural bijective correspondence between the quotient $A/\equiv$ and the stalk $\mathcal{E}_x$ at x : $s/\equiv \mapsto s(x)$. Hence, we can view the stalk at x as the collection of "germs" (i.e., equivalence classes by $\equiv$) of sections of $\mathcal{E}$, defined in open neighborhoods of x. Or equivalently, the stalk at x is the *direct limit* of the sets of sections in open neighborhoods of x. In a sense which will become precise later, all information about $\mathcal{E}$ is in fact contained in $|\mathcal{E}|$, together with the way sections restrict from larger opens to smaller ones. This point of view will be the basis of our treatment of sheaves from the next Chapter on. $\square$

Under mild restrictions, geometric sheaves are not difficult to construct. First, we set down some useful terminology.

Definition 22.13. *Let X and E be sets, $pF(X, E)$ be the set of partial maps from X to E and $S, T, \Gamma \subseteq pF(X, E)$.*

a) S is **compatible** *if for all $s, t \in S$, $s_{|Es\cap Et} = t_{|Es\cap Et}$* [3].

b) Γ **satisfies** [***comp***] *over $A \subseteq X$ if for all compatible $S \subseteq \Gamma$,*

$\bigcup_{s\in S} Es = A \;\Rightarrow\; \exists\, t \in \Gamma$, such that $Et = A$ and $t_{|Es} = s$, $\forall\, s \in S$.

c) S is **dense** *in T iff for all $t \in T$, $Et = \bigcup_{s\in S} [\![s = t]\!]$.*

It is clear that Definition 22.13 applies to a sheaf over a topological space and its sections over open subsets of the base.

[3]As in 1.2. We used *up-compatible* in 2.33 and 2.36. Since we will not use the dual concept, we drop the prefix "up" from here on.

PROPOSITION 22.14. *Let* $p : E \longrightarrow X$ *be a map, where* X *is a topological space and* E *is a set. Suppose* $\Gamma \subseteq pF(X, E)$ *satisfies*
(1) *For all* $s \in \Gamma$, $Es =_{def} dom\, s$ *is open in* X*;*
(2) *All elements of* Γ *are sections of* p*, i.e.,* $p \circ s = Id_{Es}$*;*
(3) *For all* $s, t \in \Gamma$, $[\![s = t]\!]$ *is open in* X*;*
(4) *For all* $e \in E$*, there is* $s \in \Gamma$ *such that* $e \in Im\, s$*.*
Then, there is a **unique** *topology on* E*, such that* p *is a local homeomorphism. Moreover, if* $\mathcal{E} = \langle E, p, X \rangle$ *is the sheaf so determined over* X*, then*
a) $\Gamma \subseteq |\mathcal{E}|$ *and* Γ *is dense in* $|\mathcal{E}|$*.*
b) Let B *be a basis for the topology on* X*. For* $v \in \Omega(X)$*, define*

$$\Gamma_{B,\, v} = \{s_{|_w} : w \subseteq v \text{ and } w \in B\} \quad \text{and} \quad \Gamma(v) = \{s \in \Gamma : Es = v\}.$$

If, for $u \in \Omega(X)$, $\Gamma_{B,\, u}$ *satisfies* [*comp*] *over* u*, then* $\Gamma(u) = \mathcal{E}(u)$*.*

PROOF. Let $\mathcal{B} = \{s(u \cap Es) : \langle s, u \rangle \in \Gamma \times \Omega(X)\}$; it is straightforward to show that for all $s, s' \in \Gamma$ and $u, v \in \Omega(X)$,

$$(*) \qquad s(u \cap Es) \cap s'(v \cap Es') = s([\![s = s']\!] \cap u \cap v).$$

Thus, $\mathcal{B}$ is closed under finite intersections and is a basis for a topology on E. Note that all $s \in \Gamma$ are open in this topology (by definition !). Further, from (*) one easily verifies that for all $s, s' \in \Gamma$ and $v \in \Omega(X)$,

$$s^{-1}(s'(v \cap Es') = [\![s = s']\!] \cap v,$$

that is open in X. Thus, all $s \in \Gamma$ are continuous, open and injective from the open set $Es \subseteq X$, onto the open set $Im\, s \subseteq E$. Therefore, every $s \in \Gamma$ is a homeomorphism from the open Es in X, onto the open $Im\, s$ in E. Hence, assumption (4) implies that p is a local homeomorphism and $\mathcal{E} = \langle E, p, X \rangle$ is a sheaf, with $\Gamma \subseteq |\mathcal{E}|$.

To show that Γ is dense in $|\mathcal{E}|$, let $t \in |\mathcal{E}|$, $x \in Et$ and $e = t(x)$. By (4), there is $s \in \Gamma$ such that $e = s(z) \in Im\, s$. Since t and s are sections of p, we get $x = z$. Since $\mathcal{E}$ is a sheaf, $[\![s = t]\!] \in \nu_x$. Hence, $Et = \bigcup_{s \in \Gamma} [\![s = t]\!]$, as desired.

Clearly, $\Gamma(u) \subseteq \mathcal{E}(u)$; for the reverse inclusion, fix a basis B for the topology on X and a section $t \in \mathcal{E}(u)$. By denseness, there is $S \subseteq \Gamma$, such that $Et = \bigcup_{s \in S} [\![s = t]\!]$. Since B is basis for the topology on X and all $[\![s = t]\!]$ are open in X, we can write

$$Et = \bigcup\nolimits_{i \in I} w_i,$$

where, for each $i \in I$, there is $s \in S$ such that $w_i \subseteq [\![s = t]\!]$. Thus, for $i \in I$ and $s \in S$,

$$s_{|_{w_i}} = t_{|_{w_i}}, \quad s_{|_{w_i}} \in \Gamma_u \quad \text{and} \quad Es_{|_{w_i}} = w_i.$$

Since $\Gamma_{B,\, u}$ satisfies [*comp*] over u, there is $s' \in \Gamma$ such that

$$s'_{|_{w_i}} = s_{|_{w_i}} = t_{|_{w_i}} \text{ and } Es' = u.$$

Now, item (4) in 22.8.(b) yields $s' = t$, i.e., $t \in \Gamma(u)$. $\square$

Perhaps the best way to understand 22.14 is to see it in action.

EXAMPLE 22.15. Let H be a HA and let $X = S(H)$ be the Stone space of H (19.2). Define $E = \coprod_{F \in X} H/F$ and $p : E \longrightarrow X$, by sending all elements of the quotient H/F to the prime filter F in X. Observe that p is onto X. Each $a \in H$ gives rise to a global section of p as follows :

$$a \in H \;\longmapsto\; a^* : X \longrightarrow E, \text{ where } \; a^*(F) = a/F.$$

Let $\Gamma = \{a^* : a \in H\}$; clearly, conditions (1), (2) and (4) are satisfied by Γ. Regarding (3), recall (6.10) that if $a, b \in H$, then

$$(a \leftrightarrow b) = (a \to b) \wedge (b \to a) = \max\{z \in H : a \wedge z = b \wedge z\}$$

exists in H and that for all $t \leq (a \leftrightarrow b)$, we have $t \wedge a = t \wedge b$. For a and $b \in H$,

$$\begin{aligned}[\![a^* = b^*]\!] &= \{F \in X : a/F = b/F\} \\ &= \{F \in X : \exists\, c \in F \text{ such that } a \wedge c = b \wedge c\} \\ &= \{F \in X : (a \leftrightarrow b) \in F\} = S_{a \leftrightarrow b},\end{aligned}$$

which is open in X. 22.14 applies, yielding a sheaf $\mathcal{E} = \langle E, p, S(H) \rangle$, with Γ dense in $|\mathcal{E}|$.

Next, we show that if $B = \{S_a : a \in H\}$ is the standard basis of opens for $S(H)$, then $\Gamma_{B,X}$ has [*comp*] over X. From this and 22.14.(b) we conclude that

$$\Gamma = \Gamma(X) = \mathcal{E}(X),$$

that is, the set of global sections of $\mathcal{E}$ is an isomorphic copy of H. Let $S \subseteq \Gamma$ and $u_s = S_{c_s}$, $s \in S$, be basic opens in $S(H)$ such that

$$\textstyle\bigcup_{s \in S} u_s = X \quad \text{and} \quad \{s_{|u_s} : s \in S\} \text{ is compatible in } \mathcal{E}.$$

Since X is compact (19.8), there is $T \subseteq_f S$, with $\bigcup_{t \in T} u_t = X$. Note that if we can find $a \in H$ such that $a^*_{|u_t} = t_{|u_t}$, for all $t \in T$, then, by the compatibility of S, we get $a^*_{|u_s} = s_{|u_s}$, for all $s \in S$. Hence, we may assume that S is finite, $S = \{a_i^* : 1 \leq i \leq n\}$, with $u_{a_i^*} = S_{c_i}$, $1 \leq i \leq n$. The compatibility of ${a_i^*}_{|S_{c_i}}$ means that for all $i, j \leq n$, we have

$$S_{c_i \wedge c_j} = S_{c_i} \cap S_{c_j} \subseteq [\![a_i^* = a_j^*]\!] = S_{a_i \leftrightarrow a_j}.$$

Thus, 19.5.(b) entails that for $i, j \leq n$, $c_i \wedge c_j \leq (a_i \leftrightarrow a_j)$; in particular,

$$(*) \qquad a_i \wedge (c_i \wedge c_j) \;=\; a_j \wedge (c_i \wedge c_j),$$

for all $i, j \leq n$. Define $a = \bigvee_{i=1}^n a_i \wedge c_i$; for each $i \leq n$, (*) yields

$$a \wedge c_i = (a_i \wedge c_i) \vee \textstyle\bigvee_{i \neq j} c_i \wedge c_j \wedge a_j \leq a_i \wedge c_i,$$

wherefrom it follows that $a \wedge c_i = a_i \wedge c_i$. Consequently, if F is a prime filter in H such that $c_i \in F$, then $a/F = a_i/F$, that is, a^* restricted to S_{c_i} is equal to a_i restricted to the same basic open. Hence, $\Gamma_{B,X}$ satisfies [*comp*] over X, as asserted. □

COROLLARY 22.16. *Every Heyting algebra can be identified with the set of the global sections of a sheaf of Heyting algebras over its Stone space.* □

EXAMPLE 22.17. Let R be a commutative ring with identity, which for ease of exposition, we assume to be an integral domain. For each prime ideal $P \in Spec(R)$, let R_P be the *localization* of R at P (9.41), that is, R_P is the ring of fractions a/r, such that $a \in R$ and $r \notin P$. Since R is an integral domain [4], in R_P

[4] $xy = 0$ iff $x = 0$ or $y = 0$; equivalently, (0) is a prime ideal.

$$a/r = b/s \quad \text{iff} \quad as = bt.$$

Let $E = \coprod_{P \in Spec(R)} R_P$ and define $p : E \longrightarrow Spec(R)$ by sending each element of the localization R_P to the point P in $Spec(R)$. Clearly, p is surjective. Each element of E is of the form a/b, with $a \in R$ and b outside some prime ideal in R.

Recall from Chapter 19, that $Z_r = \{P \in Spec(R) : r \notin P\}$ is the compact open associated to $r \in R$ and that $B = \{Z_r : r \in R\}$ is a basis for the Zariski topology on $Spec(R)$. Further, since R is an integral domain, $Z_r = \emptyset$ iff $r = 0$. To each pair $\langle a, r \rangle \in R \times (R - \{0\})$ we can associate a map

$$a/r : Z_r \longrightarrow E, \text{ given by } \quad P \in Z_r \mapsto a/r \in R_P,$$

that is well defined, because for all $P \in Z_r$, $r \notin P$. Hence, a/r is a section of p over its domain, Z_r.

Let $\Gamma = \{a/r : \langle a, r \rangle \in R \times (R - \{0\})\}$. The set of elements in Γ whose extent equals $Spec(R) = Z_1$ is

$$\Gamma(Spec(R)) = \{a/1 : a \in R\},$$

an isomorphic copy of R. Since R is a domain, Γ may be identified with the field of quotients of R, together with the assignment to each fraction, of the set of primes that do not contain its denominator.

Clearly, Γ satisfies conditions (1), (2) and (4) of 22.14. Further, Γ is closed under restrictions of its elements to basic opens in $Spec(R)$: if $a/r \in \Gamma$ and $Z_s \subseteq Z_r$, then $(a/r)_{|Z_s} = as/rs$, since, by 19.5.(a), we have $Z_s = Z_r \cap Z_s = Z_{rs}$. In particular, $\Gamma_{B,\,X} \subseteq \Gamma$. For condition (3), it is straightforward to see that

$$(*) \qquad [\![a/r = b/s]\!] = \begin{cases} Z_r \cap Z_s & \text{if } as = br \\ \emptyset & \text{otherwise} \end{cases}$$

Hence, $[\![a/r = b/s]\!]$ is open in $Spec(R)$, for all a/r, $b/s \in \Gamma$. By 22.14, we have a sheaf $\mathcal{E} = \langle E, p, Spec(R) \rangle$, with Γ dense in $|\mathcal{E}|$.

We now verify that if a_i/r_i, $i \in I$, is a compatible collection of sections in Γ, such that $\bigcup Z_{r_i} = Spec(R)$, then there is $a \in R$ such that $(a/1)_{|Z_{r_i}} = a_i/r_i$, for all $i \in I$. Thus, $\Gamma_{B,\,Spec(R)}$ will satisfy [*comp*] over $Spec(R)$, and

$$\Gamma(Spec(R)) = \{a/1 : a \in R\} = \mathcal{E}(R).$$

What is the meaning of a/r being compatible with b/s ? The fact that R is an integral domain implies that the zero ideal, (0), is prime. Since it is contained in every prime ideal, (0) is a generic point for $Spec(R)$, that is, $\overline{(0)} = Spec(R)$. But this means that (0) is in every non-empty open in $Spec(R)$, and so the intersection of any finite number of non-empty opens in $Spec(R)$ is non-empty. Since $r, s \neq 0$ implies $Z_r \cap Z_s \neq \emptyset$, from (*) we get that for all a, $b \in R$ and r, $s \in R - \{0\}$

$$(**) \qquad a/r \text{ is compatible with } b/s \quad \text{iff} \quad as = br.$$

Let a_i/r_i, $i \in I$, be a compatible subset of Γ, with $\bigcup Z_{r_i} = Spec(R)$. By Corollary 19.9, there is $J \subseteq_f I$, such that the ideal generated by the r_j, $j \in J$, is equal to R. Hence, there are $\alpha_j \in R$ such that

$$\textstyle\sum_{j \in J} \alpha_j r_j = 1. \tag{1}$$

Since the sections a_i/r_i are compatible, from (**) we get

$$\forall\ i, k \in I, \quad a_i r_k = a_k r_i. \tag{2}$$

Define $a = \sum_{j\in J} \alpha_j a_j$; for each $i \in I$, (1) and (2) yield

$$r_i\ a \ = \ r_i\sum_{j\in J} \alpha_j a_j \ = \ \sum_{j\in J} \alpha_j a_j r_i \ = \ \sum_{j\in J} \alpha_j r_j a_i \ = \ a_i\ (\sum_{j\in J} \alpha_j r_j) \ = \ a_i.$$

Hence, the value of $a/1$ in each localization R_P, $P \in Z_{r_i}$, is the same as a_i/r_i, and so $a/1$ is the "gluing" of the compatible family given at the outset.

With only slight modifications, one can show that the set of elements in Γ with domain Z_r is equal to $\mathcal{E}(Z_r)$, for all $r \neq 0$ in R. Further, essentially the same argument shows that an analogous construction holds for any **module** over R, with stalks that are the localizations of the module at each prime in $Spec(R)$. $\square$

COROLLARY 22.18. *Every integral domains can be identified with the set of global sections of a sheaf of local rings over its Zariski spectrum.* $\square$

Corollary 22.18 is true for **all** commutative rings with identity, as will be seen in the Example 27.21 and Theorem 27.22, due to A. Grothendieck.

EXAMPLE 22.19. Let R be a commutative regular ring (19.19). By Theorem 20.2, every prime ideal in R is maximal. Hence, R/P is a field, for all $P \in Spec(R)$, and so if e is an idempotent in R, $e/P = 0$ or $e/P = 1$.

By 19.22, $Spec(R)$ is a Boolean space, wherein $B = \{Z_r : r \in R\}$ is a basis of **clopens**. Further, if e_r is the unique idempotent of R such that $(r) = (e_r)$, then $Z_r = Z_{e_r}$. Since R has no nilpotent elements, the intersection of all primes in R is equal to $\{0\}$. The set of ideas presented below are due to R. S. Pierce ([**57**], [**6**]).

Let $E = \coprod_{P\in\ Spec(R)} R/P$ and define $p : E \longrightarrow Spec(R)$ by assigning to each element of R/P the point $P \in Spec(R)$. Each $a \in R$ defines a map (to be indicated by the same symbol)

$$a : Spec(R) \longrightarrow E, \text{ given by } P \ \mapsto \ a/P.$$

Clearly, every such map is a section of p. Let Γ be the set of sections of p over $Spec(R)$, just defined. We have

$$[\![a = b]\!] \ = \ \{P \in Spec(R) : a/P = b/P\} \ = \ \{P \in Spec(R) : (a-b) \in P\} \ = \ \text{complement of } Z_{(a-b)} \text{ in } Spec(R),$$

which is clopen in $Spec(R)$. It is now straightforward to see that Γ has properties $(1)-(4)$ of 22.14. Hence, $\mathcal{E} = \langle\, E, p, Spec(R)\,\rangle$ is a sheaf, with Γ dense in $|\mathcal{E}|$. We wish to show that $\Gamma = \mathcal{E}(Spec(R))$. As a preliminary step, we prove

Fact 1. *Let $a, b \in R$ and let e, f be idempotents in R. Then*
a) $Z_e \subseteq [\![a = b]\!]$ *iff* $ae = be$.
b) $[\![a_{|Z_e} = b_{|Z_f}]\!] = [\![a = b]\!] \cap Z_e \cap Z_f = [\![a = b]\!] \cap Z_{ef}$.
c) $a_{|Z_e}$ *is compatible with* $b_{|Z_f}$ *iff* $aef = bef$.

Proof. (a) Let P be a prime ideal in R. If $e \notin R$ and $(ae-be) = 0$, then, $(a-b) \in P$, and so $P \in [\![a = b]\!]$, proving that $Z_e \subseteq [\![a = b]\!]$. Now assume that $Z_e \subseteq [\![a = b]\!]$; we show that $ae = be$ by proving that it is in **every** prime ideal in R. We may suppose that $e \notin P$; then $P \in Z_e$, and so $a/P = b/P$, whence $(a-b) \in P$. But then, $(ae-be) = 0$, as desired. Items (b) and (c) are left to the reader [5].

[5]Compare with 22.30.(c).

Suppose $a_i \in \Gamma$ and clopens Z_{e_i}, $i \in I$, verify

(1) The restrictions of the a_i to Z_{e_i} are compatible, that is

$$\forall\, i, j \in I, \quad [\![a_{i_{|Z_{e_i}}} = a_{j_{|Z_{e_j}}}]\!] = Z_{e_i} \cap Z_{e_j}$$

(2) $\bigcup_{i \in I} Z_{e_i} = Spec(R)$.

We may suppose that each e_i is an idempotent. (1) and Fact 1 yield

(*) $$\forall\, i, k \in I, \quad a_i e_i e_k = a_k e_k e_i.$$

As in the 22.15 and 22.17, condition (2) yields a **finite** $J \subseteq I$ and $c_j \in R$, such that

(**) $$\sum_{j \in J} c_j e_j = 1.$$

Define $a = \sum_{j \in J} c_j a_j e_j$; for each $i \in I$, (*) and (**) yield

$$ae_i \;=\; e_i \sum_{j \in J} c_j a_j e_j \;=\; \sum_{j \in J} c_j a_j e_j e_i \;=\; \sum_{j \in J} c_j a_i e_i e_j \;=\; a_i e_i \sum_{j \in J} c_j e_j \;=\; a_i e_i,$$

and so, $[\![a = a_{i_{|e_i}}]\!] = Z_{e_i}$, for all $i \in I$, as desired. □

Corollary 22.20. *Every commutative regular ring can be identified with the set of global sections of a sheaf of fields over its spectrum.* □

Remark 22.21. The last three examples consist of just a glimpse of what is known as "representation of algebras by continuous sections". A classical reference is [**29**]. Recent examples also appear in [**3**]. □

A geometric sheaf $\mathcal{E} = \langle E, p, X \rangle$ is **Hausdorff** if E is a Hausdorff space. The situation in Example 22.19 is quite typical :

Lemma 22.22. *For a sheaf $\mathcal{E} = \langle E, p, X \rangle$ over a Hausdorff space X, the following are equivalent :*

(1) $\mathcal{E}$ is Hausdorff.

(2) For all $u \in \Omega(X)$ and $s, t \in \mathcal{E}(u)$, $[\![s = t]\!]$ is clopen in u.

(3) For all $s, t \in |\mathcal{E}|$, $[\![s = t]\!]$ is clopen in $Es \cap Et$.

Proof. $\underline{(1) \Rightarrow (2)}$: It is enough to show that

$$\{x \in u : s(x) \neq t(x)\} \;=\; u - [\![s = t]\!]$$

is open in u. If $x \in (u - [\![s = t]\!])$, since $s(x) = e \neq t(x) = e'$, there are $V \in \nu_e$ and $V' \in \nu_{e'}$, such that $V \cap V' = \emptyset$ and p restricted to V and V' are homeomorphisms onto $p(V)$ and $p(V')$, respectively. Hence, the intersection $p(V) \cap p(V')$ is non-empty and contained in $(u - [\![s = t]\!])$, as needed. (2) $\Rightarrow$ (1) is immediate from $s_{|[\![s=t]\!]} = t_{|[\![s=t]\!]}$ (22.30.(a)).

$\underline{(3) \Rightarrow (1)}$: Given distinct $e, e' \in E$, we have two cases :

$*$ $p(e) \neq p(e')$. Let u and v be disjoint open neighborhoods of $p(e)$ and $p(e')$, respectively. There are $U \in \nu_e$ and $V \in \nu_{e'}$, such that p is a homeomorphism of U and V onto $p(U) \subseteq u$ and $p(V) \subseteq v$, respectively. Clearly, $U \cap V = \emptyset$.

$*$ $p(e) = p(e') = x$. Again, there are $U \in \nu_e$ and $U' \in \nu_{e'}$, such that $p_{|U}$ and $p_{|U'}$ are homeomorphisms onto $p(U)$, $p(U')$. Let $v = p(U) \cap p(U')$, let s be the inverse of $p_{|U}$ and t be the inverse of $p_{|U'}$. Hence, $x \in (v - [\![s = t]\!]) = w$, with w open in V; thus, $s(w)$ and $t(w)$ are disjoint opens in E, with $e \in s(w)$ and $e' \in t(w)$. $\square$

Definition 22.23. *Let $E_i \xrightarrow{p_i} X$ be continuous maps of topological spaces, $1 \leq i \leq n$. The set*

$$\times_{i=1}^{n} E_i \;=\; \{\langle e_1, \ldots, e_n \rangle \in \prod E_i : p_i(e_i) = p_j(e_j),\ \forall\ i, j \leq n\}$$

with the topology induced by the product topology on $\prod E_i$, is the **fibered product** *of the family of E_i over the p_i. There is a natural map*

$$\times_{i=1}^{n} p_i : \times\ E_i \longrightarrow X, \quad \langle e_1, \ldots, e_n \rangle \longmapsto p_1(e_1) = \ldots = p_n(e_n).$$

For each $k \leq n$, there is a projection $\pi_k : \times\ E_i \longrightarrow E_k$, the restriction of the canonical projection onto the k^{th} coordinate. Since $\times\ E_i$ has the topology induced by $\prod E_i$, each π_k is a continuous function and $\pi_k \circ p_k = \times\ p_i$.

$$\begin{tikzcd} \times E_i \arrow[rr, "\pi_k"] \arrow[dr, "\times p_i"'] & & E_k \arrow[dl, "p_k"] \\ & X & \end{tikzcd}$$

Lemma 22.24. *If $\mathcal{E}_i = \langle E_i, p_i, X \rangle$ $1 \leq i \leq n$, are sheaves over X, then*

$$\prod_{i=1}^{n} \mathcal{E}_i = \langle \times_{i=1}^{n} E_i,\ \times\ p_i,\ X \rangle,$$

is a sheaf over X and the maps $\pi_k : \prod \mathcal{E}_i \longrightarrow \mathcal{E}_k$ in 22.23 are sheaf morphisms. The system $(\prod \mathcal{E}_i,\ \{\pi_i\}_{i \leq n})$ is the product of the $\mathcal{E}_i$ in the category $\boldsymbol{Shg(X)}$.

Proof. We prove $\times\ p_i$ is a local homeomorphism, leaving the other assertions to the reader. Let $e = \langle e_i \rangle \in (\times\ E_i)$ and $x = p_i(e_i)$; the coordinate we choose to compute x is, of course, immaterial. Since we have only a finite set of local homeomorphisms p_i, we can select $u \in \nu_x$, as well as $W_i \in \nu_{e_i}$, such that each p_i restricted to W_i is a homeomorphism onto u. Let s_i be the sections of $\mathcal{E}_i$, corresponding to the inverses of the restriction of each p_i to W_i. Define

$$\prod_{i=1}^{n} s_i : u \longrightarrow \times\ E_i, \text{ given by } z \in u \longmapsto \langle s_1(z), \ldots, s_n(z) \rangle.$$

$\prod s_i$ is a section for $\times\, p_i$; since $\times\ E_i$ is a subspace of $\prod E_i$ and $s_k = \pi_k \circ \prod s_i$ is continuous, we conclude that $\prod s_i$ is a continuous map from u to $\times\ E_i$. Clearly, $t = \prod s_i$ is injective and for all open $v \subseteq u$,

$$t(v) = \prod_{i=1}^{n} s_i(v) \cap (\times\ E_i).$$

Since sections are open maps, this last relation entails that t is open. Thus, t is a section for $\times\, p_i$, as needed. $\square$

In general, $|\mathcal{E} \times \mathcal{F}| \neq |\mathcal{E}| \times |\mathcal{F}|$. For stalks, it is clear that

$$\text{For all } x \in X, \quad (\textstyle\prod_{j=1}^{n} \mathcal{E}_j)_x = \prod_{j=1}^{n} (\mathcal{E}_j)_x.$$

that is, the stalk of a finite product is the product of the stalks of the components. Exercise 22.35 discusses the notions of *fibered product over a morphism* and *kernel* of a morphism.

By 22.2.(d), a morphism of sheaves is a local homeomorphism that commutes with the projections. Thus, given a morphism,

$$\mathcal{E} = \langle E, p, X \rangle \xrightarrow{f} \mathcal{F} = \langle F, q, X \rangle,$$

we can construct, for each open $u \subseteq X$, a map

$$f_u : \mathcal{E}(u) \longrightarrow \mathcal{F}(u), \quad s \ \mapsto \ f \circ s.$$

Further, if $v \in \Omega(u)$, then

$$\forall\ s \in \mathcal{E}(u), \quad [f_u(s)]_{|v} = f_v(s_{|v}),$$

where $_{|}$ represents restriction in $\mathcal{E}$ and $\mathcal{F}$.

$$\begin{tikzcd}
\mathcal{E}(u) \arrow[r, "f_u"] \arrow[d, "|"'] & \mathcal{F}(u) \arrow[d, "|"] \\
\mathcal{E}(v) \arrow[r, "f_v"'] & \mathcal{F}(v)
\end{tikzcd}$$

LEMMA 22.25. *Let $\mathcal{E} = \langle E, p, X \rangle$ and $\mathcal{F} = \langle F, q, X \rangle$ be sheaves over X. Let $\mathcal{B}$ be a basis for the topology on X and $\Gamma \subseteq |\mathcal{E}|$, a dense set of sections in $\mathcal{E}$. If $E \xrightarrow{f} F$ is a map, the following conditions are equivalent :*

(1) *f is a morphism of sheaves.*

(2) *For all $u \in \mathcal{B}$ and $s \in \mathcal{E}(u)$, $\quad f \circ s \in \mathcal{F}(u)$.*

(3) *For all $s \in \Gamma$, $\quad f \circ s \in |\mathcal{F}|$.*

PROOF. We know that (1) implies (2) and (3). We verify (3) implies (1), leaving (2) implies (1) to the reader. The first step is proving that $q \circ f = p$. This is equivalent to verifying that f takes $\mathcal{E}_x$ into $\mathcal{F}_x$, for all $x \in X$. Let $e \in E$, with $p(e) = x$; since Γ is dense in $|\mathcal{E}|$, there is $s \in \Gamma$, such that $s(x) = e$. Consequently, since $f \circ s$ is a section of $\mathcal{F}$, we obtain $q(f(s(x)) = q(f(e)) = x$, as needed.

It remains to show that f is continuous. As above, take $e \in E$ and $s \in \Gamma$, such that $e \in Im\, s$. Let $U \in \nu_{f(e)}$ and set $V = U \cap Im\, t$, where $t = f \circ s$. Since t is a section of $\mathcal{F}$, with $f(e) \in Im\, t$, V is an open neighborhood of $f(e)$. Consider $W = s(t^{-1}(V))$; then, $W \in \nu_e$ and $f(W) \subseteq V \subseteq U$, completing the proof. $\square$

The monics, epics and isomorphisms in $\boldsymbol{Shg(X)}$ are described in

PROPOSITION 22.26. *Let $\mathcal{E} \xrightarrow{f} \mathcal{F}$ be a morphism of sheaves and E, F be the sheaf spaces of $\mathcal{E}$ and $\mathcal{F}$, respectively.*

a) The following are equivalent :

(1) *f is a monic in $\boldsymbol{Shg(X)}$;*

(2) *f is an injection from E into F;*

(3) *For all $u \in \Omega(X)$, f_u is an injection of $\mathcal{E}(u)$ into $\mathcal{F}(u)$;*

(4) *For all s, $t \in |\mathcal{E}|$, $\quad [\![s = t]\!] = [\![fs = ft]\!]$.*

b) The following are equivalent :

(1) *f is an epic in* $\boldsymbol{Shg(X)}$*;* (2) *f is a surjection of E onto F;*

(3) *$Im\, f$ is dense in $|\mathcal{F}|$.*

c) The following are equivalent :

(1) *f is an isomorphism in* $\boldsymbol{Shg(X)}$*;*

(2) *f is a homeomorphism from E onto F.*

Proof. (a) Clearly, (2) $\Rightarrow$ (1); for the converse, we use the fibered product of $\mathcal{E}$ over f (22.35); in the notation therein, we have $f \circ d_1 = f \circ d_2$, which forces $d_1 = d_2$. But this means that $f(e) = f(e')$ implies $e = e'$, and f is an injection from E into F. The remainder of (a) is left to the reader.
(b) Clearly, (2) $\rightarrow$ (1); we prove that (3) $\Leftrightarrow$ (2), leaving (1) $\Rightarrow$ (2) as an exercise. To fix notation, assume that $\mathcal{E} = \langle\, E, p, X\,\rangle$ and $\mathcal{F} = \langle\, F, q, X\,\rangle$.
<u>(3) $\Rightarrow$ (2)</u> : Fix $y \in F$; let $x = q(y)$ and let t be a section of $\mathcal{F}$ over some open $u \in \nu_x$, such that $t(x) = y$. By (3), $Et = \bigcup_{s\in|\mathcal{E}} [\![fs\, =\, t]\!]$, and so for some $s \in |\mathcal{E}|$, $x \in [\![fs\, =\, t]\!]$. Hence, $f(s(x)) = t(x) = y$ and f is onto F.
<u>(2) $\Rightarrow$ (3)</u> : Let $t \in \mathcal{F}(u)$ be a section of $\mathcal{F}$ over u. Since f is surjective, for $x \in u$, there is $e_x \in E$, such that $f(e_x) = t(x)$. For $x \in u$, select $s_x \in |\mathcal{E}|$ such that

$* \; Es_x \in \nu_x \cap \Omega(u)$ *and* $* \; s_x(x) = e_x$.

Since $f(s_x(x)) = t(x)$, $x \in u$, we have that $[\![(f \circ s_x)\, =\, t]\!] = u_x$ are non-empty opens in X, satisfying $Et = u = \bigcup_{x\in u} u_x$, and (3) follows. Item (c) is straightforward and left to the reader. $\square$

Remark 22.27. By 22.26.(a), the subsheaves of a geometric sheaf can be identified, up to isomorphism, to the open sets of its sheaf space. Thus, the family of subsheaves of $\mathcal{E} = \langle\, E, p, X\,\rangle$ is $\{\langle\, U, p_{|U}, X\,\rangle : U \in \Omega(E)\}$. $\square$

Remark 22.28. A sheaf morphism may be an *epimorphism* in $\boldsymbol{Shg(X)}$, although the maps f_u **are not** surjections. It is this fact that produces the different Cohomology Theories that are such an important part of sheaf theory and its applications. $\square$

By 22.26.(a), if $\mathcal{E}$ is a subsheaf of $\mathcal{F}$ (written $\mathcal{E} \subseteq \mathcal{F}$), then for each $u \in \Omega(X)$, there is a natural injection, $\mathcal{E}(u) \xrightarrow{f_u} \mathcal{F}(u)$, such that for all open $v \subseteq u$, the following diagram is commutative :

$$\begin{tikzcd}
\mathcal{E}(u) \arrow[r, "f_u"] \arrow[d, "|"'] & \mathcal{F}(u) \arrow[d, "|"] \\
\mathcal{E}(v) \arrow[r, "f_v"'] & \mathcal{F}(v)
\end{tikzcd}$$

Furthermore, since $\mathcal{E}$ is a sheaf, any compatible family of sections in $|\mathcal{E}|$ can be "glued" to a section in $\mathcal{E}$.

Exercises

22.29. Let $\mathcal{E} = \langle\, E, p, X \,\rangle$ be a sheaf over X.
a) Prove that condition (4) in 22.8.(b) is equivalent to

$$\text{If } s, t \in \mathcal{E}(u) \text{ are such that } s_{|u_i} = t_{|u_i}, \text{ for } i \in I, \quad \text{then} \quad s = t.$$

b) Let $\{u_i : i \in I\} \subseteq \Omega(X)$, $u = \bigcup u_i$ and $s_i \in \mathcal{E}(u_i)$, $i \in I$. Prove that the statement in 22.8.(c) is equivalent to

$$\begin{cases} \text{If } s_{i|(u_i \cap u_j)} = s_{j|(u_i \cap u_j)}, \ \forall\, i, j \in I, \ \ \textbf{then} \\ \text{there is a } \textit{unique } s \in \mathcal{E}(u), \text{ such that } s_{|u_i} = s_i, \ \forall\, i \in I. \end{cases}$$

□

22.30. Let a, $b \in |\mathcal{E}|$, where $\mathcal{E} = \langle\, E, p, X \,\rangle$ be a sheaf over X. With notation as in 22.8,
a) $a_{|[\![a = b]\!]} = b_{|[\![a = b]\!]}$.
b) $[\![a = b]\!]$ is the *largest* open set $u \subseteq (Ea \cap Eb)$ such that $a_{|u} = b_{|u}$.
c) If u, v are opens in Ea and Eb, respectively, then

$$[\![a_{|u} = b_{|v}]\!] = u \cap v \cap [\![a = b]\!].$$

□

22.31. a) For S, $T \subseteq pF(X, E)$ show that S is dense in T iff for all $t \in T$, there is $B \subseteq S$ such that $Et = \bigcup_{b \in B} [\![s = b]\!]$.
b) If $\mathcal{E} = \langle\, E, p, X \,\rangle$ is a sheaf over X and $S, T \subseteq |\mathcal{E}|$, S is dense in T iff for all $t \in T$ there is $\{\langle\, u_i, s_i \,\rangle : i \in I\} \subseteq \Omega(X) \times S$, such that
(1) $Et = \bigcup u_i$ and (2) $t_{|u_i} = s_{i|Es_i \cap u_i}$, $\forall\, i \in I$. □

22.32. If X is a topological space, $\Omega(X)$ can be represented as the set of global sections of a geometric sheaf over X, whose stalks are $\Omega(X)/\nu_x$, $x \in X$. □

22.33. Let $\mathcal{E}$ be the sheaf of 22.19, with the convention of not distinguishing between elements of Γ and R. Show that
a) If $s \in \mathcal{E}(Z_r)$, then $s = a_{|Z_r}$, for some $a \in \Gamma$.
b) For all $s, t \in \mathcal{E}$, if Es and Et are clopen in $Spec(R)$, then $[\![s = t]\!]$ is clopen in $Spec(R)$.
c) If P is a prime ideal in a commutative regular ring R, then the localization of R at P (9.41), R_P, is naturally isomorphic to the residue field R/P. □

22.34. a) In the situation of 22.24, show that for each $u \in \Omega(X)$, there is a bijective map

$$\eta_u : [\textstyle\prod \mathcal{E}_i](u) \longrightarrow \prod E_i(u),$$

such that for all open $v, u \in \Omega(X)$, if $v \subseteq u$, then the following diagram is commutative, where $_|$ is the restriction map in the sheaf $\prod \mathcal{E}_i$, while $_|{}^n$ is the product of the restriction maps on each $\mathcal{E}_i$:

$$\begin{tikzcd}
{[\prod E_i](u)} \arrow[r, "\eta_u"] \arrow[d, "|"'] & \prod E_i(u) \arrow[d, "|^{n}"] \\
{[\prod E_i](v)} \arrow[r, "\eta_v"'] & \prod E_i(v)
\end{tikzcd}$$

b) Is it possible to use 22.14 to give a proof of 22.24 ? □

22.35. Let $\mathcal{E} = \langle E, p, X \rangle \xrightarrow{f} \mathcal{F} = \langle F, q, X \rangle$ be a morphism of sheaves over X. Define the **fibered product of *E* over *f*** by

$$E \times_f E = \{\langle e, e' \rangle \in E \times E : f(e) = f(e')\},$$

with the topology induced by $E \times E$. Note that for all $\langle e, e' \rangle \in E \times_f E$ we have $p(e) = p(e')$. Thus, $\langle e, e' \rangle \mapsto p(e)$ defines a map from $E \times_f E$ to X, written ρ. We also have projections d_1, d_2 from $E \times_f E$ to E, given by the restriction of the projections defined on $E \times E$ onto the first and the second coordinates, respectively. Clearly, these are continuous functions. Then :

a) $E \times_f E$ is open in $E \times E$.

b) ρ is the restriction of $p \times p$ to $E \times_f E$. Conclude that $\mathcal{E} \times_f \mathcal{E} = \langle E \times_f E, \rho, X \rangle$ is a sheaf over X.

c) d_1 and d_2 are the restriction to $E \times_f E$ of the canonical projections from $E \times E \longrightarrow E$, described in 22.24. Conclude that they are sheaf morphisms from $\mathcal{E} \times_f \mathcal{E}$ to $\mathcal{E}$. Verify that $f \circ d_1 = f \circ d_2$.

In the above situation, if $f = Id_{\mathcal{E}}$ then we obtain a sheaf, indicated by $\Delta_{\mathcal{E}}$, which corresponds to the graph of the identity relation on $\mathcal{E}$. □

22.36. a) Let K be a closed set in a topological space Y. Let $Z = Y \cup (\{*\} \times K)$ be the disjoint union Y and K, with the following topology :

(1) The topology on Y is its original topology.

(2) An open neighborhood of $\langle *, z \rangle \in (\{*\} \times K)$ is an open neighborhood V of z in Y, with $\langle *, z \rangle$ in place of z : $(V - \{z\}) \cup \{\langle *, z \rangle\}$.

Let $\langle Y, p, X \rangle$ be a sheaf over X.

a) Define $Z \xrightarrow{q} X$ by $q_{|Y} = p$ and $q(*, z) = p(z)$; then $\langle Z, q, X \rangle$ is a sheaf over X and $Y \xrightarrow{h} Z$, given by $h(y) = y$, is a sheaf morphism.

b) Show that $k : Y \longrightarrow Z$, given by

$$k(z) = \begin{cases} z & \text{if } z \notin K \\ \langle *, z \rangle & \text{if } z \in K, \end{cases}$$

is a sheaf morphism and prove that (1) ⇔ (2) in 22.26.(b). □

The following Exercise indicate how Examples 22.15, 22.17 and 22.19 do in fact yield algebraic representation results.

22.37. Notation as in 22.15, let $\mathcal{E} = \langle E, p, S(H) \rangle$ be the sheaf constructed therein. There are global sections $\top^*$ and $\perp^*$, defined, for $F \in S(H)$, by

$$\top^*(F) = \top/F \quad \text{and} \quad \bot^*(F) = \bot/F.$$

Define $\wedge, \vee : \mathcal{E} \times \mathcal{E} \longrightarrow \mathcal{E}$ as follows :

For $\langle e, e' \rangle \in (E \times E)$, with $p(e) = p(e') = F \in S(H)$, write $e = a/F$ and $e' = b/F$, with $a, b \in H$ and set

$$\begin{cases} \wedge(e, e') &= a/F \wedge b/F &= (a \wedge b)/F \\ \vee(e, e') &= a/F \vee b/F &= (a \vee b)/F. \end{cases}$$

Define $\leq = \{\langle a/F, b/F \rangle \in E \times E : a/F \leq b/F \text{ in } H/F\}$, together with a projection on $S(H)$, corresponding to the restriction of $p \times p$ to $\leq$. Then :

a) $\leq$ is a subsheaf of $\mathcal{E} \times \mathcal{E}$, and the maps $\wedge$, $\vee$ are well-defined sheaf morphisms.

b) For each open u in $S(H)$, the set of sections over u, $\mathcal{E}(u)$, inherits

(i) a binary relation $\leq(u)$, the set of sections of $\leq$ over u;

(ii) two binary operations, namely $\wedge_u$ and $\vee_u$.

(iii) two distinguished elements ${\top^*}_{|u}$, ${\bot^*}_{|u}$.

With this structure, $\mathcal{E}(u)$ is a HA. Further, the restriction map $\mathcal{E}(u) \longrightarrow \mathcal{E}(v)$, $v \subseteq u$, are HA-morphisms.

c) For each $a \in H$ there is a *HA-isomorphism*, $\gamma_a : \mathcal{E}(S_a) \longrightarrow H/a^{\rightarrow}$, such that for all $b \leq a$ the following diagram is commutative :

$$\begin{tikzcd} \mathcal{E}(S_a) \arrow[r, "\gamma_a"] \arrow[d, "|"'] & H/a^{\rightarrow} \arrow[d, "\alpha_{ab}"] \\ \mathcal{E}(S_b) \arrow[r, "\gamma_b"'] & H/b^{\rightarrow} \end{tikzcd}$$

where α_{ab} is the HA-morphism $\alpha_{ab}(x/a^{\rightarrow}) = x/b^{\rightarrow}$, $x \in H$.

e) The map constructed in 22.15 is a lattice isomorphism from H onto the *lattice* of global sections of $\mathcal{E}$. $\square$

22.38. With Exercise 22.37 as a model, discuss Examples 22.17 and 22.19, to show that integral domains and commutative regular rings are *ring isomorphic* to the rings of global sections of the sheaves constructed in those Examples. $\square$

22.39. How would one define a first-order structure in the category of geometric sheaves ? $\square$

CHAPTER 23

Sheaves and Presheaves

In this Chapter we present the objects in the title from a functorial viewpoint. This has proven to be more useful and flexible than to deal directly with geometric objects. The results of Chapter 16 will be of constant use. The references for this Chapter are, essentially, the same as in Chapter 22. Deviating a bit from a classical presentation, we introduce the notation of [**15**] at an early stage.

From Example 16.4, we know that the poset $\Omega(X)$, of opens in a topological space X, can be considered as a category.

Definition 23.1. *Let X be a topological space and let $\mathcal{C}$ be a category. A* **$\mathcal{C}$-presheaf over X** *is contravariant functor, $P : \Omega(X) \longrightarrow \mathcal{C}$. For $u \subseteq v$ in $\Omega(X)$*
$*$ *$P(u)$ are the* **sections** *of P over u;*
$*$ *$p_{vu} =_{def} P(\langle u, v \rangle) : P(v) \longrightarrow P(u)$ is the* **restriction morphism** *from $P(v)$ to $P(u)$.*
The expression presheaf of $\mathcal{C}$-objects *is synonymous with $\mathcal{C}$-presheaf over X.*

If P and Q are $\mathcal{C}$-presheaves over X, a **morphism** *is a natural transformation, $\eta : P \longrightarrow Q$. $\mathcal{C}$-Presheaves over X and their morphism are a category, written* ***pSh*$(X, \mathcal{C})$**.

Remark 23.2. A morphism $\eta : P \longrightarrow Q$ of $\mathcal{C}$-presheaves is a natural transformation, that is,
$*$ $\eta = \{\eta_u \in [P(u),\ Q(u)]_{\mathcal{C}} : u \in \Omega(X)\}$ [1];
$*$ For all $v \leq u$ in $\Omega(X)$, the following diagram is commutative :

$$\begin{tikzcd} P(u) \arrow[r, "\eta_u"] \arrow[d, "(\cdot)_{|v}"'] & Q(u) \arrow[d, "(\cdot)_{|v}"] \\ P(v) \arrow[r, "\eta_v"'] & Q(v) \end{tikzcd}$$

We shall latter see how to relate this "graded" notion of morphism with a more "set-theoretical" concept (23.19). $\square$

[1] Recall that $[A, B]_{\mathcal{C}}$ is the set of morphisms from A to B in $\mathcal{C}$.

EXAMPLE 23.3. If $\mathcal{C}$ has a final object, $\mathbf{1}$, we may define a presheaf over X, also indicated by $\mathbf{1}$, as follows : for all $u \in \Omega(X)$,

$\mathbf{1}(u) = \mathbf{1}$ and the restriction maps are the identity morphism of $\mathbf{1}$.

It is clear that $\mathbf{1}$ is the final object in $\boldsymbol{pSh(X, \mathcal{C})}$. Similar comments apply when $\mathcal{C}$ has an initial object or a zero. ☐

REMARK 23.4. It is clear that a presheaf of $\mathcal{C}$-objects can also be defined as a *covariant* functor from $\Omega(X)^{op}$ to $\mathcal{C}$, where $*^{op}$ is the dual category as in 16.5. Many texts on the subject adopt this definition.

If $P : \Omega(X)^{op} \longrightarrow \mathcal{C}$ is a presheaf, then with the notation of 23.1, observe that if $u \subseteq v \subseteq w$ in $\Omega(X)$, then functoriality entails

(1) $p_{uu} = Id_{P(u)}$ and (2) $p_{wu} = p_{vu} \circ p_{wv}$.

Moreover, $P(\emptyset)$ must be an object in $\mathcal{C}$ such that $[P(u), P(\emptyset)] \neq \emptyset$, for all $u \in \Omega(X)$. Whenever $\mathcal{C}$ *has a final object*, $\mathbf{1}$, it is standard to set

$$P(\emptyset) = \mathbf{1},$$

a convention we shall also adopt. The following table exemplifies the terminology in 23.1, with notation as in 16.3 :

Usual Name	Functor from $\Omega(X)^{op}$ to
Presheaf of sets over X	**Set**
Presheaf of partial orders over X	**Po**
Presheaf of distributive lattices	$\boldsymbol{\mathcal{D}}$
Presheaf of Abelian groups over X	**AbGr**
Presheaf of rings over X	**CR**
Presheaf of L-structures over X	**L-mod**

There is something in common with all the examples in the preceding table : all are *set-based categories* (16.1), the only type of category that will be target of presheaves in this book. ☐

23.5. **Conventions.** *a) From here on, the term* **category** *will mean* **set-based category***, that is, sets underlie its objects and maps underlie its morphisms.*

b) If $P : \Omega(X)^{op} \longrightarrow \mathcal{C}$ is a $\mathcal{C}$-presheaf over X, then

(1) $P(\emptyset) = \{*\}$, a fixed singleton;

(2) If $u \in \Omega(X)$ and $s \in P(v)$, define $s_{|u} = p_{v,u\cap v}(s)$.

Note that if $u \subseteq v$ in $\Omega(X)$ and $x \in P(v)$, then $x_{|u} = p_{vu}(s)$. Hence, the operation $(\cdot)_{|u}$ generalizes the restriction morphisms in a presheaf.

(3) For $u, v \in \Omega(X)$, $\boldsymbol{u \leq v}$ stands for $v \in \Omega(X)$ and $u \in \Omega(v)$. ☐

DEFINITION 23.6. *Let $P : \Omega(X)^{op} \longrightarrow \mathcal{C}$ be a $\mathcal{C}$-presheaf over X.*

a) The **domain** *of P is the* **set** $|P| = \coprod_{u\in\Omega(X)} P(u)$.

b) For $s \in |P|$, the **extent** *of s in P is defined as*

$$E_P s = \textit{the unique } u \in \Omega(X) \textit{ such that } s \in P(u).$$

c) If $S \subseteq |P|$, the **support of S** *is defined as*

$$E_P S = \bigcup_{s \in S} Es.$$

d) For $s, t \in |P|$, define [2]

$$[\![s = t]\!]_P \ = \ \bigcup \{u \leq (Es \cap Et) : s_{|u} = t_{|u}\},$$

called the **equality** *of P. Note that $[\![s = t]\!]_P \in \Omega(X)$.*

e) P is **extensional** [3] *if for all $s, t \in |P|$,*

$$[ext] \qquad E_P s = E_P t = [\![s = t]\!]_P \quad \Leftrightarrow \quad s = t.$$

Context allowing, we omit P from the notation of extent, support and equality.

Proposition 23.7. *Let P be a $\mathcal{C}$-presheaf over X, let $s, t, z \in |P|$ and let $u, v \in \Omega(X)$.*

a) Restriction in P verifies the following properties :

$$[rest\ 1] : \ s_{|Es} = s; \qquad [rest\ 2] : \ Es_{|u} = u \cap Es;$$

$$[rest\ 3] : \ (s_{|u})_{|v} = s_{|u \cap v}; \qquad [E] : \ Es = [\![s = s]\!].$$

b) The equality of P satisfies the following properties :

$[= \ 1] : \ [\![s = t]\!] \ = \ [\![t = s]\!]$;

$[= \ 2] : \ [\![s = t]\!] \cap [\![t = z]\!] \ \subseteq \ [\![s = z]\!]$.

$[E =] : \ [\![s = t]\!] \ \leq \ (Es \cap Et)$.

c) $[\![s_{|u} = s]\!] = u \cap Es$

d) $[\![s_{|u} = t_{|v}]\!] = u \cap v \cap [\![s = t]\!]$.

e) If P is extensional, then $s_{|[\![s=t]\!]} = t_{|[\![s=t]\!]}$ and

$$[\![s = t]\!] \ = \ \max \{u \leq (Es \cap Et) : s_{|u} = t_{|u}\}.$$

f) Consider the conditions

$$(1) \ s_{|u} = t_{|u}; \qquad (2) \ u \leq (Es \cup Et) \to [\![s = t]\!].$$

Then, (1) $\Rightarrow$ (2). If P is extensional, these conditions are equivalent.

Proof. a) All equations in (a) are straightforward consequences of the definitions. Relations $[= \ 1]$ and $[E =]$ in (b) are clear; the proof of $[= \ 2]$ is exactly like that given in 22.8.(b).(2) for the corresponding relation in geometric sheaves.
c) Since $Es_{|u} = u \cap u$ ($[rest\ 2]$), $[E =]$ in (b) yields

$$[\![s_{|u} = s]\!] \leq Es_{|u} \cap Es = u \cap Es \cap Es = u \cap Es.$$

To verify equality, note that $[rest\ 1]$ and $[rest\ 3]$ in (a) imply,

$$(s_{|u})_{|u \cap Es} = s_{|u \cap u \cap Es} = s_{|u \cap Es},$$

and so $u \cap Es \leq [\![s = t]\!]$, as needed.
d) Since $Es_{|u} = u \cap Es$ and $Et_{|v} = v \cap Et$, $[E =]$ entails

[2]Compare 22.8.(a); and keep in mind 23.5.(b).(3).
[3]*Separated* or a *mono-presheaf*.

$$[\![s_{|u} = t_{|v}]\!] \leq u \cap v \cap Es \cap Et.$$

In particular,

$$Es \cap Et \cap u \cap v \cap [\![s_{|u} = t_{|v}]\!] \;=\; [\![s_{|u} = t_{|v}]\!]. \tag{I}$$

Item (c), (I) and the transitive property [= 2] of equality yield

$$\begin{aligned} u \cap v \cap [\![s = t]\!] \;&\supseteq\; u \cap v \cap [\![s = s_{|u}]\!] \cap [\![s_{|u} = t_{|v}]\!] \cap [\![t_{|v} = t]\!] \\ &=\; u \cap v \cap Es \cap Et \cap [\![s_{|u} = t_{|v}]\!] \\ &=\; [\![s_{|u} = t_{|v}]\!]. \end{aligned}$$

We may once again use transitivity to write

$$[\![s_{|u} = t_{|v}]\!] \;\supseteq\; [\![s_{|u} = s]\!] \cap [\![s = t]\!] \cap [\![t = t_{|v}]\!],$$

and item (c) applies to yield $u \cap v \cap [\![s = t]\!] \;\leq\; [\![s_{|u} = t_{|v}]\!]$, establishing the desired equality.

e) By the preceding items we have

$$Es_{|[\![s=t]\!]} \;=\; Es \cap [\![s = t]\!] \;=\; [\![s = t]\!] \;=\; [\![s_{|[\![s=t]\!]} = t_{|[\![s=t]\!]}]\!],$$

and so if P is extensional, the desired equality follows. The remaining assertion is now clear.

f) If (1) holds, then

$$Es_{|u} = Es \cap u = Et_{|u} = Et \cap u.$$

Thus, if $v = Es \cap u = Et \cap u$, we have $v \leq (Es \cap Et)$ and $s_{|v} = t_{|v}$, by the properties in item (a). It follows that $v \leq [\![s = t]\!]$. Hence,

$$u \cap (Es \cup Et) = (u \cap Es) \cup (u \cap Et) = v \leq [\![s = t]\!],$$

and the adjointness [$\rightarrow$] in 6.1 entails $u \leq (Es \cup Et) \rightarrow [\![s = t]\!]$, establishing (2).
Conversely, suppose that P is extensional and that (2) holds. Then,

$$u \cap (Es \cup Et) \leq [\![s = t]\!] \leq Es \cap Et.$$

Hence,

$$u \cap [\![s = t]\!] \;\subseteq\; u \cap Es \;\subseteq\; u \cap (Es \cup Et) \;\subseteq\; u \cap [\![s = t]\!],$$

and so $u \cap Es = u \cap [\![s = t]\!]$; similarly, one has $u \cap Et = u \cap [\![s = t]\!]$. Since P is extensional, items (a) and (e) yield

$$\begin{aligned} s_{|u} \;&=\; s_{|Es\cap u} \;=\; s_{|[\![s=t]\!]\cap u} \;=\; (s_{|[\![s=t]\!]})_{|u} \;=\; (t_{|[\![s=t]\!]})_{|u} \;=\; t_{|[\![s=t]\!]\cap u} \\ &=\; t_{|Et\cap u} \;=\; t_{|u}, \end{aligned}$$

ending the proof. □

Definition 23.8. *Let P be a $\mathcal{C}$-presheaf over X and $S \subseteq |P|$.*

a) S is **compatible** *if for all $s, t \in S$,* $s_{|Et} = t_{|Es}$.

b) P is a **sheaf** *if for all* compatible $S \subseteq |P|$, *there is a* unique $t \in |P|$ *such that*

(1) $Et = \bigcup_{s \in S} Es$; (2) *For all* $s \in S$, $t_{|Es} = s$.

A morphism of $\mathcal{C}$-sheaves over S is a morphism of the underlying presheaves. $\mathcal{C}$-sheaves and their morphisms are a category, written $\boldsymbol{Sh(X, \mathcal{C})}$.

Example 23.9. If $\mathcal{E} = \langle E, p, X \rangle$ is a geometric sheaf over X, the discussion in Chapter 22 (22.6) indicates that $\mathcal{E}$ originates a sheaf as in 23.8,

$$u \in \Omega(X) \mapsto \mathcal{E}(u), \text{ with restriction maps } (\cdot)_{|uv}.$$

Note that the equality defined therein coincides with the one set down here. Moreover, Lemma 22.25 and the paragraphs that precede it, show that morphisms of geometric sheaves originate morphisms of sheaves as defined above. $\square$

REMARK 23.10. One should beware of confusing the support of a presheaf with the largest open over which it has sections. The support of the covering space of the circle by the real line (22.11) is S^1, but it has no global sections. Note that this gives an example of a **sheaf** with this property, and so the question here is not lack of completeness. $\square$

EXAMPLE 23.11. Let I be a set and $\{M_i : i \in I\}$ be a non-empty family of sets indexed by I. Endow I with the discrete topology (all points are open). Define a presheaf $\mathcal{M} : \Omega(I) \longrightarrow$ **Set**, as follows :

$*$ For $u \subseteq I$, $\mathcal{M}(u) = \prod_{i \in u} M_i$;

$*$ If $u \leq v \subseteq I$, the restriction map, $\rho_{vu} : \mathcal{M}(v) \longrightarrow \mathcal{M}(u)$, is the projection that forgets the coordinates outside u.

It is easily verified that $\mathcal{M}$ is a sheaf over I, the **discrete sheaf** generated by the family $\{M_i : i \in I\}$. For $s, t \in |\mathcal{M}|$, $Es = dom\, s$ and

$$[\![s = t]\!] \;=\; \{i \in Es \cap Et : s(i) = t(i)\}.$$

$\mathcal{M}$ has a special property : all restriction maps are *surjective*. In particular, every section is the restriction of a global section. Presheaves satisfying the latter property are called **flabby** (31.10.(e)). $\square$

EXAMPLE 23.12. If P is a $\mathcal{C}$-presheaf over X and $u \in \Omega(X)$, define a functor $P_{|u} : \Omega(X)^{op} \longrightarrow \mathcal{C}$, by

$$P_{|u}(v) = \begin{cases} P(v) & \text{if } v \leq u \\ \emptyset & \text{otherwise,} \end{cases}$$

with restrictions induced by P; $P_{|u}$ is a $\mathcal{C}$-presheaf over X, called the **restriction of P to u**. Note that

$*$ $P_{|u}$ is a sheaf whenever P is a sheaf; $\quad *$ $|P_{|u}| = \{x \in |P| : Ex \leq u\}$;

$*$ For $S \subseteq |P|$, $S \subseteq |P_{|u}|$ iff $ES \leq u$. $\square$

EXAMPLE 23.13. For opens $v \leq u$ in $\Omega(X)$, there is a frame map

$$\delta_{uv} : \Omega(u) \longrightarrow \Omega(v), \text{ given by } \delta_{uv}(w) \;=\; w \cap v.$$

The contravariant functor $\widetilde{\Omega} : \Omega(\mathrm{X}) \longrightarrow$ **Frame**, given by

$$u \longmapsto \Omega(u) \quad \text{and} \quad \iota_{v,u} \longmapsto \delta_{uv},$$

is a sheaf of frames over X, called **the sheaf of opens** of X and written $\widetilde{\Omega}(X)$. The reader may check that for $p \in \Omega(u)$ and $q \in \Omega(v)$,

$$[\![p = q]\!] \;=\; (p \leftrightarrow q) \cap u \cap v. \qquad \square$$

EXAMPLE 23.14. Recall that $\mathbb{C}(X, Y)$ is the set of continuous functions from X to Y. If $v \leq u$ are opens in X, we have a natural restriction

$$\cdot_{|v} : \mathbb{C}(u, Y) \longrightarrow \mathbb{C}(v, Y),\ f \;\longmapsto\; f_{|v}. \tag{*}$$

The associations

$$u \in \Omega(X) \longmapsto \mathbb{C}(u, Y) \quad \text{and} \quad v \leq u \longmapsto \cdot_{|v},$$

define a sheaf of sets over X, the **sheaf of continuous functions** of X into Y, also written $\mathbb{C}(X, Y)$. The fact that $\mathbb{C}(X, Y)$ is indeed a sheaf comes from an application of 1.2 to a compatible family of continuous maps, defined on open sets of X. Since continuity is a local property, 1.2 applies *ipsis litteris*, once continuity is added to its hypotheses and conclusion.

For all sections s, t in $\mathbb{C}(X, Y)$,

$$[\![s = t]\!] \;=\; \text{int}\,\{x \in X : s(x) = t(x)\}.$$

If $Y = \mathbb{R}$ [4], $\mathbb{C}(X)$ is an algebra over $\mathbb{R}$, and the restriction maps in (*) are algebra homomorphisms. Hence, $\mathbb{C}(X)$ is a sheaf of algebras over X. $\square$

Example 23.15. Let X, Y be topological spaces. Recall that (see [Cb] in page 194)

$$\mathbb{C}_b(X, Y) = \{f \in \mathbb{C}(X, Y) : \overline{Im\, f} \text{ is compact in } Y\}.$$

If $u \leq v$ in X, the natural restrictions of 23.14 also yield restrictions from $\mathbb{C}_b(v, Y)$ to $\mathbb{C}_b(u, Y)$, that constitute an extensional presheaf of sets over X, written $\mathbb{C}_b(X, Y)$. Equality in this case is exactly as in Example 23.14.

If $Y = \mathbb{R}$, $\mathbb{C}_b(X)$ is the algebra of **bounded** continuous real valued functions on X, an algebra over $\mathbb{R}$, associated to which we have an extensional presheaf of algebras over X, $\mathbb{C}_b(X)$, that in general **is not** a sheaf (if $X = \mathbb{R}$, for instance).

Clearly, $\mathbb{C}_b(X) = \mathbb{C}(X)$, when X is a compact space. The Hausdorff spaces for which $\mathbb{C}_b(X) = \mathbb{C}(X)$ are called **pseudo-compact**. The reader may wish to try his hand at proving that the first uncountable ordinal, ω_1, with the order topology, is an example of a pseudo compact space that is not compact. $\square$

Example 23.16. Let A be a set. The associations

$$u \in \Omega(X) \mapsto A \quad \text{and} \quad \emptyset \neq v \leq u \mapsto Id_A$$ [5]

define an extensional presheaf of sets over X, called the **constant presheaf** A on X. The reader might wish to verify that A is a sheaf iff X is irreducible (12.8).

We can consider A as a topological space, with the discrete topology (all points are open). The sheaf of continuous functions $\mathbb{C}(X, A)$, denoted by $\boldsymbol{A}(X)$, is called the **constant sheaf** of stalk A on X. When X is clear from context, its mention will be omitted. For $s, t \in |\boldsymbol{A}|$,

$$[\![s = t]\!] \text{ is a disjoint union of \textbf{clopens} in } Es \cap Et.$$

We also have an extensional presheaf $\boldsymbol{A}_b(X) = \mathbb{C}_b(X, A)$ (as in 23.15). Since A has the discrete topology, *compact* is the same as *finite*, and so for all $u \leq X$,

$$\boldsymbol{A}_b(u) = \{f \in \mathbb{C}(u, A) : Im\, f \text{ is a finite subset of } A\},$$

with restrictions induced by $\boldsymbol{A}$. Clearly, $\boldsymbol{A}(u) = \boldsymbol{A}_b(u)$ when u is compact and $\boldsymbol{A} = \boldsymbol{A}_b$ if A is finite. In this case, for $s, t \in |\boldsymbol{A}_b|$,

$$[\![s = t]\!] \text{ is \textbf{clopen} in } Es \cap Et.$$

[4] Or whenever Y is a topological ring or algebra

[5] Since $\{0\}$ is the final object in **Set**, $A(\emptyset) = \{0\}$, as accorded in 23.4.

Our convention is that the constant sheaf is denoted by a bold letter corresponding to the set from which it is constructed. Thus, if $n \geq 0$ is positive integer, $\boldsymbol{n}(X)$ is the constant sheaf of stalk $\{0, 1, \ldots, n\}$ over the topological space X. We have already registered that when allowed we drop mention to X. The constant sheaf with empty stalk is $\mathbf{0}$, while $\mathbf{1}$ is the constant $\{0\}$ sheaf. $\square$

EXAMPLE 23.17. Let X be a topological space. Notation as in 23.16, if A is a first-order structure, then $\boldsymbol{A}$ will a sheaf of first-order structures as well. To see this, fix a first-order language L and the L-structure A. We endow $T = \mathbb{C}(X, A)$ with the L-structure from A^X, as in 17.9. Explicitly, if $\overline{f} = \langle f_1, \ldots, f_n \rangle \in T^n$ and $x \in X$, set

$$\overline{f}(x) = \langle f_1 x, \ldots, f_n x \rangle.$$

Then, if $R \in rel(n, L)$, $\omega \in op(n, L)$, $c \in Ct(L)$ and $\overline{f} \in T^n$,

[R] : $\mathbb{C}(X, A) \models R[\overline{f}]$ iff $\forall\, x \in X, \quad A \models R[\overline{f}(x)]$;

[ω] : $\omega^T(\overline{f}) = \omega^A \circ \langle f_1, \ldots, f_n \rangle$;

[c] : The interpretation of c is the constant map $\widehat{c}$, of value c.

The equation in [ω] means that for all $x \in X$, $[\omega^T(\overline{f})](x) \;=\; \omega^A(\overline{f}(x))$.
To see that this indeed defines an element of T, it suffices to show that each $x \in X$ has a neighborhood on which $\omega^T(\overline{f})$ is constant. Fix $x \in X$ and select $u_1, \ldots, u_n \in \nu_x$ such that f_k is constant in u_k. Consider $u = \bigcap_{k=1}^n u_k$; all f_k are constant in u and so for all $y \in u$, $\omega^A(\overline{f}(y)) = \omega^A(\overline{f}(x))$, as needed.

It is straightforward that for opens $v \leq u$ in X, the restriction maps

$$f \in \mathbb{C}(u, A) \mapsto f_{|v} \in \mathbb{C}(v, A)$$

are L-morphisms. Thus, $\boldsymbol{A}$ is a presheaf of L-structures over X. It is easily established that $\boldsymbol{A}$ is actually a *sheaf*, as asserted. Similarly, $\boldsymbol{A}_b$ is an extensional presheaf of L-structures over X. In section 24.5 we present some of the fundamental properties of constant sheaves, as well as other examples. $\square$

REMARK 23.18. $\mathbb{C}(\cdot, \cdot)$ is a *bifunctor* from **Top** $\times$ **Top** to **Set** (16.15), contravariant in the first coordinate and covariant in the second. Hence, continuous maps $\alpha : X' \longrightarrow X$ and $\beta : Y \longrightarrow Y'$ induce

$$\begin{cases} \langle \beta, \alpha \rangle : \mathbb{C}(X, Y) \longrightarrow \mathbb{C}(X', Y'), \quad f \mapsto \beta \circ f \circ \alpha \\ \qquad\qquad X' \xrightarrow{\alpha} X \xrightarrow{f} Y \xrightarrow{\beta} Y'. \end{cases}$$

For $v \in \Omega(X')$, set $\alpha_v = \alpha_{|\alpha^{-1}(v)}$. Then, for all $u \leq v$ in $\Omega(X')$, the following diagram commutes :

$$\begin{tikzcd} \mathbb{C}(\alpha^{-1}(v), Y) \arrow[r, "\langle \alpha_v, \beta \rangle"] \arrow[d, "\cdot_{|\alpha^{-1}(u)}"'] & \mathbb{C}(v, Y') \arrow[d, "\cdot_{|u}"] \\ \mathbb{C}(\alpha^{-1}(u), Y) \arrow[r, "\langle \alpha_u, \beta \rangle"'] & \mathbb{C}(u, Y') \end{tikzcd}$$

This example, as well as 20.9, show that it is important to study *morphisms between sheaves with different bases*, giving an indication on how to define such a concept (see Definitions 29.1 and 29.3). □

Let $\eta : P \longrightarrow Q$ be a morphism of $\mathcal{C}$-presheaves (23.2). We can view η as a map from $|P|$ to $|Q|$, as follows :

$$\text{For } s \in |P|, \text{ set } \quad \eta s = \eta_{Es}(s).$$

The following Lemma summarizes the relations between the "natural transformation" concept of morphism and that just introduced.

LEMMA 23.19. *Let P and Q be $\mathcal{C}$-presheaves over X. Consider the following conditions :*
(1) *There is a* unique *$\mathcal{C}$-presheaf morphism, $\eta : P \longrightarrow Q$, such that $\eta_{Es}(s) = f(s)$, $s \in |P|$.*
(2) *For all $s \in |P|$ and $u \in \Omega(X)$,* $\begin{cases} (i) \quad E_Q f(s) = E_P s; \\ (ii) \quad f(s_{|u}) = f(s)_{|u}. \end{cases}$
(3) *For all $s, t \in |P|$,* $\begin{cases} [mor\ 1] : \quad E_Q f(s) = E_P s; \\ [mor\ 2] : \quad [\![s = t]\!]_P \ \subseteq \ [\![f(s) = f(t)]\!]_Q. \end{cases}$

Then,
a) (1) $\Rightarrow$ (2) $\Rightarrow$ (3).
b) If Q is extensional, then (3) $\Rightarrow$ (2).
c) If P and Q are presheaves of sets then (3) *and* (1) *are equivalent. If Q is extensional, all three conditions are equivalent.*

PROOF. We write $E(\cdot)$ for the extension of sections in P and in Q, the context making it clear to which presheaf the section belongs.
a) (1) $\Rightarrow$ (2) is an immediate consequence of the definitions.
<u>(2) $\Rightarrow$ (3) :</u> Clearly it suffices to prove [mor 2]. For $s, t \in |P|$, the properties of restriction in 23.7.(a) and (2).(ii) yield

$$(*) \quad [\![f(s_{|[\![s=t]\!]}) = f(t_{|[\![s=t]\!]})]\!] \ = \ [\![f(s)_{|[\![s=t]\!]} = f(t)_{|[\![s=t]\!]}]\!] \ = \ [\![f(s) = f(t)]\!] \cap [\![s = t]\!].$$

On the other hand, $s_{|[\![s=t]\!]} = t_{|[\![s=t]\!]}$ and $Es_{|[\![s=t]\!]} = [\![s = t]\!]$. Thus, from (2).($i$) and (*) comes

$$[\![s = t]\!] \ = \ [\![f(s) = f(t)]\!] \cap [\![s = t]\!],$$

which is equivalent to [mor 2].
b) If $\langle\, s, v \,\rangle \in P(u) \times \Omega(u)$, then

$$Ef(s_{|v}) = Es_{|v} = Es \cap v = Ef(s) \cap v = E(fs)_{|v}. \tag{A}$$

Thus, [mor 2] and [$rest$ 3] in 23.7.(a) yield

$$[\![f(s_{|v}) = (fs)_{|v}]\!] \ = \ [\![f(s_{|v}) = fs]\!] \cap v \ \supseteq \ [\![s_{|v} = s]\!] \cap v \ = \ Es \cap v.$$

By (A), $[\![f(s_{|v}) = (fs)_{|v}]\!] \ \subseteq \ Es \cap v$, and so

$$[\![f(s_{|v}) = (fs)_{|v}]\!] \;=\; Es \cap v \;=\; Ef(s_{|v}) = E(fs)_{|v}.$$

Now, the extensionality of Q yields $f(s_{|v}) = (fs)_{|v}$.

c) First note that if $s \in P(u)$, then $f(s) \in Q(u)$ ([*mor* 1]). Therefore, we may define, for $u \in \Omega(X)$,

$$\eta_u : P(u) \longrightarrow Q(u) \quad \text{by} \quad s \mapsto f(s).$$

It is straightforward that the family of maps η_u, $u \in \Omega(X)$, is a morphism of presheaves of **sets**; its uniqueness is clear from the construction. $\square$

Remark 23.20. Whenever possible, and based on 23.19, we shall write a morphism without explicit mention of its grading. For instance, if $\eta : P \longrightarrow Q$ is a morphism and $S \subseteq |P|$, we write

$$\eta S = \{\eta s : s \in S\}$$

instead of $\{\eta_{Es} s : s \in S\}$. $\square$

The examples we have given above are all sheaves and extensional presheaves. There is good reason for this : if the category $\mathcal{C}$ is sufficiently rich, there is a natural way to associate to a $\mathcal{C}$-presheaf over X an extensional one. This produces a functor, from the category of $\mathcal{C}$-presheaves to the subcategory of extensional $\mathcal{C}$-presheaves that reflects all categorical constructions. We shall prove this for *presheaves of first-order structures*, a result that is sufficiently general for our purposes.

Theorem 23.21. *Let L be a first-order language with equality and let P be a presheaf of L-structures over X. There is an extensional* **L-mod***-presheaf over X, εP, and a morphism of presheaves of L-structures, $\varepsilon : P \longrightarrow \varepsilon P$, that satisfies the following conditions :*

a) $|\varepsilon P| = \{\varepsilon s : s \in |P|\}$.

b) For all $s, t \in |P|$, $[\![\varepsilon s = \varepsilon t]\!] = [\![s = t]\!]$.

c) If $\eta : P \longrightarrow Q$ is a morphism and Q is an extensional presheaf of L-structures over X, there is a **unique** *morphism, $\varepsilon\eta : \varepsilon P \longrightarrow Q$, making the following diagram commute :*

$$\begin{tikzcd} P \arrow[rr, "\varepsilon"] \arrow[dr, "\eta"'] & & \varepsilon P \arrow[dl, "\varepsilon\eta"] \\ & Q & \end{tikzcd}$$

The presheaf εP is the **extensionalization** *of P.*

Proof. We start with

Fact 1. *Let ω be a n-ary relation symbol in L and $u \in \Omega(X)$. For $\overline{s} = \langle s_1, \ldots, s_n \rangle$, $\overline{t} = \langle t_1, \ldots, t_n \rangle$ in $P(u)^n$, $\bigcap_{k=1}^{n} [\![s_k = t_k]\!] \;\le\; [\![\omega(\overline{s}) = \omega(\overline{t})]\!]$.*

Proof. [6] Let $u_1, \ldots, u_n \in \Omega(X)$ be such that for $1 \leq k \leq n$,
$$u_k \leq (Es_k \cap Et_k) \quad \text{and} \quad s_{k|u_k} = t_{k|u_k},$$
and set $u = \bigcap_{k=1}^n u_k$. It follows from [*rest* 3] in 23.7.(a) that
$$\text{For all } 1 \leq k \leq n, \quad s_{k|u} = t_{k|u}. \tag{I}$$
Since P is a presheaf of L-structures, the restriction maps are L-morphisms, that is,
$$\omega(\overline{s})_{|u} = \omega(s_{1|u}, \ldots, s_{n|u}). \tag{II}$$
(I) and (II) yield $\omega(\overline{s})_{|u} = \omega(\overline{t})_{|u}$, showing that $u \subseteq [\![\omega(\overline{s}) = \omega(\overline{t})]\!]$. Since $\Omega(X)$ is a $[\wedge, \bigvee]$-lattice, 8.4 guarantees that
$$\bigcap_{k=1}^n [\![s_k = t_k]\!] = \bigcup \{\bigcap_{k=1}^n u_k : u_k \leq Es_k \cap Et_k \text{ and } s_{k|u_k} = t_{k|u_k}\},$$
and the inequality in the statement follows.

For $s, t \in |P|$ define
$$s \; \theta \; t \quad \text{iff} \quad Es = Et = [\![s = t]\!].$$
It is clear that θ is an equivalence relation *on* $|P|$.

Fact 2. *For $u \in \Omega(X)$, set $\theta_u = \theta \cap (P(u) \times P(u))$. Then, θ_u is a congruence on the L-structure $P(u)$.*

Proof. Clearly, θ_u is an equivalence relation on $P(u)$. Let ω be a n-ary operation symbol in L. It must be shown that for $\overline{s} = \langle s_1, \ldots, s_n \rangle$ and $\overline{t} = \langle t_1, \ldots, t_n \rangle$ in $P(u)^n$, we have
$$s_k \; \theta_u \; t_k, \text{ for } 1 \leq k \leq n \quad \Rightarrow \quad \omega(\overline{s}) \; \theta_u \; \omega(\overline{t}). \tag{III}$$
It is clear that $E\omega(\overline{s}) = E\omega(\overline{t}) = u$, because ω is an operation on $P(u)$. From the hypothesis in (III) and Fact 1, we obtain
$$\bigcup_{k=1}^n [\![s_k = t_k]\!] = \bigcup_{k=1}^n Es_k = u \leq [\![\omega(\overline{s}) = \omega(\overline{t})]\!],$$
and so $[E =]$ in 23.7.(b) entails $[\![\omega(\overline{s}) = \omega(\overline{t})]\!] = u$, ending the proof of Fact 2.

For $u \in \Omega(X)$, let
$$\boldsymbol{\varepsilon}P(u) = P(u)/\theta_u,$$
be the *quotient* L-structure of $P(u)$ by θ_u, as in 17.21; write
$$\boldsymbol{\varepsilon}_u : P(u) \longrightarrow P(u)/\theta_u$$
for the canonical quotient L-morphism. For $s \in |P|$, write $\boldsymbol{\varepsilon}s =_{def} s/\theta_{Es}$. For opens $u \leq v$ in X, define
$$\rho_{vu} : \boldsymbol{\varepsilon}P(v) \longrightarrow \boldsymbol{\varepsilon}P(u), \text{ by } \rho_{vu}(\boldsymbol{\varepsilon}s) = \boldsymbol{\varepsilon}(s_{|u}).$$

Fact 3. *For opens $u \leq v$ in X, ρ_{vu} is a L-morphism such that*
(1) If $u \leq v \leq w$ are opens in X, $\rho_{wu} = \rho_{vu} \circ \rho_{wv}$;
(2) $\rho_{uu} = Id_{\boldsymbol{\varepsilon}P(u)}$.
*Moreover, $\boldsymbol{\varepsilon}P : \Omega(X) \longrightarrow$ **L-mod** defined by*
$$u \longmapsto P(u) \quad \text{and} \quad \iota_{uv} \longmapsto \rho_{vu}$$
is a presheaf over X.

Proof. Fix opens $u \leq v$ in X and let $f = \boldsymbol{\varepsilon}_u \circ (\cdot)_{|u}$.

[6] In fact, we should use $\omega^{P(u)}$, but this would overload notation.

$$P(v) \xrightarrow{(\cdot)_{|u}} P(u) \xrightarrow{\varepsilon_u} \boldsymbol{\varepsilon}P(u).$$

By construction, f is a L-morphism. We now verify that

$$\theta_v \subseteq \theta_f = \{\langle s, t\rangle \in P(v)^2 : fs = ft\},$$

that is, for all s, $t \in P(v)$, $\ s\ \theta_v\ t \ \Rightarrow\ s_{|u}\ \theta_u\ t_{|u}$. Since $u = Es = Et = [\![s\ =\ t]\!]$, 23.7.(d) yields

$$[\![s_{|u}\ =\ t_{|u}]\!] \ = \ u \cap [\![s\ =\ t]\!] \ = \ u$$

and the conclusion follows from [*rest* 2] in 23.7.(a). By the universal property in 17.21, there is a *unique* L-morphism, $\widehat{f} : \boldsymbol{\varepsilon}P(v) \longrightarrow \boldsymbol{\varepsilon}P(u)$, such that

$$\widehat{f} \circ \boldsymbol{\varepsilon}_v \ = \ f \ = \ \boldsymbol{\varepsilon}_u \circ (\cdot)_{|u}.$$

Since $\widehat{f}$ is pointwise identical with ρ_{vu}, the latter is a L-morphism, as desired. The verification of conditions (1) and (2) is straightforward and $\boldsymbol{\varepsilon}P$ is a presheaf of L-structures over X.

Fact 4. *For all s, $t \in |P|$,*
(1) $E\boldsymbol{\varepsilon}s = Es$;
(2) $[\![\boldsymbol{\varepsilon}s\ =\ \boldsymbol{\varepsilon}t]\!] = [\![s\ =\ t]\!]$;
(3) $\boldsymbol{\varepsilon}P$ *is an extensional presheaf of L-structures over X.*

Proof. (1) is clear, because if $s \in P(u)$, then $\boldsymbol{\varepsilon}s \in \boldsymbol{\varepsilon}P(u)$. For (2), if $u \leq Es \cap Et$ is such that $s_{|u} = t_{|u}$, then

$$\rho_{Es,u}(\boldsymbol{\varepsilon}s) = \boldsymbol{\varepsilon}(s_{|u}) = \boldsymbol{\varepsilon}(t_{|u}) = \rho_{Et,u}(\boldsymbol{\varepsilon}t),$$

and $u \leq [\![\boldsymbol{\varepsilon}s\ =\ \boldsymbol{\varepsilon}t]\!]$. Hence, $[\![s\ =\ t]\!] \subseteq [\![\boldsymbol{\varepsilon}s\ =\ \boldsymbol{\varepsilon}t]\!]$. Conversely, if $v \leq Es \cap Et$, satisfies $\rho_{Es,v}(\boldsymbol{\varepsilon}s) = \rho_{Et,v}(\boldsymbol{\varepsilon}t)$, the definition of the restriction maps ρ_{**} entails

$$\boldsymbol{\varepsilon}(s_{|v}) = \boldsymbol{\varepsilon}(t_{|v}),$$

that is, $s_{|v}\ \theta_v\ t_{|v}$. Hence, items (a) and (d) in 23.7 yield

$$v \cap Es \ = \ Es_{|v} \ = \ Et_{|v} \ = \ v \cap Et \ = \ [\![s_{|v}\ =\ t_{|v}]\!] \ = \ v \cap [\![s\ =\ t]\!].$$

Since $v \leq Es \cap Et$, the preceding equations entail $v = v \cap [\![s\ =\ t]\!]$, that is, $v \leq [\![s\ =\ t]\!]$. It follows that $[\![\boldsymbol{\varepsilon}s\ =\ \boldsymbol{\varepsilon}t]\!] \subseteq [\![s\ =\ t]\!]$, completing the verification of (2). Item (3) is immediate from (1) and (2).

Up to now we have proven (a) and (b) (Fact 3.(2)) in the statement of the Theorem. It remains to verify (c).

If $\eta = \{\eta_u \in [P(u),\ \ Q(u)] : u \in \Omega(X)\}$ is a natural transformation from P to Q (23.2), define $\boldsymbol{\varepsilon}\eta : \boldsymbol{\varepsilon}P \longrightarrow Q$ as follows :

For $u \in \Omega(X)$, $(\boldsymbol{\varepsilon}\eta)_u : \boldsymbol{\varepsilon}P(u) \longrightarrow Q(u)$ is given by

$$(\boldsymbol{\varepsilon}\eta)_u(\boldsymbol{\varepsilon}s) = \eta_u s.$$

To see that this well defined, let $Es = Et = [\![s\ =\ t]\!]$, s, $t \in P(u)$. By Lemma 23.19, we have

$$[\![s\ =\ t]\!] \ \subseteq \ [\![\eta_u s\ =\ \eta_u t]\!],$$

and so $u = E\eta_u s = E\eta_u t = [\![\eta_u s = \eta_u t]\!]$. Since Q is extensional, we conclude that $\eta_u s = \eta_u t$, as desired. It is straightforward that $\varepsilon\eta$ is the unique L-morphism making the displayed diagram commutative, ending the proof [7]. $\square$

23.22. **Convention.** *From here on, unless explicit mention to the contrary, "presheaf" is synonymous with "extensional presheaf".* $\square$

That morphisms preserve restriction, compatibility and the gluing of compatible families is left as an exercise for the reader :

Lemma 23.23. *Let $P \xrightarrow{\eta} Q$ be a morphism of presheaves and $\{x\} \cup S \subseteq |P|$ be a set of sections in P. Then, with notation as in 23.20,*
a) For all $u \in \Omega(X)$, $\eta(s_{|u}) = (\eta s)_{|u}$.
b) If S is compatible in P, $\eta(S) = \{\eta s : s \in S\}$ is compatible in Q.
c) If $t \in |P|$ satisfies $Et = \bigcup_{s \in S} Es$ and $[\![t = s]\!] = Es$, for all $s \in S$, then, ηt satisfies the same conditions with respect to $\eta(S)$. $\square$

Remark 23.24. Let P be a presheaf of sets over X. For $s, t \in |P|$, define

$$s \leq t \quad \text{iff} \quad Es \subseteq Et \text{ and } t_{|Es} = s.$$

yielding a partial order on $|P|$ (because of extensionality), with which $|P|$ is a meet semilattice :

$$\inf\{s,\, t\} \;=\; s \wedge t \;=\; s_{|[\![s=t]\!]} \;=\; t_{|[\![s=t]\!]}.$$

When P is a sheaf, then for all $S \subseteq |P|$,

$$\bigvee S \text{ exists in } |P| \quad \text{iff} \quad S \text{ has an upper bound} \quad \text{iff} \quad S \text{ is compatible.}$$

In fact, for any $S \subseteq |P|$, even when P is a presheaf :

$$S \text{ has an upper bound} \quad \text{iff} \quad \bigvee S \text{ exists in } |P|,$$

simply because the restriction of an upper bound of S to $\bigcup_{s \in S} Es$ will be $\bigvee S$ in P. Thus, when P is a presheaf, we cannot use the existence of $\bigvee$ to define compatibility. To describe it, we introduce a new binary operation on $|P|$, &, defined as follows :

$$\text{For } s, t \in |P|, \quad s \,\&\, t = s_{|Et}.$$

Note that

$$*\; E(s \,\&\, t) = Es \cap Et; \qquad *\; s \,\&\, s = s; \qquad *\; s \wedge t \leq s \,\&\, t.$$

Since successive restrictions to open sets is the same as restricting once to their intersection ([rest 2] in 23.7.(a)), & is associative. Moreover, for $r, s, t, z \in |P|$,

$$r \leq t \text{ and } s \leq z \;\Rightarrow\; (r \,\&\, s) \leq (t \,\&\, z) \text{ and } (s \,\&\, r) \leq (z \,\&\, t),$$

that is, & is increasing in both variables. The reader will have noticed that & is **not** commutative. In fact, a subset $S \subseteq |P|$ is **compatible** iff for all $s, t \in S$, $s \,\&\, t = t \,\&\, s$. Hence, we may define compatibility from &, without appeal to completeness. Observe that

$$s \wedge t = s \,\&\, t \quad \text{iff} \quad s \,\&\, t = t \,\&\, s \quad \text{iff} \quad \{s,\, t\} \text{ is compatible.}$$

[7]The last part of the proof also follows from the universal property in 17.21.

The fact that P is a sheaf can be expressed as : subsets of $|P|$ that &-commute have supremum. It is left to the reader to check that, if P is a sheaf, then for all compatible $S \subseteq |P|$ and $t \in |P|$,

$$t \,\&\, \bigvee S = \bigvee_{s\in S} (t \,\&\, s) \quad \text{and} \quad (\bigvee S) \,\&\, t = \bigvee_{s\in S} (s \,\&\, t).$$

What might come as a surprise is that the maps

$$(\cdot) \,\&\, s, \quad s \,\&\, (\cdot) \;\;:\;\; |P| \longrightarrow |P|,$$

might not have right adjoints. This is due to the fact that a sheaf is not a complete lattice, but a semilattice in which only certain types of subsets (compatible ones) have supremum. In any case, sheaves and presheaves are close to the concept of *quantale* and their elementary counterparts ([**61**]). $\square$

REMARK 23.25. Let X be the one-point topological space. A presheaf over X can be canonically identified with a set, namely, its set of global sections. This identification yields natural isomorphisms between the categories $\boldsymbol{pSh(X, \mathbf{Set})}$, $\boldsymbol{Sh(X, \mathbf{Set})}$ and $\mathbf{Set}$. $\square$

Exercises

23.26. Let P be a $\mathcal{C}$-presheaf over X. P is extensional iff for all $u \in \Omega(X)$ and $s,\, t \in P(u)$

$$\begin{array}{l} \text{If } u_i,\ i \in I, \text{ is an open covering of } u, \\ \text{such that } p_{uu_i}(s) = p_{uu_i}(t),\ \forall\ i \in I \end{array} \quad \Rightarrow \quad s = t. \qquad \square$$

23.27. Let P be a $\mathcal{C}$-presheaf over X.

a) If P is a sheaf, then it is extensional.

b) P is a sheaf iff for all $s_i \in P(U_i)$, $i \in I$, if $u = \bigcup u_i$, then

$$\begin{array}{l} p_{u_i,\, u_i \cap u_j}(s_i) = p_{u_j,\, u_i \cap u_j}(s_j), \\ \forall\ i, j \in I \end{array} \quad \Rightarrow \quad \begin{array}{l} \exists\ \textit{unique}\ s \in P(u), \text{ such that} \\ p_{uu_i}(s) = s_i,\ \forall\ i \in I. \end{array} \qquad \square$$

CHAPTER 24

Presheaves of Sets

In this chapter we discuss sheaves and presheaves **of sets**. The reasons for this are twofold :
$*$ Without a doubt, presheaves of sets underlie very important mathematical constructions and are frequently used as an introduction to the subject;
$*$ Since we are ultimately interested in **First-Order Logic**, presheaves of first-order structures are our main concern. Consider how the classical concept of L-structure is defined. The basic building blocks are sets and constructions with sets. Thus, a L-structure is a set, together with maps, elements and certain distinguished subsets of its finite powers. It seems, reasonable to proceed by analogy to arrive at the notion of L-structures in presheaves, replacing the category of sets by that of presheaves. If this path is taken, elementary "set theory" must be internalized in the category of presheaves, that is, one has to establish the analogues of unions, intersections, the power set, maps, elements and relations, among other things.

It will turn out that "L-structure in the category of presheaves" comes to the same thing as "presheaves of L-structures", but the conceptual role of working inside the category of presheaves over a topological space is important to understand other situations, in which such a correspondence is false, as is the case in the general framework of Topoi, originating with A. Grothendieck and his school, to treat Algebraic Geometry, just to mention one notable example.

Throughout this Chapter, X is a topological space and $\Omega = \Omega(X)$ is the frame of opens in X. All presheaves will be *extensional*. Write $\boldsymbol{pSh(X)}$ and $\boldsymbol{Sh(X)}$ for the categories of presheaves and sheaves of sets over X, respectively.

We employ Proposition 23.7 and Lemma 23.19, without necessarily mentioning it. The definition of extensionality, $[ext]$, appears in 23.6.(c).

1. Categorical Constructions

This section is dedicated to the description of some of the basic categorical constructions in $\boldsymbol{pSh(X)}$ and $\boldsymbol{Sh(X)}$.

24.1. **Initial and Final Objects.** These are the constant sheaves $\mathbf{0}$ and $\mathbf{1}$, respectively (see 23.16). It is clear that for each presheaf P on X, there are unique morphisms from $\mathbf{0}$ to P and from P to $\mathbf{1}$ [1]. $\square$

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[1]The empty map is the unique map with empty domain, into any set.

24.2. **Products.** Let P_i, $i \in I$, be a family of presheaves over X. For $u \in \Omega(X)$, define

$$(\prod_{i \in I} P_i)(u) = \prod_{i \in I} P_i(u),$$

and for an open subset v of u, a restriction

$$(\prod_{i \in I} P_i)(u) \longrightarrow (\prod_{i \in I} P_i)(v), \quad \langle s_i \rangle \ \mapsto \ \langle s_{i_{|v}} \rangle$$

It is straightforward that this defines a presheaf over X, which is a sheaf if the same is true of each component. The canonical projections

$$\pi_{iu} : (\prod_{i \in I} P_i)(u) \longrightarrow P_i(u),$$

are presheaf morphisms, $\pi_i : (\prod_{i \in I} P_i) \longrightarrow P_i$. The pair

$$\langle \prod_{i \in I} P_i, \ \{\pi_i\}_{i \in I} \rangle$$

is the **product** of the family P_i in the categories $\boldsymbol{pSh(X)}$ or $\boldsymbol{Sh(X)}$.

Write $P = \prod_{i \in I} P_i$; the domain of P is given by

$$|P| = \{\langle s_i \rangle \in \prod_{i \in I} |P_i| : \forall\, i, j \in I, \ \ Es_i = Es_j\}.$$

Note the distinction between $|P|$ and the product of the $|P_i|$: only i-tuples with constant extent are in $|P|$. If $\overline{s} = \langle s_i \rangle$, $\overline{t} = \langle t_i \rangle$ are sections in $|P|$, then [2]

$$[\![\overline{s} = \overline{t}]\!] \ = \ \bigwedge_{i \in I} [\![s_i = t_i]\!] \ = \ \mathrm{int}\left(\bigcap_{i \in I} [\![s_i = t_i]\!]\right).$$

In particular, $E\overline{s} \ = \ \bigwedge_{i \in I} Es_i \ = \ Es_i$, for any $i \in I$. The projections π_i may be written as maps, $\pi_i : |P| \longrightarrow |P_i|$, $\ \pi_i \overline{s} = s_i$. If $\overline{s}$, $\overline{t}$ are sections in P, then

$$E\overline{s} \ = \ E\overline{t} \ = \ [\![\overline{s} = \overline{t}]\!],$$

implies that for all $i \in I$,

$$Es_i = Et_i = [\![s_i = t_i]\!]$$

and the extensionality of each component entails $\overline{s} = \overline{t}$. Hence, P is an extensional presheaf. When $I = \emptyset$, we have

$$P^{\emptyset} =_{def} P^0 = \mathbf{1},$$

where $\mathbf{1}$ is the constant sheaf $\{0\}$, as in 23.16. The reason for this is that in any category, the empty product is its final object. In particular, in **Set**, the empty product is the final object $1 = \{0\}$. ☐

24.3. **Fibered product over a map.** Let $\lambda : P \longrightarrow Q$ be a morphism of presheaves over X. Define, for $u \in \Omega(X)$,

$$(P \times_\lambda P)(u) \ = \ \{\langle s, t \rangle \in P(u) \times P(u) : \lambda_u(s) = \lambda_u(t)\},$$

with restrictions induced by the product $P \times P$. $P \times_\lambda P$ is a presheaf, called the **fibered product** of P over λ. If P is a sheaf, the same is true of $P \times_\lambda P$. There are morphisms,

$$\delta_1, \delta_2 : P \times_\lambda P \longrightarrow P,$$

given by the restriction to $(P \times_\lambda P)(u) \ \subseteq \ P(u) \times P(u)$, of the projections of the product presheaf $P \times P$ on each coordinate. Clearly, $\lambda \circ \delta_1 = \lambda \circ \delta_2$.

If $\lambda = Id_P$, the fibered product of P over λ is the graph of the identity relation on P, Δ_P; when P is clear from context, write Δ for this presheaf. ☐

24.4. **Equalizers.** Let λ, β be morphisms from P to Q. For each $u \in \Omega(X)$, define

[2] Recall that $\bigwedge$ in the frame $\Omega(X)$ is the *interior* of the intersection; 2.11.(b).

$$Eq(\lambda,\beta)(u) = \{s \in P(u) : \lambda_u(s) = \beta_u(s)\}.$$

If $v \leq u$ [3], since λ and β are morphisms, we may define a restriction

$$Eq(\lambda,\beta)(u) \longrightarrow Eq(\lambda,\beta)(v) \quad \text{by} \quad s \;\mapsto\; s_{|v}.$$

Then, $Eq(\lambda,\beta)$ is a presheaf, and a sheaf whenever P is a sheaf. For $u \in \Omega(X)$, there is a map $\eta_u : Eq(\lambda,\beta)(u) \longrightarrow P(u)$, the canonical embedding of $Eq(\lambda,\beta)(u)$ as a subset of $P(u)$. Clearly, $\eta = \langle\, \eta_u \,\rangle_{u\in\Omega(X)}$ is a morphism and $\lambda \circ \eta = \beta \circ \eta$. $\langle\, Eq(\lambda,\beta),\ \eta \,\rangle$ is the equalizer of the pair $\langle\, \lambda,\beta \,\rangle$ in $\boldsymbol{pSh(X)}$ or $\boldsymbol{Sh(X)}$. $\square$

24.5. **Coproducts.** Let P_i, $i \in I$, be a family of presheaves over X. Define a presheaf, $C = \coprod_{i\in I} P_i$, by the following prescriptions [4] :

(i) For $u \in \Omega(X)$, $C(u) = \coprod P_i(u)$;

(ii) If $u \leq v$ and $\langle\, i,s \,\rangle \in C(u)$, then $\langle\, i,s \,\rangle_{|v} = \langle\, i, s_{|v} \,\rangle$.

It is straightforward that C is a presheaf over X, which is a sheaf whenever the same is true of each component. For $i \in I$, there is a morphism of presheaves,

$$\lambda_i : P_i \longrightarrow C, \text{ given by } \lambda_i s = \langle\, i,s \,\rangle.$$

It is straightforward that if Q is a presheaf over X and $f_i : P_i \longrightarrow Q$ are morphisms, then there is a **unique** morphism, $f : C \longrightarrow Q$, such that $f \circ \lambda_i = f_i$, $\forall\ i \in I$.

$$\begin{tikzcd} P_i \arrow[rr, "\lambda_i"] \arrow[dr, "f_i"'] & & C \arrow[dl, "f"] \\ & Q & \end{tikzcd}$$

Hence, the pair $\langle\, C,\ \{\lambda_i\ : i \in I\} \,\rangle$ is **the coproduct** of the family P_i in $\boldsymbol{pSh(X)}$ or $\boldsymbol{Sh(X)}$. Domain and equality in C are given, respectively by

$*$ $|C| = \{\langle\, i,s \,\rangle : i \in I \text{ and } s \in |P_i|\}$;

$*$ For $\langle\, i,s \,\rangle$, $\langle\, j,t \,\rangle \in |C|$,

$$[\![\langle\, i,s \,\rangle \;=\; \langle\, j,t \,\rangle]\!] \;=\; \begin{cases} \emptyset & \text{if } \; i \neq j \\ [\![s \;=\; t]\!] & \text{otherwise} \end{cases}$$

It is clear that C is an extensional presheaf over X. $\square$

2. Monics and Epics. The Structure of Subpresheaves

The notions of monic and epic in a category are defined in 16.8. The following result should be compared with Proposition 22.26.

Proposition 24.6. *Let P and Q be sheaves or presheaves over X and let $P \xrightarrow{\eta} Q$ be a morphism.*

a) The following are equivalent in $\boldsymbol{pSh(X)}$ or $\boldsymbol{Sh(X)}$:

[3]Recall (23.5.(3)) that $v \leq u$ means $u \in \Omega$ and $v \in \Omega(u)$.

[4]The definition of disjoint union of sets is in 1.5.

(1) η *is monic;* (2) *For all* $u \in \Omega(X)$, η_u *is injective;*

(3) *For all* $s, t \in |P|$, $[\![\eta s = \eta t]\!] = [\![s = t]\!]$.

b) The following are equivalent in $\boldsymbol{pSh(X)}$ *or* $\boldsymbol{Sh(X)}$ *:*

(1) η *is epic.*

(2) *For all* $t \in |Q|$, *there are* $u_i \subseteq \Omega(X)$ *and* $s_i \in P(u_i)$, $i \in I$, *such that*

(i) $\bigcup_{i \in I} u_i = Et;$ (ii) *For all* $i \in I$, $t_{|u_i} = \eta_{u_i}(s_i)$.

(3) *For all* $t \in |Q|$, $Et = \bigcup_{s \in |P|} [\![t = s]\!]$.

c) The following are equivalent in $\boldsymbol{pSh(X)}$ *or* $\boldsymbol{Sh(X)}$ *:*

(1) η *is an isomorphism.* (2) *For all* $u \in \Omega(X)$, η_u *is bijective.*

Proof. We treat sheaves and presheaves at the same time, asking the reader to notice that, when P and Q are sheaves, the objects we need, fibered product over a morphism (24.3) and the sheaf of opens of X (23.13), are sheaves.

(a) Clearly, (2) $\Rightarrow$ (1), while (2) $\Rightarrow$ (1) can be handled exactly as in Proposition 22.26, by considering the two maps originating in the fibered product of P over η with values in P.

(2) $\Rightarrow$ (3) : If $s, t \in |P|$, then $[\![s = t]\!] \subseteq [\![\eta s = \eta t]\!]$ ([*mor* 2]). To prove the reverse inclusion, set $w = [\![\eta s = \eta t]\!]]$; note that [*mor* 1] implies $w \subseteq Et \cap Es$. Thus, both $s_{|w}$ and $t_{|w}$ are in $P(w)$. Hence,

$$E\eta(s_{|w}) = Es_{|w} = Es \cap w = w;$$

similarly, we can show that $E\eta(t_{|w}) = w$. Moreover,

$$[\![\eta(s_{|w}) = \eta(t_{|w})]\!] = [\![(\eta s)_{|w} = (\eta t)_{|w}]\!] = [\![\eta s = \eta t]\!] \cap w = w.$$

Thus, [ext] yields $\eta(s_{|w}) = \eta(t_{|w})$ in $Q(w)$. Since η_w is injective, we get $s_{|w} = t_{|w}$. Now, [rest 3] entails $w = [[\![s_{|w} = t_{|w}]\!] = [\![s = t]\!] \cap w$, and so $w \subseteq [\![s = t]\!]$, as desired.

(3) $\Rightarrow$ (2) : Suppose $s, t \in P(u)$ satisfy $\eta s = \eta t$. Then,

$$E\eta s = E\eta t = [\![\eta s = \eta t]\!].$$

Hence, $Es = Et = [\![s = t]\!]$, and the conclusion follows by extensionality.

b) It is easily seen that (2) iff (3); we prove (1) $\Rightarrow$ (2), leaving the converse, as well as item (c) as exercises. Recall that $\widetilde{\Omega}$ (23.13) is the sheaf of opens of X. We define two morphisms, $\alpha,\ \beta : Q \longrightarrow \widetilde{\Omega}$, as follows : for $t \in Q(u)$, set

$*$ $\alpha_u(t) = \langle\, u, u \,\rangle$; $*$ $\beta_u(t) = \langle \bigcup_{s \in |P|} [\![\eta s = t]\!],\ u \,\rangle$.

Clearly, α_u, β_u are maps from $Q(u)$ into $\widetilde{\Omega}(u) = \{\langle\, v, u \,\rangle : v \leq u\}$. We shall verify that β is a morphism, omitting the (much simpler) case of α. For $t \in Q(u)$ and $v \leq u$, it must be shown that

$$\beta_v(t_{|v}) = \beta_u(t)_{|v}. \tag{I}$$

It suffices to check that the *first coordinate* of these pairs are the same in $\widetilde{\Omega}(v)$, since both second coordinates are v. We have

$$\begin{aligned}\textstyle\bigcup_{s\in|P|}\ [\![\eta s\ =\ t_{|v}]\!] &= \textstyle\bigcup_{s\in|P|}\ v\cap[\![\eta s\ =\ t]\!]\\ &= v\cap\textstyle\bigcup_{s\in|P|}\ [\![\eta s\ =\ t]\!].\end{aligned}\tag{II}$$

Since restriction to v in $\widetilde{\Omega}$ is given by intersection with v in *both coordinates*, it is clear that (II) entails (I) and so β is a morphism of presheaves. Note that for $u\in\Omega(X)$ and $s\in P(u)$, $\beta_u(\eta_u(s))=\langle\, u,u\,\rangle$, because

$$\textstyle\bigcup_{a\in|P|}\ [\![\eta a\ =\ \eta s]\!]\ \ =\ \ E\eta s=u.$$

Hence, $\alpha\circ\eta=\beta\circ\eta$; since η is epic, this entails $\alpha=\beta$, completing the proof. $\square$

Example 24.7. Important example of sheaves come from the theory of analytic functions. For an open set u in complex plane $\mathbb{C}$, let $\mathcal{H}(u)$ be the ring of holomorphic functions in u, with the canonical restriction maps if $v\leq u$. If T is an open set in $\mathbb{C}$, let $\mathcal{H}_T$ be the sheaf of analytic functions in opens of T. There is a morphism, $\mathcal{H}_T\xrightarrow{exp}\mathcal{H}_T$, given by $f\mapsto e^f$. It is clear that *exp* is monic. The Riemann mapping theorem gives a necessary and sufficient condition for an analytic map to a have a logarithm : its domain must be simply connected. Since $\mathbb{C}$ has a basis consisting of simply connected opens, 24.6.(b) entails that *exp* is an epic. However, exp_u is onto iff u is simply connected. This shows that it is impossible to do better than conditions (2) and (3) in the statement of 24.6, even if dealing with sheaves. Similar examples are obtained from the distinction between exact and closed differential forms in open sets that are not simply connected. This example also shows that a morphism of sheaves may be monic and epic without being an isomorphism. $\square$

Item (a) in Proposition 24.6.(a) is the basis of the following

Definition 24.8. *If P is a presheaf over X, a* **subpresheaf** *of P is a presheaf Q over X such that for all $u\leq v$ in $\Omega(X)$,*

* *$Q(u)\subseteq P(u)$;*
* *The restriction of Q is that induced by P.*

Or equivalently, a subpresheaf Q of P is a subset $|Q|\subseteq|P|$ that is closed under the restriction map of P.

If P is a sheaf, a **subsheaf** *of P is a subpresheaf Q that has gluing of all compatible families, i.e., for all compatible $S\subseteq|Q|$,*

$$\exists\,!\ t \text{ in } |Q|,\ \textit{such that } Et=\textstyle\bigcup_{s\in S}\ Es\quad\textit{and}\quad t_{|Es}=s,\ \forall\ s\in S.$$

Write $Q\subseteq P$ to indicate that Q is a subpresheaf of P and $\mathcal{P}P(X)$ for the set of subpresheaves of P.

Remark 24.9. It is clear that $\langle\,\mathcal{P}P(X),\subseteq\,\rangle$ is a partially ordered set. Moreover, if $S,T\in\mathcal{P}P(X)$, then

$$S\subseteq T\quad\text{iff}\quad|S|\subseteq|T|,$$

a fact that will be used repeatedly below without further comment. $\square$

24.10. **Intersections.** If P_α, $\alpha\in A$, are subpresheaves of P, set, for $u\in\Omega(X)$,

$$[\textstyle\bigcap_{\alpha\in A}\ P_\alpha](u)=\bigcap_{\alpha\in A}\ P_\alpha(u),$$

With the restriction maps induced by P, $\bigcap P_\alpha$ is a subpresheaf of P, the **intersection** of the P_α. Further, $\bigcap P_\alpha$ is a sheaf, if each P_α is a sheaf.

The presheaf $\bigcap P_\alpha$ is the *meet* of the family P_α in the poset $\mathcal{P}P(X)$, which is, therefore, a complete lattice. It is left to reader to describe domain, extent and equality of the intersection of subpresheaves. $\square$

A description of "unions", or more precisely of *joins*, in the complete lattice $\mathcal{P}P(X)$ is the content of Exercise 24.71.

Now consider the analogous problem for *subsheaves*. There is no problem with intersection, as noted in 24.10. For joins, it is another story. With notation as in 24.71, the point is that there may be many compatible subsets of the union of the P_α that have no gluing in it. To treat this problem, we introduce :

DEFINITION 24.11. *Let P be a presheaf over X, $C, B \subseteq |P|$ and T a subpresheaf of P.*
a) C is **dense** *in B iff for all $b \in B$,* [5] *$Eb = \bigcup_{c\in C} [\![b = c]\!]$, i.e., if for all $b \in B$, there is an open covering u_i of Eb, together with $c_i \in C$, such that $b_{|u_i} = c_{i|u_i}$, for all $i \in I$.*
b) T is a **closed subpresheaf** *of P, if for all $t \in |P|$,*

$$Et = \bigcup_{a\in T} [\![a = t]\!] \quad \Rightarrow \quad t \in |T|.$$

Write $\mathfrak{P}P(X)$ for the set of closed subpresheaves of P.
c) The **subpresheaf generated by** *B in P is*

$$pB \;=\; \bigcap \{Q \subseteq P : Q \textit{ is a presheaf and } B \subseteq |Q|\}.$$

d) The **closed subpresheaf generated by** *B in P is*

$$\overline{B} \;=\; \bigcap \{Q \subseteq P : Q \textit{ is a closed subpresheaf and } B \subseteq |Q|\}.$$

$\overline{B}$ is called the **closure** *of B in P* [6].

Clearly, $B \subseteq |pB| \subseteq |\overline{B}|$ and pB is a subpresheaf of $\overline{B}$. Moreover, $\langle \mathfrak{P}P(X), \subseteq \rangle$ is a poset.

REMARK 24.12. If P is a presheaf and $S \cup \{x\} \subseteq |P|$, then the following conditions are equivalent :

$$(1)\ Ex \;\le\; \bigcup_{s\in S} [\![s = x]\!]; \qquad (2)\ Ex \;=\; \bigcup_{s\in S} [\![s = x]\!].$$

To see this, just recall that $[\![y = x]\!] \le Ex$, for all $y \in |P|$, by $[E =]$ in 23.7.(b). Similarly, if $S \subseteq T \subseteq |P|$, then (1) entails $Ex = \bigcup_{t\in T} [\![t = x]\!]$, because

$$\bigcup_{s\in S} [\![s = x]\!] \;\subseteq\; \bigcup_{t\in T} [\![t = x]\!].$$

These relations will be used frequently, without comment. $\square$

LEMMA 24.13. *If P is a sheaf over X and $Q \subseteq P$, then Q is a closed subpresheaf of P iff Q is a subsheaf of P. In particular, $\mathfrak{P}P(X)$ is the complete lattice of* subsheaves *of P.*

PROOF. Suppose that Q is a *subsheaf* of P and $t \in |P|$ verifies $Et = \bigcup_{q\in|Q|} [\![q = t]\!]$. Consider the family

$$S = \{q_{|[\![q=t]\!]} : q \in |Q|\}.$$

[5] Compare with 22.13.(c).
[6] Here there is need of Exercise 24.72.(b).

Since Q is a presheaf, it is closed under restrictions. Hence, $S \subseteq |Q|$. Next, observe that S is *compatible*, because for all $q \in |Q|$

$$q_{|[\![q=t]\!]} \;=\; t_{|[\![q=t]\!]},$$

by 23.7.(e). Alternatively, since $Eq_{|[\![q=t]\!]} \;=\; [\![q\,=\,t]\!]$, for $q,\, q' \in |Q|$,

$$\begin{aligned}(q_{|[\![q=t]\!]})_{|[\![q'=t]\!]} \;&=\; (t_{|[\![q=t]\!]})_{|[\![q'=t]\!]} \;=\; t_{|[\![q=t]\!]\cap[\![q'=t]\!]} \;=\; (t_{|[\![q'=t]\!]})_{|[\![q=t]\!]}\\ &=\; (q'_{|[\![q'=t]\!]})_{|[\![q=t]\!]},\end{aligned}$$

verifying the compatibility of S. Since Q is a sheaf, there is $s \in |Q|$ such that

$$Es = \bigcup_{q\in|Q|} [\![q\,=\,t]\!] \quad \text{and} \quad s_{|[\![q=t]\!]} = q_{|[\![q=t]\!]} = t_{|[\![q=t]\!]}.$$

Hence, $Es = Et$ and the transitive property of equality ([= 2] in 23.7.(b)) entails that for all $q \in |Q|$,

$$\begin{aligned}[\![s\,=\,t]\!] \;&\supseteq\; [\![s\,=\,t_{|[\![q=t]\!]}]\!] \cap [\![t_{|[\![q=t]\!]} \,=\, t]\!] \;=\; [\![s\,=\,t_{|[\![q=t]\!]}]\!] \cap [\![q\,=\,t]\!]\\ &=\; [\![s_{|[\![q=t]\!]} \,=\, t_{|[\![q=t]\!]}]\!] \;=\; [\![q\,=\,t]\!].\end{aligned}$$

Thus, $[\![s\,=\,t]\!] \;\supseteq\; \bigcup_{q\in|Q|} [\![q\,=\,t]\!] \;=\; Es \;=\; Et$, and extensionality guarantees that $s = t \in |Q|$, as desired.

Conversely, suppose that Q is a closed subpresheaf of P and that $S \subseteq |Q|$ is compatible. Then, S has a gluing t in $|\boldsymbol{P}|$ (it is a sheaf). Conditions (1) and (2) in 23.8.(b) yield

$$(1)\; Et = \bigcup_{s\in S} Es; \qquad\qquad (2)\; t_{|Es} = s, \;\; \forall\, s \in S.$$

But then, $\bigcup_{s\in S} [\![s\,=\,t]\!] \;=\; \bigcup_{s\in S} [\![t_{|Es} \,=\, t]\!] \;=\; \bigcup_{s\in S} Es \cap [\![t\,=\,t]\!]$

$$=\; Et \cap \bigcup_{s\in S} Es \;=\; Et \cap Et \;=\; Et,$$

and so we must have (24.12) $Et = \bigcup_{q\in|Q|} [\![q\,=\,t]\!]$. Since Q is a closed subpresheaf of P, we obtain $t \in Q$, as needed. □

Lemma 24.14. *Let $\mu,\, \eta : P \longrightarrow Q$ be presheaf morphisms and let D be a dense subset of sections in P. The following are equivalent :*

$$(1)\; \mu = \eta; \qquad\qquad (2)\; \mu_{|D} = \eta_{|D}.$$

Proof. We prove that (2) ⇒ (1). For $s \in |P|$ we may write

$$Es \;=\; \bigcup_{d\in D} [\![s\,=\,d]\!]. \tag{I}$$

The transitive law for equality, [*mor* 1] and [*mor* 2] in 23.7.(3) yield, for $d \in D$,

$$\begin{aligned}[\![\mu s\,=\,\eta s]\!] \;&\supseteq\; [\![\mu s\,=\,\mu d]\!] \cap [\![\mu d\,=\,\eta d]\!] \cap [\![\eta d\,=\,\eta s]\!]\\ &=\; [\![\mu s\,=\,\mu d]\!] \cap Ed \cap [\![\eta d\,=\,\eta s]\!]\\ &\supseteq\; [\![s\,=\,d]\!] \cap [\![s\,=\,d]\!] \;=\; [\![s\,=\,d]\!],\end{aligned}$$

and so, taking joins with respect to $d \in D$, (I) entails

$$Es \;=\; E\mu s \;=\; E\eta s \;=\; [\![\mu s\,=\,\eta s]\!],$$

and extensionality implies $\mu s = \eta s$, as desired. □

Theorem 24.15. *Let P be a presheaf over X and $\Gamma \subseteq |P|$. For $u \in \Omega(X)$, set $\Gamma(u) = \{s \in \Gamma : Es = u\}$.*

a) $p\Gamma$ is the subpresheaf of P whose domain is given by

$$|p\Gamma| = \{t_{|u} \in |P| : t \in \Gamma \text{ and } u \in \Omega(X)\}.$$

Moreover, Γ is dense in $p\Gamma$ and the embedding $\Gamma \subseteq p\Gamma$ has the following morphism extension property :

If Q is a presheaf and $\Gamma \xrightarrow{f} Q$ is a map such that for all s, s' in Γ

$$(i)\ Ef(s) = Es; \qquad (ii)\ [\![s = s']\!] \subseteq [\![fs = fs']\!],$$

*then, there is a **unique** presheaf morphism, $pf : p\Gamma \longrightarrow Q$, such that $pf(s) = f(s)$, $\forall\ s \in \Gamma$.*

b) $\overline{\Gamma}$ is the subpresheaf whose domain is given by

$$|\overline{\Gamma}| = \{t \in |P| : Et = \bigcup_{s\in\Gamma} [\![t = s]\!]\}.$$

Further, Γ is dense in $\overline{\Gamma}$ and satisfies the following morphism extension property :

*If Q is a sheaf and $\Gamma \xrightarrow{f} Q$ a presheaf morphism, then f has a **unique** extension, $\overline{f} : \overline{\Gamma} \longrightarrow Q$.*

Proof. a) It is clear that $p\Gamma$ is closed under restriction and contained in any subpresheaf of P that contains Γ. Hence, it must be the subpresheaf generated by Γ in P. Since for all $\langle t, u\rangle \in \Gamma \times \Omega(X)$

$$Et_{|u} = u \cap Et = [\![t = t_{|u}]\!],$$

we have $Et_{|u} = \bigcup_{s\in\Gamma} [\![s = t_{|u}]\!]$, and Γ is dense in $p\Gamma$.

For the extension property, assume that $f : \Gamma \longrightarrow Q$ satisfies (i) and (ii). If $s \in |p\Gamma|$, there is $\langle t, u\rangle \in \Gamma \times \Omega(X)$, such that $s = t_{|u}$. Define $pf(s) = (fs)_{|u}$; if this in fact defines a **map**, then it is easily established that f is a morphism. Clearly, pf extends f to $p\Gamma$. So assume that $s = t_{|u} = z_{|v}$, with $t, z \in \Gamma$ and $u, v \in \Omega(X)$. We wish to show that $(ft)_{|u} = (fz)_{|v}$ in Q. Note that

$$Es = Et \cap u = Ez \cap v = [\![t_{|u} = z_{|v}]\!] = [\![z = t]\!] \cap u \cap v. \qquad (*)$$

From $Efs = Es$ and $Eft = Et$, we get

$$E(ft)_{|u} = Eft \cap u = Et \cap u \ \text{ and } \ E(fz)_{|v} = Ez \cap v.$$

Consequently, by (*),

$$E(ft)_{|u} = E(fz)_{|v} = Et \cap u = Ez \cap v. \qquad (**)$$

But then, (i) yields

$$[\![(ft)_{|u} = (fz)_{|v}]\!] = [\![ft = fz]\!] \cap u \cap v \supseteq [\![t = z]\!] \cap u \cap v,$$

and so, (*) and (**) entail $[\![(ft)_{|u} = (fz)_{|v}]\!] = E(ft)_{|u} = E(fz)_{|v}$. Extensionality now entails $(ft)_{|u} = (fz)_{|v}$, as desired. Uniqueness is clear from 24.14.

b) For $\overline{\Gamma}$ to be a closed subpresheaf of P, we must show that

(1) If $\langle t, u\rangle \in |c\Gamma| \times \Omega(X)$, then $t_{|u} \in |c\Gamma|$; (it is a subpresheaf)

(2) For all $t \in |P|$, $Et = \bigcup_{s\in|c\Gamma|} [\![s = t]\!] \ \Rightarrow\ t \in |c\Gamma|$.

For (1), note that if $\langle t, u\rangle \in |c\Gamma| \times \Omega(X)$, then

$$Et_{|u} = Et \cap u = \bigcup_{s\in|\Gamma|} ([\![s = t]\!] \cap u) = \bigcup_{s\in|\Gamma|} [\![s = t_{|u}]\!],$$

and $t_{|u} \in |c\Gamma|$. For (2), note that for $a \in \Gamma$ and $s \in |c\Gamma|$, transitivity of equality yields $[\![s = t]\!] \cap [\![s = a]\!] \subseteq [\![a = t]\!]$. Hence, taking unions with respect to $a \in \Gamma$ on both sides, we arrive at

$$[\![s = t]\!] \cap \bigcup_{a\in\Gamma} [\![s = a]\!] \subseteq \bigcup_{a\in\Gamma} [\![a = t]\!]. \qquad (+)$$

Since $s \in |c\Gamma|$, $Es = \bigcup_{a\in\Gamma} [\![a = s]\!]$, which substituted into (+) yields

$$[\![s = t]\!] \cap Es = [\![s = t]\!] \subseteq \bigcup_{a\in\Gamma} [\![a = t]\!]. \qquad (++)$$

Since (++) holds for all $s \in |c\Gamma|$, the hypothesis in (2) entails

$$Et = \bigcup_{s\in|c\Gamma|} [\![s = t]\!] \subseteq \bigcup_{a\in\Gamma} [\![a = t]\!],$$

wherefrom it follows that $Et = \bigcup_{a\in\Gamma} [\![a = t]\!]$ and $t \in |c\Gamma|$, as needed. Clearly, $\overline{\Gamma}$ is contained in any closed subpresheaf of P that contains Γ, being therefore the closure of Γ in P. The preceding argument also shows that Γ is dense in $\overline{\Gamma}$.

For the extension property, let f be a morphism from Γ to a sheaf Q; for $t \in \overline{\Gamma}$, consider the following family of sections in Q :

$$S = \{(fs)_{|[\![t=s]\!]} : s \in \Gamma\}.$$

We contend that S is compatible in Q and $\bigcup \{Ex : x \in S\} = Et$. For $s, z \in |\Gamma|$, [*mor* 2] (23.19) and [= 2] (23.7) yield

$$\begin{aligned} [\![(fs)_{|[\![t=s]\!]} = (fz)_{|[\![z=t]\!]}]\!] &= [\![fs = fz]\!] \cap [\![s = t]\!] \cap [\![z = t]\!] \\ &\supseteq [\![s = t]\!] \cap [\![s = t]\!] \cap [\![z = t]\!] \\ &= [\![s = t]\!] \cap [\![z = t]\!]. \end{aligned}$$

By [*mor* 1] (23.19), $E(fs)_{|[\![s=t]\!]} = Efs \cap [\![s = t]\!] = [\![s = t]\!]$; similarly, $E(fz)_{|[\![z=t]\!]} = [\![t = z]\!]$, and S is a compatible family in Q. Since Γ is dense in $c\Gamma$, we have $Et = \bigcup \{[\![s = t]\!] : s \in |\Gamma|\}$. Thus,

$$\bigcup \{E(fs)_{|[\![s=t]\!]} : s \in \Gamma\} = \bigcup_{s\in\Gamma} [\![t = s]\!] = Et.$$

The unique extension of f to $\overline{\Gamma}$ is given by $\overline{f}t$ = the unique gluing of S in $|Q|$. $\square$

LEMMA 24.16. *Let P be a presheaf over X and S, T, $B \subseteq |P|$.*
a) If S is dense in T and $B \subseteq T$, then S is dense in B.
b) S dense in T and T dense in B $\Rightarrow$ S dense in B.
c) If $S \subseteq T$ and S is dense in T, then for all $x \in |P|$

$$\bigcup_{s\in S} [\![s = x]\!] = \bigcup_{t\in T} [\![t = x]\!].$$

d) If $S \subseteq T$, then S is dense in T iff $\overline{S} = \overline{T}$.

PROOF. Item (a) is clear. For (b), if $x \in B$, denseness and transitivity of equality yield

$$\begin{aligned} Ex &= \bigcup_{t\in T} [\![x = t]\!] = \bigcup_{t\in T} ([\![x = t]\!] \cap Et) \\ &= \bigcup_{t\in T} \left([\![x = t]\!] \cap \bigcup_{s\in S} [\![s = t]\!]\right) \\ &= \bigcup_{t\in T} \bigcup_{s\in S} [\![x = t]\!] \cap [\![s = t]\!]) \subseteq \bigcup_{s\in S} [\![x = s]\!], \end{aligned}$$

and the conclusion follows from 24.12.

c) Since $S \subseteq T$, it is clear that the left-hand side of (c) is contained in its right-hand side for any $x \in |P|$. For the reverse inclusion, denseness and 8.4 yield

$$\begin{aligned}\bigcup_{t\in T} [\![t = x]\!] &= \bigcup_{t\in T} ([\![t = x]\!] \cap Et)\\ &= \bigcup_{t\in T} \left([\![t = x]\!] \cap \bigcup_{s\in S} [\![t = s]\!]\right)\\ &= \bigcup_{t\in T} \bigcup_{s\in S} [\![t = x]\!] \cap [\![t = s]\!]\\ &\subseteq \bigcup_{t\in T} \bigcup_{s\in S} [\![x = s]\!] = \bigcup_{s\in S} [\![x = s]\!],\end{aligned}$$

and we conclude by 24.12.

d) If $\overline{S} = \overline{T}$, since S is dense in $\overline{S}$ (24.15.(b)), we have S dense in $\overline{T}$ and the conclusion follows from (a). Conversely, if S is dense in T, then 24.15.(b) entails $T \subseteq \overline{S}$. Since $\overline{T}$ is the least closed subpresheaf of P containing T, we obtain $\overline{T} \subseteq \overline{S}$. The reverse inclusion follows immediately from $S \subseteq T$. ☐

Here is how denseness propagates to finite products :

Proposition 24.17. *Let $D \subseteq |P|$ be a dense subset of a presheaf P over X and $n \geq 1$ be an integer. For $\overline{d} = \langle d_1, \ldots, d_n \rangle \in D^n$, set*

$$E\overline{d} = \bigcap_{i=1}^{n} Ed_i \quad \textit{and} \quad \overline{d}_{|E\overline{d}} = \langle {d_1}_{|E\overline{d}}, \ldots, {d_n}_{|E\overline{d}} \rangle.$$

Then, $D_n = \{\overline{d}_{|E\overline{d}} : \overline{d} \in D^n\}$ is dense in P^n.

Proof. First note that D_n is indeed a subset of the *domain* of P^n, because for all $i \in \underline{n}$ [7], $E({d_i}_{|E\overline{d}}) = E\overline{d} \cap Ed_i = E\overline{d}$.

For $\overline{t} = \langle t_1, \ldots, t_n \rangle \in |P^n|$ and $\overline{d} \in D^n$, we have

$$\bigcap_{i\leq n} E\overline{d} \cap [\![d_i = t_i]\!] = \bigcap_{i\leq n} Ed_i \cap [\![d_i = t_i]\!] = \bigcap_{i\leq n} [\![d_i = t_i]\!].$$

Hence, since $E\overline{t} = Et_i$, for all $i \in \underline{n}$, 8.4 yields

$$\begin{aligned}\bigcup_{\overline{s}\in D_n} [\![\overline{s} = \overline{t}]\!] &= \bigcup_{\overline{d}\in D^n} [\![\overline{d}_{|E\overline{d}} = \overline{t}]\!] = \bigcup_{\overline{d}\in D^n} \bigcap_{i\leq n} [\![{d_i}_{|E\overline{d}} = t_i]\!]\\ &= \bigcup_{\overline{d}\in D^n} \bigcap_{i\leq n} E\overline{d} \cap [\![d_i = t_i]\!]\\ &= \bigcup_{\overline{d}\in D^n} \bigcap_{i\leq n} [\![d_i = t_i]\!] = \bigcap_{i\leq n} \bigcup_{\overline{d}\in D^n} [\![d_i = t_i]\!]\\ &= \bigcap_{i\leq n} Et_i = E\overline{t},\end{aligned}$$

and D_n is dense in P^n. ☐

We can now describe joins in $\mathfrak{P}P(X)$:

Corollary 24.18. *If P is a presheaf and P_α, $\alpha \in A$, are closed subpresheaves of P, then their join in $\mathfrak{P}P(X)$ is given by*

$$\bigvee_{\alpha\in A} P_\alpha = \overline{\bigcup_{\alpha\in A} P_\alpha},$$

that is, $\bigvee P_\alpha$ is the subpresheaf of P whose domain is

$(*)$ $\qquad |\bigvee P_\alpha| = \{z \in |P| : Ez = \bigcup \{[\![z = s]\!] : s \in \bigcup_{\alpha\in A} |P_\alpha|\}.$

Moreover, $\bigcup_{\alpha\in A} |P_\alpha|$ is dense in $\bigvee_{\alpha\in A} P_\alpha$. ☐

The extension properties in 24.15 yield

Corollary 24.19. *a) Let $P \xrightarrow{f} Q$ and $P \xrightarrow{g} T$ be monics in* $\boldsymbol{pSh(X)}$ *and let Γ be a set of sections in P. Then, there is a* unique *isomorphism, $pf(\Gamma) \xrightarrow{\gamma} pg(\Gamma)$, making the diagram below-left commutative :*

[7] Recall that $\underline{n} = \{1, 2, \ldots, n\}$ (page 15).

$$\begin{tikzcd} \Gamma \arrow[rr, "f_{|\Gamma}"] \arrow[dr, "g_{|\Gamma}"'] & & pf(\Gamma) \arrow[dl, "\gamma"] \\ & pg(\Gamma) & \end{tikzcd} \qquad \begin{tikzcd} P \arrow[rr, "f"] \arrow[dr, "g"'] & & \overline{f(P)} \arrow[dl, "\gamma"] \\ & \overline{g(P)} & \end{tikzcd}$$

b) Let $P \xrightarrow{f} Q$ *and* $P \xrightarrow{g} T$ *be monics in* $\boldsymbol{pSh(X)}$*, with* Q *and* T *sheaves over* X*. There is a* unique *isomorphism,* $\gamma : \overline{f(P)} \longrightarrow \overline{g(P)}$*, making the diagram above-right commutative.* $\square$

By Corollary 24.19, the closure or the presheaf generated by a set S of sections is, up to isomorphism, independent of the presheaf in which S is embedded.

We now state a result guaranteeing that every presheaf can be embedded in a sheaf, called its **completion** or **sheafification**.

THEOREM 24.20. *If* P *is a presheaf over* X*, there is a sheaf* cP *over* X *and a morphism,* $c : P \longrightarrow cP$*, such that*
a) c *is monic and epic.*
b) If Q *is a sheaf over* X *and* $\lambda : P \longrightarrow Q$ *is a presheaf morphism, then there is a* **unique** *sheaf morphism* $c\lambda : cP \longrightarrow Q$ *making the following diagram commutative :*

$$\begin{tikzcd} P \arrow[rr, "c"] \arrow[dr, "\lambda"'] & & cP \arrow[dl, "c\lambda"] \\ & Q & \end{tikzcd}$$

Theorem 24.20 is a consequence of Theorem 27.9, to be proven in Chapter 27. Note that the universal property in 24.20.(b) implies that cP is unique up to isomorphism and that completion is left adjoint to the natural forgetful functor from $\boldsymbol{Sh(X)}$ to $\boldsymbol{pSh(X)}$.

We now discuss the structure of the complete lattices $\mathcal{P}P(X)$ and $\mathfrak{P}P(X)$.

PROPOSITION 24.21. *Let* $\mathbf{1}$ *be the constant* $\{0\}$*-sheaf* $(23.16, 24.1)$.
a) Any set of sections in $\mathbf{1}$ *is compatible.*
b) S *is a subsheaf of* $\mathbf{1}$ *iff* $S = \mathbf{1}_{|ES}$.
c) The map $u \in \Omega(X) \mapsto \mathbf{1}_{|u} \in \mathfrak{P}\mathbf{1}(X)$ *is an isomorphism of complete lattices.*

PROOF. a) Is immediate from the fact that $|\mathbf{1}|$ is the set of continuous maps from opens in X to the one point set $\{0\}$.

b) For $S \subseteq |\mathbf{1}|$, 23.12 implies that $S \subseteq \mathbf{1}_{|ES}$. Since all sets of sections in $\mathbf{1}$ are compatible, $|S|$ is compatible and must glue to the unique map from ES to $\{0\}$. But then $S = \mathbf{1}_{|ES}$. The converse is clear, since all restrictions of a sheaf to an open set are sheaves.
c) By (b), the map in (c) is bijective. To show that it is an isomorphism of complete lattices, it is enough to prove that it is an isomorphism of posets, that is, for all u, $v \in \Omega(X)$,

$$u \leq v \quad \text{iff} \quad \mathbf{1}_{|u} \subseteq \mathbf{1}_{|v}.$$

One direction is obvious; the other follows from $E\mathbf{1}_{|u} = u$ and the fact that support is increasing, i.e., $S \subseteq T \;\Rightarrow\; ES \leq ET$, completing the proof. □

Proposition 24.21 shows that, in general, $\mathfrak{P}P(X)$ will not be a Boolean algebra, since $\Omega(X)$ cannot be a Boolean algebra except in (very) special circumstances. Generalizing 24.21, we have

THEOREM 24.22. *$\mathcal{P}P(X)$ and $\mathfrak{P}P(X)$ are frames, with $\bot = \emptyset$ (the empty sheaf) and $\top = P$.*

PROOF. We treat the case of $\mathfrak{P}P(X)$, leaving the other as an exercise. Let T, S_α, $\alpha \in A$, be closed subsheaves of P. By 8.7, it is enough to prove that

$$T \cap \bigvee S_\alpha \subseteq \bigvee_{\alpha\in A} (T \cap S_\alpha).$$

Since the restriction is that induced by P, this relation is equivalent to the inclusion of the respective domains. Let $z \in |T \cap \bigvee S_\alpha|$; then, $(*)$ in 24.18 yields

$$\begin{aligned} Ez &= \bigcup \{[\![z = s]\!] : s \in |T| \cap |\bigvee S_\alpha|\} \\ &= \bigcup \{[\![z = s]\!] : s \in |T| \cap \textstyle\bigcup_{\alpha\in A} |S_\alpha|\} \\ &= \bigcup \{[\![z = s]\!] : s \in \textstyle\bigcup_{\alpha\in A} |T| \cap |S_\alpha|\} \end{aligned}$$

and so $z \in |\bigvee_{\alpha\in A} T \cap S_\alpha|$, as desired. □

Explicit formulas for implication and negation are the content of

PROPOSITION 24.23. *For S, $T \in \mathfrak{P}P(X)$, P a presheaf over X, we have* [8]
a) $|S \to T| \;=\; \{x \in |P| : \forall\, u \in \Omega(X)\ (x_{|u} \in |S| \Rightarrow x_{|u} \in |T|)\}$. [9]
b) $|\neg S| \;=\; \{x \in |P| : \forall\, u \in \Omega(X),\ x_{|u} \notin |S|\}$.

PROOF. a) The first step is verifying that the right-hand side of the equality, A, is a closed subpresheaf of P. If $z \in |P|$ is such that

$$Ez \;=\; \bigcup_{a\in A} [\![a = z]\!],$$

and $u \in \Omega(X)$, assume that $z_{|u} \in |S|$. For each $a \in A$, 23.7.(e) and [*rest* 3] in 23.7.(a) yield

$$a_{|u\cap[\![a=z]\!]} \;=\; z_{|u\cap[\![a=z]\!]},$$

[8]Exactly the same formulas hold for S, $T \in \mathcal{P}P(X)$.
[9]The $\Rightarrow$ here is *classical material implication* !

and so, S being a presheaf, it follows that $a_{|u \cap [\![a = z]\!]} \in |S|$. Since $a \in A$, we conclude that $a_{|u \cap [\![a = z]\!]} \in |T|$. But then,

$$\begin{aligned} \textstyle\bigcup_{a \in A} [\![x_{|u} = a_{|u \cap [\![a = z]\!]}]\!] &= \textstyle\bigcup_{a \in A} u \cap [\![a = z]\!] \cap [\![a = z]\!] \\ &= u \cap \textstyle\bigcup_{a \in A} [\![a = z]\!] = u \cap Ez = Ez_{|u}, \end{aligned}$$

and $z_{|u} \in |T|$ because T is closed. Hence, A is a closed subpresheaf of P. We now verify that for all $Z \in \mathfrak{P}P(X)$

$$S \cap Z \subseteq T \quad \text{iff} \quad Z \subseteq A,$$

and so $A = (S \to T)$. Assume that $S \cap Z \subseteq T$, $x \in |Z|$ and that $z_{|u} \in |S|$, for $u \in \Omega(X)$; then, since Z is a presheaf, $z_{|u} \in |S| \cap |Z| = |S \cap Z|$. Hence, $x_{|u} \in |T|$ and $x \in |A|$. The converse is clear. Item (b) follows from (a) and the fact that $\neg S = (S \to \mathbf{0})$. □

24.24. **Image and Restriction of Morphisms** Let $P \xrightarrow{\eta} Q$ be a morphism of presheaves. If T is a subpresheaf of P, the **restriction** of η to T, $\eta_{|T} : T \longrightarrow Q$, is given by $t \in |T| \ \mapsto\ \eta t \in |Q|$. If one prefers to view it as a natural transformation, then

$$\eta_{|T} = \{\eta_{u_{|T(u)}} : u \in \Omega(X)\}.$$

The **image** of η is defined as

$$Im\,\eta = \overline{\{\eta x \ :\ x \in |P|\}},$$

that is, the *closure* of the set-theoretical image of η. The set-theoretical image is a subpresheaf of Q, but it will not be a subsheaf, even if P and Q are sheaves. Hence, we prefer to use closure to uniformly obtain subobjects in $\boldsymbol{pSh(X)}$ and $\boldsymbol{Sh(X)}$. In general, if $S \subseteq |P|$,

$$\eta_* S = \overline{\{\eta s \ :\ s \in S\}},$$

is the **image of $\boldsymbol{S}$ by $\boldsymbol{\eta}$**. □

LEMMA 24.25. *If $\eta : P \longrightarrow Q$ is a morphism of presheaves, the map $\eta_* : \mathfrak{P}P(X) \longrightarrow \mathfrak{P}Q(X)$* [10] *is a $\bigvee$-morphism.*

PROOF. Let $\{S_i : i \in I\} \subseteq \mathfrak{P}P(X)$; it is clear that η_* is increasing, and so it is enough to check that

$$\eta_*(\textstyle\bigvee_{i \in I} S_i) \subseteq \textstyle\bigvee_{i \in I} \eta_* S_i.$$

Since $\bigcup |S_i|$ is dense in $\bigvee S_i$ (Corollary 24.18), 24.16.(c) yields, for $y \in |\eta_*(\bigvee_{i \in I} S_i)|$,

$$\begin{aligned} Ey &= \textstyle\bigcup_{s \in \bigvee S_i} [\![y = \eta s]\!] = \textstyle\bigcup_{s \in \bigcup |S_i|} [\![y = \eta s]\!] \\ &= \textstyle\bigcup_{i \in I} \bigcup_{s \in |S_i|} [\![y = \eta s]\!] = \textstyle\bigcup_{i \in I} \bigcup_{z \in |\eta_* S_i|} [\![y = z]\!] \ ^{11} \\ &= \textstyle\bigcup_{t \in \bigcup |\eta_* S_i|} [\![y = t]\!], \end{aligned}$$

and so $y \in |\bigvee_{i \in I} \eta_* S_i|$, as needed. □

[10]Defined in 24.24.

[11]Because $\{\eta s : s \in |S_i|\}$ is dense in $\eta_* S_i$ and 24.16.(c).

24.26. **Inverse Image.** Let $P \xrightarrow{\eta} Q$ be a morphism of presheaves. If S is a subpresheaf of Q, the map
$$u \in \Omega(X) \;\; \mapsto \;\; \eta_u^{-1}(S(u)) \subseteq P(u),$$
with the restrictions induced by P, is a subpresheaf of P, the **inverse image** of S by η and written $\eta^{-1}(S)$. Notice that the domain of the inverse image is given by
$$|\eta^{-1}(S)| = \{s \in |P| : \eta s \in |S|\},$$
that is, the set-theoretic inverse image. It is well-known that in the case of sets, inverse image is well-behaved, preserving all set-theoretic operations. The analogous properties in our present setting are described in Proposition 24.27. $\square$

PROPOSITION 24.27. *Let $\eta : P \longrightarrow Q$ be a morphism of presheaves.*
a) For $S,\, T \subseteq |Q|$, $\;S$ dense in $T \;\Rightarrow\; \eta^{-1}S$ is dense in $\eta^{-1}T$. Moreover,
$$(1)\ \eta^{-1}(\overline{T}) = \overline{\eta^{-1}T}; \qquad\qquad (2)\ T \in \mathfrak{P}Q(X) \;\Rightarrow\; \eta^{-1}T \in \mathfrak{P}P(X).$$
b) The map $\eta^* : \mathfrak{P}Q(X) \longrightarrow \mathfrak{P}P(X)$, $S \mapsto \eta^{-1}(S)$, is an **open** *frame-morphism.*
c) The pair $\langle\, \eta_*, \eta^* \,\rangle$ satisfies, for $S \in \mathfrak{P}P(X)$ and $T \in \mathfrak{P}Q(X)$,
$$[adj] \qquad\qquad \eta_* S \subseteq T \quad \textit{iff} \quad S \subseteq \eta^* T,$$
being therefore an adjoint pair (4.6, 7.8), with η_* left adjoint to η^*.
d) η^* is monic iff η is epic.

PROOF. a) Let $S \subseteq Q$ and $x \in \eta^{-1}T$; then $\eta x \in T$ and so
$$E\eta x \;=\; \bigcup\nolimits_{s \in S} [\![\eta x \,=\, s]\!]. \tag{I}$$
Let $A \;=\; \{x_{|[\![\eta x = s]\!]} : s \in S\}$; then, for all $s \in S$, condition (ii) in 23.19.(2) and 23.7.(e) yield $\eta(x_{|[\![\eta x = s]\!]}) \;=\; (\eta x)_{|[\![\eta x = s]\!]} \;=\; s_{|[\![\eta x = s]\!]} \in |S|$, and so $y_{|[\![\eta x = s]\!]} \in \eta^{-1}S$, i.e., $A \subseteq \eta^{-1}S$. Hence, since $E\eta x = Ex$, (I) entails
$$\begin{aligned}\bigcup\nolimits_{s \in S} [\![x \,=\, y_{|[\![\eta x = s]\!]}]\!] \;&=\; \bigcup\nolimits_{s \in S} [\![\eta x \,=\, s]\!] \cap [\![x \,=\, x]\!] \;=\; \bigcup\nolimits_{s \in S} [\![x \,=\, s]\!] \\ &=\; E\eta x \;=\; Ex,\end{aligned}$$
showing that $x \in \overline{A} \subseteq \overline{\eta^{-1}S}$, as desired. To get the full conclusion in (a), recall that a subset of $|P|$ is dense in the presheaf it generates and that denseness is transitive (24.15.(a), 24.16.(b)). For (1), since T is dense in $\overline{T}$ and $T \subseteq \overline{T}$, we have
$$\eta^{-1}T \text{ dense in } \eta^{-1}(\overline{T}) \quad \text{and} \quad \eta^{-1}T \subseteq \eta^{-1}(\overline{T}),$$
and we conclude by 24.16.(d). Item (2) follows immediately from (1).
b) Let $\{T_i : i \in I\} \subseteq \mathfrak{P}Q(X)$. Since (set-theoretical) inverse image preserves arbitrary intersections, it is clear that η^* is a $\bigwedge$-morphism. For joins, we have, using (a).(1)
$$\begin{aligned}\eta^*(\bigvee T_i) \;&=\; \eta^*(\overline{\bigcup\nolimits_{i \in I} |T_i|}) \;=\; \overline{\eta^*(\bigcup\nolimits_{i \in I} |T_i|)} \\ &=\; \overline{\bigcup\nolimits_{i \in I} \eta^* T_i} \;=\; \bigvee \eta^* T_i.\end{aligned}$$
Since η^* preserves meets, if $S,\, T \in \mathfrak{P}Q(X)$, then
$$\eta^* S \cap \eta^*(S \to T) \;=\; \eta^*(S \cap (S \to T)) \;\subseteq\; \eta^* T,$$
and so $\eta^*(S \to T) \;\subseteq\; (\eta^* S \to \eta^* T)$ [12]. To prove the reverse inclusion, it is enough to show, by the adjointness condition $[\to]$ in 6.1, that for $A \in \mathfrak{P}P(X)$,

[12] Note : this argument is valid for *any* $\wedge$-semilattice morphism between HAs.

$$A \cap \eta^* S \subseteq \eta^* T \;\;\Rightarrow\;\; A \subseteq \eta^*(S \to T). \qquad (*)$$

For $x \in |A|$, assume that $(\eta x)_{|u} \in |S|$, $u \in \Omega(X)$. From $\eta x_{|u} = \eta(x_{|u})$, it follows that $x_{|u} \in |A| \cap |S| = |A \cap S|$. The hypothesis in (*) entails $x_{|u} \in \eta^* T$, that is, $\eta(x_{|u}) \in T$. This reasoning (and 24.23.(a)) proves that $\eta x \in |S \to T|$, completing the proof of (b).

Item (c) is straightforward and left to the reader. For (d), observe that 7.9.(a) guarantees that η^* is monic (i.e., injective) iff η_* is onto. Hence, it is enough to show that η_* is onto iff η is an epic. If η_* is onto $\mathfrak{P}Q(X)$, then

$$\eta_* P = Q = \overline{\{\eta x \;:\; x \in |P|\}},$$

showing that the set-theoretical image of $|P|$ by η is dense in Q. By 24.6.(b), η is an epic. If η is epic and $S \in \mathfrak{P}Q(X)$, it is left to the reader to show that $\eta_* \eta^* S \;=\; S$, completing the proof. $\square$

Since we have products and the notion of subobject in $\boldsymbol{pSh(X)}$, we can generalize Definition 18.15, as follows :

DEFINITION 24.28. *Let A_i, $i \in I$, be a family of presheaves over X and let $A = \prod_{i \in I} A_i$. Let P be a presheaf and $A \xrightarrow{f} P$ be a morphism. Let S be a subpresheaf of A and let J be a subset of I.*

*a) f **depends only on $\boldsymbol{J}$** iff for all $u \in \Omega(X)$, $f_u : A(u) \longrightarrow P(u)$ depends only on J, that is, if $s = \langle s_i \rangle$ and $t = \langle t_i \rangle$ are sections in $A(u)$, then*

$$J \subseteq \{i \in I : s_i = t_i\} \quad \text{implies} \quad f_u(s) = f_u(t).$$

*b) S **depends only on $\boldsymbol{J}$** iff for all $u \in \Omega(X)$ and all $s = \langle s_i \rangle$, $t = \langle t_i \rangle$ in $A(u)$,*

$$J \subseteq \{i \in I : s_i = t_i\} \text{ and } s \in S(u) \quad \text{implies} \quad t \in S(u).$$

24.29. **Notation.** Let $n \in \mathbb{N}$ and let A be a non-empty set. For $\overline{s} = \langle s_i \rangle \in A^n$, for $k \in \underline{(n+1)}$ and $a \in A$, define $\langle a/k;\, \overline{s} \rangle \in A^{n+1}$, by

$$\langle a/k;\, \overline{s} \rangle(i) = \begin{cases} s_i & \text{if } i < k \\ a & \text{if } i = k \\ s_{(i-1)} & \text{if } i > k. \end{cases}$$

Hence, $\langle a/k;\, \overline{s} \rangle$, corresponds to constructing the $(n+1)$-tuple that has s_i in the first $(k-1)$ coordinates, has a in the k^{th} coordinate, and has s_{i-1} in the coordinates indexed by $i > k$. This notation applies both to sections over an open set in powers of a presheaf, as well as to sections in powers of the domain of a presheaf. $\square$

24.30. **Quantifiers.** Let P be sheaf over X and let S be a subsheaf of P^n, $n \in \mathbb{N}$. For $k \in \underline{n}$, let $\widehat{\pi}_k : P^n \longrightarrow P^{(n-1)}$, be the projection that forgets the k^{th} coordinate. What are the meanings of $\exists x_k\, S$ and $\forall x_k\, S$? The classical answers in **Set** are

$$\text{(I)} \qquad \begin{cases} \exists x_k\, S &= \;\; \text{Im}\; \widehat{\pi}_k; \\ \forall x_k\, S &= \;\; \text{largest } A \subseteq P^{(n-1)}, \text{ such that } \widehat{\pi}_k^{-1}(A) \subseteq S \\ &= \;\; \{\overline{s} \in P^{(n-1)} : \text{For all } t \in P,\ \langle t/k;\, \overline{s} \rangle \in S\}. \end{cases}$$

Because these answers are "structural", it is natural to use the same ideas to define these concepts in $\boldsymbol{Sh(X)}$. Thus, in view of 24.24, define,

$$[\exists x_k\ S](u) = \{\overline{z} \in P(u)^{(n-1)} : \left.\begin{array}{c} \exists\ u_i \leq u \text{ and } t_i \in P(u_i),\ i \in I, \\ \text{such that } \bigcup u_i = u \text{ and} \\ \langle\, t_i/k;\ \overline{z}_{|u_i}\,\rangle \in S(u_i),\ i \in I. \end{array}\right\} \qquad [\exists]$$

Then, $u \longmapsto [\exists x_k\ S](u)$, with restrictions induced by $P^{(n-1)}$, is a subpresheaf of $P^{(n-1)}$, the **existential quantification** along the k^{th} component.

For the universal quantifier, one might be tempted to consider

$$A(u) = \{\overline{z} \in [P(u)]^{(n-1)} : \text{For all } t \in P(u),\ \ \langle\, t/k;\ \overline{z}\,\rangle \in S(u)\}.$$

But observe that, for $v \subseteq u$, if the restriction map of P is not onto, then $u \mapsto A(u)$ *will not* be a presheaf. Thus, we are led to define

$$[\forall x_k\ S](u) = \{\overline{z} \in P(u)^{(n-1)} : \left.\begin{array}{c} \text{For all } v \in \Omega(u) \text{ and } t \in P(v), \\ \langle\, t/k;\ \overline{z}_{|v}\,\rangle \in S(v) \end{array}\right\} \qquad [\forall]$$

which, with restrictions induced by $P^{(n-1)}$, is a subpresheaf, the **universal quantification** of S along the k^{th} component. The reader can then verify that we have

FACT 24.31. *If $S \subseteq P^n$ depends only on α, then $(\exists x_k\, S)$ and $(\forall x_k\, S)$ depend only on $\alpha - \{k\}$.* $\square$

By 24.75, S can be considered as a subpresheaf of P^α, namely $\pi_\alpha(S)$, where $\pi_\alpha : P^n \longrightarrow P^\alpha$ is the map that forgets the coordinates *outside* α. Moreover,

$$\pi_\alpha{}^{-1}(\pi_\alpha(S)) = S. \qquad \text{(II)}$$

Fact 24.31 and (II) imply that if $k \notin \alpha$, then

$$(\exists x_k\ S) = (\forall x_k\ S) = S. \qquad \text{(III)}$$

Moreover, if $S \subseteq P^n$, after quantifying, existentially or universally, with respect to all components on which S depends, we obtain a subsheaf of the final object $\mathbf{1}$, that is, an open set in X (24.21). $\square$

3. Relations and Quotients

In this section we develop an elementary theory of relations in $\boldsymbol{pSh(X)}$. A full blown theory will emerge when we discuss characteristic maps in Part 7.

As is well known in the case of sets, a relation from A to B is a subset of $A \times B$. This can, of course, be generalized to any set of components. In particular, if n is a positive integer, a n-ary relation on a set A is a subset of A^n. In $\boldsymbol{pSh(X)}$, the concept is analogous, with one subtlety :

DEFINITION 24.32. *If P, Q are presheaves over X, a* **relation from $\boldsymbol{P}$ to $\boldsymbol{Q}$** *is a* **closed** *subpresheaf of $P \times Q$. If $n \geq 1$ is an integer, a* **$\boldsymbol{n}$-ary relation on $\boldsymbol{P}$** *is a* **closed** *subpresheaf of P^n.*

Thus, a relation from P to Q consists of

* For each $u \in \Omega(X)$, a subset $R(u) \subseteq P(u) \times Q(u)$;
* For $v \leq u$, $\ \langle\, s, t\,\rangle \in R(u)\ \ \Rightarrow\ \ \langle\, s_{|v}, t_{|v}\,\rangle \in R(v)$;

$*$ If $\langle x, y \rangle \in |P \times Q|$ is such that
$$Ex = Ey = \bigcup_{\langle s,t\rangle \in |P\times Q|} [\![x = s]\!] \cap [\![y = t]\!],$$
then $\langle x, y \rangle \in |R|$ [13]. Write Δ for Δ_2, the diagonal of P^2.

If R is a relation from P to Q, its **inverse**, R^{-1} is defined by
$$R^{-1}(u) = \{\langle t, s \rangle \in Q(u) \times P(u) : \langle s, t \rangle \in R(u)\},$$
with restriction induced by $Q \times P$; clearly, R^{-1} is a closed subpresheaf of $Q \times P$.

Binary relations can be composed. Care must be exercised in transcribing the usual set-theoretic definition, so as to furnish *closed* subpresheaves. This idea is used so often that we take time to explain it.

Let R be relation from P to Q and S a relation from Q to T. By analogy with the set-theoretic case, consider, for each $u \in \Omega(X)$
$$C(u) = \left\{\langle s, t \rangle \in P(u) \times T(u) : \begin{array}{c} \exists\, z \in Q(u), \text{ such that} \\ \langle s, z \rangle \in R(u) \text{ and } \langle z, t \rangle \in S(u) \end{array}\right\}$$
with restrictions induced by $P \times T$; C is a subpresheaf of $P \times T$, but will not, in general, be closed. We are then led to define the **composition of *R* and *S*** as the *closure* of C in $P \times T$. Hence, for $u \in \Omega(X)$ and $\langle s, t \rangle \in P(u) \times T(u)$
$$\langle s, t \rangle \in (S \circ R)(u)$$
$$\text{iff}$$
$$\exists\, u_i \leq u \text{ and } z_i \in Q(u_i) \text{ such that } u = \bigcup u_i \text{ and } \forall\, i \in I,$$
$$\langle s_{|u_i}, z_i \rangle \in R(u_i) \text{ and } \langle z_i, t_{|u_i} \rangle \in S(u_i).$$
Hence, the distinction between C and $(S \circ R)$ is that one must use the set-theoretical idea *locally*. One way to further understand this is to realize that $(S \circ R)$ is the *image* by a morphism of the relation
$$\{\langle s, z, t \rangle \in |P \times Q \times T| : \langle s, z \rangle \in |R| \text{ and } \langle z, t \rangle \in |S|\}.$$

Next, we discuss equivalence relations in $\boldsymbol{pSh(X)}$.

DEFINITION 24.33. *If P is a presheaf over X, an* **equivalence relation** *on P is a binary relation E on P such that :*

$[ER1]$: $\Delta \subseteq E$; $\qquad [ER2]$: $E = E^{-1}$; $\qquad [ER3]$: $E \circ E \subseteq E$.

The following observations are straightforward :

I. For each $u \in \Omega(X)$, $E(u)$ is a (set-theoretic) equivalence relation on $P(u)$;

II. The intersection of a family of equivalence relations on P is an equivalence relation on P;

III. $P \times P$ is the largest and Δ is the smallest equivalence relation on P; the set of equivalence relations on a presheaf is a complete lattice under inclusion.

Item (III) allows us to set down

DEFINITION 24.34. *If P is a presheaf over X and $R \subseteq P \times P$, the* **equivalence relation generated** *by R is*
$$E(R) = \bigcap \{E \subseteq P \times P : E \text{ is an equivalence relation and } R \subseteq E\}.$$

[13] Notation as in 24.2.

Example 24.35. Let $P \xrightarrow{\lambda} Q$ be a morphism. The set

$$|ker(\lambda)| \;\; = \;\; \{\langle\, s, t\,\rangle \in |P \,\times\, P| : \lambda s = \lambda t\}$$

is the domain of an equivalence relation on P, called the **kernel** of λ. Note that $ker(\lambda)$ is the inverse image of the diagonal of $Q \,\times\, Q$ by the canonical morphism

$$\lambda \,\times\, \lambda : P \,\times\, P \longrightarrow Q \,\times\, Q,$$

and so a closed subpresheaf of $P \,\times\, P$, by 24.27.(a). Clearly, $ker(\lambda)$ is the equalizer (24.4) of the pair $\langle\, \lambda, \lambda\,\rangle$. Moreover, if $R \subseteq P \,\times\, P$ is a binary relation on P, then

(*) $$\forall\, \langle\, s, t\,\rangle \in |R|, \;\; \lambda s = \lambda t \;\; \text{iff} \;\; E(R) \subseteq ker(\lambda).$$ ☐

Let E be an equivalence relation on P and let $v \leq u$ be opens in X. Since E is a subpresheaf of $P \,\times\, P$, for all $s, t \in P(u)$, if s and t are equivalent with respect to $E(u)$, then $s_{|_v}$ and $t_{|_v}$ are equivalent with respect to $E(v)$. Thus, if

$$[P/E](u) = P(u)/E(u) = \{s/E(u) : s \in P(u)\},$$

is the set of equivalence classes of $P(u)$ with respect to the $E(u)$, the map

$$[P/E](u) \longrightarrow [P/E](v), \;\;\; s/E(u) \;\; \mapsto \;\; s_{|_v}/E(v)$$

is a well defined restriction on P/E, making it a presheaf over X, the **quotient** of P by the equivalence E. The natural map, $\pi_E : P \longrightarrow P/E$, defined by

$$s \in P(u) \;\; \longmapsto \;\; s/E(u) \in P(u)/E(u)$$

is a morphism of presheaves, the **projection or quotient morphism**. Since $E = ker(\pi_E)$, every equivalence relation is the kernel of a morphism (24.35).

As an application, we can construct, in $\boldsymbol{pSh(X)}$, the coequalizer of a pair of morphisms, $\lambda,\, \mu : P \longrightarrow Q$. Let E be the equivalence relation on Q, generated by the subpresheaf R of $Q \,\times\, Q$, whose domain is given by

$$|R| = \{\langle\, \lambda s, \mu s\,\rangle : s \in |P|\}.$$

By (*) in 24.35, $\pi_E \circ \lambda = \pi_E \circ \mu$, where $\pi_E : Q \longrightarrow Q/E$ is the quotient morphism. $\langle\, Q; \pi_E\,\rangle$ is the coequalizer of the pair $(\lambda,\, \mu)$, that is, it has the following universal property (16.30) :

If $Q \xrightarrow{f} T$ is a presheaf morphism and $f \circ \lambda = f \circ \mu$, then there is a **unique** $g : Q/E \longrightarrow T$, such that $f = g \circ \pi_E$:

$$\begin{tikzcd} Q \arrow[rr, "f"] \arrow[dr, "\pi_E"'] & & Q/E \arrow[dl, "g"] \\ & T & \end{tikzcd}$$

Write Coeq$(\lambda,\, \mu)$ for the coequalizer of the pair $(\lambda,\, \mu)$.

In general, it is not true that if P/E is a sheaf when P and E are sheaves. However, Theorem 24.20 applies to yield a completion of P/E, which is *defined* as the quotient of the sheaf P by the equivalence relation E in the category $\boldsymbol{Sh(X)}$.

Similarly, the coequalizer of a pair of morphisms in $\boldsymbol{Sh(X)}$ is the completion of the construction presented above for presheaves.

Theorem 16.31, together with the constructions in this section and in section 24.1, yield

THEOREM 24.36. *$\boldsymbol{Sh(X)}$ and $\boldsymbol{pSh(X)}$ are complete and cocomplete categories.*

4. The Sheaf of Subsheaves. Exponentiation

In the category **Set**, an important object is the family of subsets of a set and the family of maps between two sets. These concepts are of course related, because the set $\{0, 1\} = 2$ has a special role : there is a natural and bijective correspondence between subsets of A and functions from A to 2.

In this section, we discuss what sheaf plays the role of 2 in $\boldsymbol{Sh(X)}$, as well as exponentiation. We shall focus, mainly, on the category $\boldsymbol{Sh(X)}$, because it is the most important case. To treat these themes for presheaves or closed presheaves all that is needed are minor modifications and/or the use of Theorem 24.20.

To construct an analogue of the power set in $\boldsymbol{Sh(X)}$, we must find a **sheaf** that fulfills that role. The next definition sets down the concept we need.

DEFINITION 24.37. *Let Q be a sheaf over the topological space X. With notation as in* 23.12, *for $u \in \Omega(X)$, set*

$$\mathfrak{P}Q(u) = \{T \subseteq Q : T \text{ is a subsheaf of } Q_{|u}\}.$$

For $v \leq u$ in X, there is a natural **restriction map**

$$_{|v} : \mathfrak{P}Q(u) \longrightarrow \mathfrak{P}Q(v), \text{ given by } \ T \longmapsto T_{|v}.$$

The association $u \in \Omega(X) \longmapsto \mathfrak{P}Q(u)$, together with the restriction maps $_{|v}$ defined above, constitute a **presheaf** *written $\mathfrak{P}Q$.*

Note that the set of global sections of $\mathfrak{P}Q$ is precisely $\mathfrak{P}Q(X)$, which is a frame by Theorem 24.22, but in general, not a cBa. This points to a fundamental difference between the categories **Set** and $\boldsymbol{Sh(X)}$: the analogue of the power set is a frame, *not* a cBa.

Recall (23.6.(c)) that the **support** of a sheaf Q is the open set

$$EQ = \bigcup \{u \in \Omega(X) : Q(u) \neq \emptyset\}.$$

Clearly, $EQ_{|u} \subseteq u$. Hence, an alternate description of $\mathfrak{P}Q$ is :

i) $|\mathfrak{P}Q| = \{\langle S, u\rangle \in \mathfrak{P}Q(X) \times \Omega(X) : ES \subseteq u\}$;

ii) For $\langle S, u\rangle \in |\mathfrak{P}Q|$ and $v \in \Omega(X)$, set $\langle S, u\rangle_{|v} = \langle S_{|u\cap v}, v\rangle$;

iii) For $\langle S, u\rangle, \langle T, v\rangle \in |\mathfrak{P}Q|$,

$$\begin{aligned}[\![\langle S, u\rangle = \langle T, v\rangle]\!] &= \bigcup \{w \in \Omega(u \cap v) : S_{|w} = T_{|w}\} \\ &= \bigcup \{w \in \Omega(u \cap v) : S(w) = T(w)\}.\end{aligned}$$

Perhaps a comment is in order regarding the equalities in (iii). To say that two sheaves are the **same or equal** is to say that the corresponding **functors** are the same. However, if S and T are subsheaves of a sheaf Q, since restrictions are those induced by Q, S will be equal to T if and only if they have the same set of sections over all opens in X. This leads from the second to the third term in (iii). The last description can be seen to be equivalent to the (classical) one in 24.37, by realizing that $\mathfrak{P}Q(u)$ can be identified with the set of pairs $\langle S, u \rangle$, where S is a subsheaf of Q with support contained in u.

Note that for all $S \in \mathfrak{P}Q(u)$, $ES = [\![S = S]\!] = u$.

PROPOSITION 24.38. *Let Q be sheaf over X.*
a) For $u \in \Omega(X)$, $\mathfrak{P}Q(u)$ is a frame, with $Q_{|u}$ as its top and $\emptyset$ as its bottom. Implication and pseudo-complementation in $\mathfrak{P}Q(u)$ are given by the following prescriptions, where $w \in \Omega(u)$ and the restrictions are those induced by Q :

$$\begin{aligned} (S \to T)(w) &= \{s \in Q(w) : \forall\, v \leq w,\ s_{|v} \in S(v) \ \Rightarrow\ s_{|v} \in T(v)\}; \\ \neg\ S(w) &= \{s \in Q(w) : \forall\, v \leq w,\ v \neq \emptyset \ \Rightarrow\ s_{|v} \notin S(v)\}. \end{aligned}$$

b) The restriction maps, $_{|v} : \mathfrak{P}Q(u) \longrightarrow \mathfrak{P}Q(v)$, $S \longmapsto S_{|v}$, are open frame morphisms.
c) $\mathfrak{P}Q$ is a sheaf over X.

PROOF. Left to the reader; the main ingredients, besides the comments just made on equality and restriction in $\mathfrak{P}Q$, are the computations used in the proof of 24.22 and 24.23. $\square$

The sheaf $\mathfrak{P}Q$ is the sheaf of subsheaves of Q.

If $Q \in \boldsymbol{Sh(X)}$, the sheaf $\mathfrak{P}Q$ has a special property, called **flabbyness**, whose definition is item (a) of

PROPOSITION 24.39. *Let Q and P be sheaves over X.*
a) For all $u \in \Omega(X)$, the restriction map from $\mathfrak{P}Q(X)$ to $\mathfrak{P}Q(u)$ is surjective [14]*. In particular, $\mathfrak{P}P(X)$ is dense in $\mathfrak{P}Q$.*
b) Let λ, μ be morphisms from $\mathfrak{P}Q$ to P. Then, $\lambda = \mu$ iff $\lambda_X = \mu_X$.
c) There is a natural bijective correspondence between the set of morphisms from $\mathfrak{P}Q$ to P and the set of maps from $\mathfrak{P}Q(X)$ to $P(X)$, given by $\lambda \mapsto \lambda_X$.

PROOF. a) Since $Q_{|u}$ is a subsheaf of Q, all of its subsheaves are subsheaves of Q. The density assertion is now obvious. Item (b) is an immediate consequence of the fact that global sections are dense and 24.14.
c) Let $\mathfrak{P}Q \xrightarrow{\lambda} P$ be a sheaf morphism. If $\langle S, u \rangle$ is a section in $\mathfrak{P}Q$, then

$$\lambda_u(\langle S, u \rangle) \ = \ \lambda_X(\langle S, X \rangle)_{|u},$$

and so the values of λ are entirely determined by those of λ_X. The remaining statements follow from this and item (b). $\square$

[14]Or equivalently, every section in $\mathfrak{P}Q$ can extended to a **global** section (i.e., an element of $\mathfrak{P}Q(X)$); see also 31.10.(e)

Our next result shows that the sheaf $\widetilde{\Omega}$ of opens in X (23.13) is a subobject classifier. Before the statement, we need the following

24.40. **Notation.** Let Q be a sheaf over X.
a) For $S \in \mathfrak{P}Q(X)$ and $u \in \Omega(X)$, define

$$\gamma_{Su} : Q(u) \longrightarrow \Omega(u), \text{ by } \gamma_{Su}(s) = \{v \leq u : \exists\, t \in Q(v), \text{ with } s_{|v} = t\}.$$

The family $\gamma_S = \{\gamma_{Su} : u \in \Omega(X)\}$ is a morphism from Q to $\widetilde{\Omega}$, the **characteristic morphism** of S.
b) Let $\lambda : Q \longrightarrow \widetilde{\Omega}$ be a morphism. For $u \in \Omega(X)$, define

$$Q_\lambda(u) = \{s \in Q(u) : \lambda_u s = u\}.$$

With restrictions induced by Q, Q_λ is a *subsheaf* of Q, the **characteristic subsheaf** of λ. $\square$

With these preliminaries, the reader is invited to prove

THEOREM 24.41. *Let Q be a sheaf over X. With notation as above, the maps*

$$\begin{cases} S \in \mathfrak{P}Q(X) & \mapsto & \gamma_S : Q \longrightarrow \widetilde{\Omega} \\ \lambda : Q \longrightarrow \widetilde{\Omega} & \mapsto & Q_\lambda \in \mathfrak{P}Q(X), \end{cases}$$

are inverse bijective correspondences between $\mathfrak{P}Q(X)$ and the set of all sheaf morphisms from Q to $\widetilde{\Omega}$. $\square$

One of the fundamental laws in **Set** is the existence of a natural bijective correspondence between $A^{B \times C}$ and A^{B^C}, i.e., between $[B \times C,\, A]$ and $[C,\, A^B]$, where $[X, Y]$ is the set of morphisms from X to Y, as in Chapter 16. An important ingredient in this adjunction is the concept of "evaluation at a point" :

$$ev : A^B \times B \longrightarrow A, \text{ given by } \langle\, f, b\,\rangle \mapsto f(b).$$

Indeed, the correspondence mentioned above can be constructed as follows : for a morphism $C \xrightarrow{\lambda} A^B$, consider

$$\lambda \times Id_B : C \times B \longrightarrow A^B \times B, \text{ defined by } \langle\, c, b\,\rangle \mapsto \langle\, \lambda(c), b\,\rangle;$$

we associate to λ the map $g = ev \circ (\lambda \times Id_B)$ from $C \times B$ to A :

$$\begin{tikzcd} C \times B \arrow[rr, "\lambda \times Id_B"] \arrow[dr, "g"'] & & A^B \times B \arrow[dl, "ev"] \\ & A & \end{tikzcd}$$

Note that the construction is entirely categorical. This same law holds in $\boldsymbol{Sh(X)}$, once the appropriate concepts have been defined.

Recall that if $\lambda : P \longrightarrow Q$ is a morphism of sheaves and S is a subsheaf of P, the restriction of λ to S is the morphism

$$\lambda_{|S} : S \longrightarrow Q, \text{ given by } s \mapsto \lambda s.$$

If $u \in \Omega(X)$, **define** $\lambda_{|u}$ as the restriction of λ to the subsheaf $P_{|u}$. Observe that, in fact, $\lambda_{|u} : P_{|u} \longrightarrow Q_{|u}$. Further, if $v \leq u$ in X, then $(\lambda_{|u})_{|v} = \lambda_{|v}$. In particular, if $Q^P(u)$ is the set of morphisms from $P_{|u}$ to $Q_{|u}$ and $v \leq u$ in X, there is a map

$$_{|v} : Q^P(u) \longrightarrow Q^P(v), \text{ given by } \lambda \mapsto \lambda_{|v}.$$

These assignments constitute a contravariant functor, Q^P, from $\Omega(X)$ to **Set**, i.e., a presheaf over X. For $\lambda \in Q^P(u)$ and $\mu \in Q^P(v)$, we have

$$\begin{aligned} [\![\lambda = \mu]\!] &= \bigcup \{w \in \Omega(u \cap v) : \lambda_{|w} = \mu_{|w}\} \\ &= \bigcup \{w \in \Omega(u \cap v) : \forall\ s \in P(w),\quad \lambda s = \mu s\}. \end{aligned}$$

It is now straightforward that Q^P is a **sheaf** over X.

Definition 24.42. *Let P and Q be sheaves over X. With notation as above, the sheaf Q^P is defined to be the* **sheaf of morphisms** *from P to Q.*

Define the **evaluation morphism** *$ev : Q^P \times P \longrightarrow Q$, as the natural transformation $ev = \{ev_u : u \in \Omega(X)\}$, where*

$$ev_u : Q^P(u) \longrightarrow Q(u) \text{ is given by } ev_u(\lambda, s) = \lambda s.$$

Proposition 24.38 now yields

Corollary 24.43. *The correspondences in Theorem* 24.41 *extend to* **isomorphisms** *between the sheaves $\mathfrak{P}Q$ and $\widetilde{\Omega}^Q$.* □

With the same proof as in **Set**, we have

Theorem 24.44. *In $\boldsymbol{Sh(X)}$, exponentiation and Cartesian product are adjoints that is, if P, Q and R are sheaves, there is a natural sheaf isomorphism between $Q^{P\times R}$ and Q^{P^R}, given by*

$$\lambda \in [R, Q^P] \quad \mapsto \quad ev \circ (\lambda \times Id_R) \in [P \times R, Q],$$

where ev is the evaluation morphism of 24.42. □

Remark 24.45. Since $\boldsymbol{Sh(X)}$ has finite products and the adjunction in 24.44, it is, like **Set**, a **Cartesian closed** category (see [**44**], section IV.6, p.95 ff). □

In Part 7 we shall return to the theme of 24.41, describing a theory of **characteristic functions**, fundamental in treating first-order structures in $\boldsymbol{pSh(X)}$.

5. Constant Sheaves

Let A be a set and $\boldsymbol{A}$ be the constant A sheaf over X, as in 23.16 and 23.17. By definition, for each $u \in \Omega(X)$, $\boldsymbol{A}(u)$ consists of the set of continuous maps, $s : u \longrightarrow A$, where A has the discrete topology (all points as open).

We can identify A with the set of constant maps from X to A : for $a \in A$, write $\widehat{a}$ for the constant map of value a on X. Write $\widehat{A}$ for the image of A in $\boldsymbol{A}(X)$ by the injection $a \longmapsto \widehat{a}$.

For $s \in |\boldsymbol{A}|$ and $a \in A$, $\;s^{-1}(a) = [\![s = \widehat{a}]\!]$, which is **clopen** in Es. Thus,

$$[\![s = \widehat{a}]\!] \neq \emptyset \quad \text{iff} \quad a \in Im\, s.$$

Furthermore, if $a \neq b$ in A, then $[\![s = \widehat{a}]\!] \cap [\![s = \widehat{b}]\!] = \emptyset$. We summarize some of the basic properties of sections of constant sheaves in

LEMMA 24.46. *Let A be a set, $f \in A^X$, $s, t \in |\boldsymbol{A}|$ and $u \in \Omega(X)$.*
a) $f \in |\boldsymbol{A}|$ iff for all $a \in A$, $[\![f = \widehat{a}]\!]$ is clopen in X.
b) $Es = \bigcup_{a \in Ims} [\![s = \widehat{a}]\!]$, this union being composed of pairwise disjoint non-empty clopens in Es. Moreover,

(1) *For all $a \in Im\, s$, $s_{|[\![s=\hat{a}]\!]} = \widehat{a}_{|[\![s=\hat{a}]\!]}$;* (2) *$\widehat{A}$ is dense in $\boldsymbol{A}$.*

c) $[\![s = t]\!] = \{x \in Es \cap Et : s(x) = t(x)\}$.

d) $[\![s = t]\!]$ is clopen in $Es \cap Et$.

e) If $\overline{f} = \langle f_1, \ldots, f_n \rangle \in \boldsymbol{A}(u)^n$, then there are non-empty pairwise disjoint clopens in u, *$\{u_i : i \in I\}$, such that $u = \bigcup_{i \in I} u_i$ and each f_k is constant on u_i, for all $i \in I$ and $1 \leq k \leq n$.*

f) Let $\langle t_i, u_i \rangle \in \boldsymbol{A}(X) \times \Omega(u)$, $i \in I$, be such that
(1) *$\{u_i : i \in I\}$ are pairwise disjoint non-empty clopens in u;*
(2) *$u = \bigcup_{i \in I} u_i$;*
Then, there is $s \in \boldsymbol{A}(u)$, such that for all $i \in I$, $s_{|u_i} = t_{i|u_i}$.

g) If u is clopen in X, then the restriction map from $\boldsymbol{A}(X)$ to $\boldsymbol{A}(u)$ is a surjection.

PROOF. (a) is just a restatement of the fact that f is *continuous*. Item (b) is clear; to verify (c), recall that

$$\begin{aligned}[\![s = t]\!] &= \bigcup \{w \in \Omega(Es \cap Et) : s_{|w} = t_{|w}\} \\ &\subseteq \{x \in Es \cap Et : s(x) = t(x)\}.\end{aligned}$$

Now, if $s(x) = t(x) = a$, then $x \in [\![s = \widehat{a}]\!] \cap [\![t = \widehat{a}]\!] = w$, a clopen subset of $Es \cap Et$, such that $s_{|w} = t_{|w}$. But this implies the equality in (c).

d) To show that $[\![s = t]\!]$ is clopen in $Es \cap Et$, let $x \in Es \cap Et$ satisfy $s(x) = a \neq t(x)$. Then, $x \in [\![s = \widehat{a}]\!] \cap Et$, a clopen in Et, and

$$[\![s = \widehat{a}]\!] \subseteq \{x \in Es \cap Et : s(y) \neq t(y)\}.$$

Thus, the complement of $[\![s = t]\!]$ is open in $Es \cap Et$, as needed. Item (e) follows from (b) and 8.4.

f) Just observe that the family $\{t_{i|u_i} : i \in I\}$ is compatible and so can be glued to a section s over u.

g) Fix $s \in \boldsymbol{A}(u)$ and $a \in A$; since the sections $\widehat{a}_{|(X-u)}$ and s have disjoint extents whose union is X, (d) implies that there is $t \in \boldsymbol{A}(X)$ such that $t_{|Es} = s$, as desired. □

REMARK 24.47. It is well known (and easy to verify) that if Y_i are topological spaces, $i \in \underline{n}$, then [15]

$$\mathbb{C}(X, \prod Y_i) \text{ is naturally isomorphic to } \prod \mathbb{C}(X, Y_i),$$

[15] $\underline{n} = \{1, \ldots, n\}$, as in page 15.

where $\prod Y_i$ has the product topology. Since the **finite** product of discrete spaces is discrete, the finite product of constant A_i sheaves is naturally isomorphic to the constant $\prod A_i$ sheaf. $\square$

REMARK 24.48. A map $A \xrightarrow{f} B$ induces a morphism (23.18)

$$\boldsymbol{A} \xrightarrow{f} \boldsymbol{B}, \text{ given by } s \longmapsto f \circ s.$$

This produces a covariant functor from **Set** to the category of constant sheaves over X, that preserves monics and epics. $\square$

The next result characterizes morphisms whose domain is a constant sheaf.

PROPOSITION 24.49. *Let Q be a presheaf over X, let A be a set and let λ, μ: $\boldsymbol{A} \longrightarrow Q$ be presheaf morphisms.*
a) The following are equivalent : [16]

$$(1)\ \lambda = \mu. \qquad\qquad (2)\ \lambda_{X_{|\widehat{A}}} = \mu_{X_{|\widehat{A}}}.$$

b) If Q is a sheaf, then the map $\tau : [\boldsymbol{A}, Q] \longrightarrow Q(X)^A$, $\lambda \mapsto \lambda_{X_{|\widehat{A}}}$, is a natural bijective correspondence between the morphisms from $\boldsymbol{A}$ to Q and the set of maps from A to the set of global sections of Q, $Q(X)$.

PROOF. a) Since $\widehat{A}$ is dense in $\boldsymbol{A}$, (1) $\Leftrightarrow$ (2) follows from 24.14.
b) It is enough to verify that every $f \in Q(X)^{\hat{A}}$ gives rise to a morphism, whose restriction to $\widehat{A}$ is f (24.14). For $s \in |\boldsymbol{A}|$, consider the family

$$T \;=\; \{f\widehat{a}_{|[\![s=\hat{a}]\!]} : a \in A\} \;\subseteq\; |Q|.$$

Since $\widehat{A}$ is a dense set of global sections in $\boldsymbol{A}$,

$$Es \;=\; \bigcup_{a\in A} [\![s = \widehat{a}]\!] \;=\; \bigcup_{a\in A} f\widehat{a}_{|[\![s=\hat{a}]\!]}.$$

Hence, T is compatible, and there is a unique $t \in |Q|$, such that for all $a \in A$,

$$Et \;=\; Es \;\text{ and }\; t_{|[\![s=\hat{a}]\!]} \;=\; f\widehat{a}_{|[\![s=\hat{a}]\!]}.$$

Define $\lambda s = t$. It is straightforward that λ is a morphism, such that $\lambda_{X_{|\hat{A}}} = f$. $\square$

In particular, we have

COROLLARY 24.50. *Let P be a presheaf over X. There is a natural bijective correspondence between the morphisms from $\mathbf{1}$ to P and the set of global sections of P, $P(X)$, given by*

$$\lambda \in [\mathbf{1}, P] \;\mapsto\; \lambda_X(\widehat{0}) \in A(X).$$

where $\widehat{0}$ is the constant map with value 0 [17].

We now return to the consideration of constant sheaves **of $\boldsymbol{L}$-structures** as in 23.17. Let X be a topological space, let L be a first-order language with equality and M a L-structure. We know from 23.17 that $\boldsymbol{M}$ and $\boldsymbol{M_b}$ are presheaves of L-structures, with relations, constants and operations defined pointwise. We know

[16]Recall that $\lambda = \{\lambda_u : u \in \Omega(X)\}$.
[17]$\widehat{0}$ is the unique element of $\mathbf{1}(X)$.

that $\boldsymbol{M}$ is a sheaf over X. To characterize the type of completeness of $\boldsymbol{M_b}$ we introduce

DEFINITION 24.51. *Let $\mathcal{C}$ be a category. A $\mathcal{C}$-presheaf P over X is* **finitely complete** *if all* **finite** *compatible sets of sections in P have a unique gluing. That is, if $S \subseteq_f |P|$ is compatible, then there is a* unique *$t \in |P|$ such that*

$$Et = \bigcup_{s \in S} Es \quad \text{and} \quad t_{|Es} = s, \text{ for all } s \in S.$$

PROPOSITION 24.52. *$\boldsymbol{M_b}$ is a finitely complete presheaf over X.*

PROOF. If $S \subseteq_f |\boldsymbol{M_b}|$ is compatible, since continuity is a local property, 1.2 yields a continuous $t : \bigcup_{s \in S} \longrightarrow M$, whose restriction to each Es is $s \in S$. Since S is finite and all sections in S have finite image, the same must be true of t. $\square$

DEFINITION 24.53. *Let $\phi(v_1, \ldots, v_n)$ be a formula in L, in the free variables $v_1, \ldots, v_n$. Let $u \in \Omega(X)$ and $\overline{f} = \langle f_1, \ldots, f_n \rangle \in \boldsymbol{M}(u)^n$. Define*

$$\mathfrak{v}_u(\phi)(\overline{f}) = \{x \in u : M \models \phi[f_1(x), \ldots, f_n(x)]\}$$

called the **Feferman-Vaught value of $\boldsymbol{\phi}$ at $\overline{\boldsymbol{f}}$** [18]. *When u is clear from context, its mention will be omitted.*

Recall (17.4) that a formula is *primitive* if it is of the form $\exists \overline{x}\, \phi$, where ϕ is a conjunction of atomic formulas. Note that the set of primitive formulas is closed under conjunction.

PROPOSITION 24.54. *Let $u \in \Omega(X)$ and $\overline{f} \in \boldsymbol{M}(u)^n$. Notation as above,*

a) If $\phi(v_1, \ldots, v_n)$ is a formula in L, then $\mathfrak{v}_u(\phi)(\overline{f})$ is clopen in u.

b) Let Δ be the set of formulas $\phi(v_1, \ldots, v_n)$ in L such that

$[\Delta]$ *For all* $\overline{f} \in \boldsymbol{M}(u)^n, \quad \boldsymbol{M}(u) \models \phi[\overline{f}] \quad \text{iff} \quad \mathfrak{v}_u(\phi)(\overline{f}) = u.$

Then, Δ is closed under conjunction and contains all primitive formulas in L.

c) Let Γ be the set of formulas $\phi(v_1, \ldots, v_n)$ in L such that

$[\Gamma]$ *For all* $\overline{f} \in \boldsymbol{M}(u)^n, \quad \mathfrak{v}_u(\phi)(\overline{f}) = u \quad \Rightarrow \quad \boldsymbol{M}(u) \models \phi[\overline{f}].$

Then,

(1) $\phi, \psi \in \Delta \quad \Rightarrow \quad (\phi \to \psi) \in \Gamma$; (2) $\phi \in \Gamma \quad \Rightarrow \quad \forall z\, \phi \in \Gamma.$

d) Let σ be a sentence of the form $\forall z\, (\phi \to \exists \overline{y}\, \psi)$, where ϕ, ψ are conjunctions of atomic formulas. Then,

$$M \models \sigma \quad \Rightarrow \quad \boldsymbol{M}(u) \models \sigma.$$

PROOF. a) By 24.46.(e), there is a partition of u into clopens, $\{u_i : i \in I\}$, such that all f_k are constant in u_i, $i \in I$. Write a^i_k for the constant value of f_k on u_i. Consequently

$$\begin{cases} \mathfrak{v}(\phi)(\overline{f}) & = & \bigcup \{u_i : M \models \phi[a^i_1, \ldots, a^i_n]\} \\ & \text{and} & \\ \mathfrak{v}(\neg\,\phi)(\overline{f}) & = & \bigcup \{u_i : M \models \neg\,\phi[a^i_1, \ldots, a^i_n]\}. \end{cases}$$

[18] In honor of the seminal [**14**]; see Theorem 24.55.

Hence, u is the union of the disjoint opens $\mathfrak{v}(\phi)(\overline{f})$ and $\mathfrak{v}(\neg\,\phi)(\overline{f})$, and both these values are clopen in u.
b) Note that atomic formulas are in Δ because $\boldsymbol{M}(u)$ is, by definition, a substructure of M^u with the L-interpretation given in 17.9. It is straightforward that Δ is closed under conjunctions. Next, we show

(*) $$\phi \in \Delta \;\; \Rightarrow \;\; \exists z\phi \in \Delta.$$

For $\langle\, g; \overline{f}\,\rangle \in \boldsymbol{M}(u)^{n+1}$, suppose that $\boldsymbol{M}(u) \models \phi[g; \overline{f}]$. Then, for all $x \in u$,

$$M \models \phi[g(x); \overline{f}(x)]$$

and so $\mathfrak{v}(\exists z\phi)(\overline{f}) = u$. If this equality holds, let u_i, $i \in I$, be a clopen partition of u, such that f_k is constant in u_i, $1 \leq k \leq n$ (24.46.(e)). Let $a_1^i, \ldots, a_n^i$ be the values of $f_1, \ldots, f_n$ on u_i. Since $M \models \exists z\phi[z;\, a_1^i, \ldots, a_n^i]$, there are $b_i \in M$, such that

$$\text{For all } i \in I, \quad M \models \phi[b_i;\, a_1^i, \ldots, a_n^i].$$

Let $g : u \longrightarrow M$ be the map that on each u_i has constant value b_i. Then, $g \in \boldsymbol{M}(u)$ and $\mathfrak{v}(\phi)(g; \overline{f}) = u$. Since $\phi \in \Delta$, we get $\boldsymbol{M}(u) \models \phi[g; \overline{f}]$, and so $\boldsymbol{M}(u) \models \exists z\phi[\overline{f}]$, completing the proof of (*) and of item (b).
c) (1) Suppose ϕ, $\psi \in \Delta$ and $\mathfrak{v}(\phi \to \psi)(\overline{f}) = u$. If $\boldsymbol{M}(u) \models \phi[\overline{f}]$, then $\mathfrak{v}(\phi)(\overline{f}) = u$ and so $\mathfrak{v}(\psi)(\overline{f}) = u$. But then, the hypothesis that $\psi \in \Delta$ yields $\boldsymbol{M}(u) \models \psi[\overline{f}]$, as needed.
(2) Suppose that $\phi \in \Gamma$ and $\mathfrak{v}(\forall z\phi)(\overline{f}) = u$. For $g \in \boldsymbol{M}(u)$ and $x \in u$, we have

$$M \models \phi[g(x); \overline{f}(x)]$$

because $M \models \forall z\phi[z; \overline{f}(x)]$. Hence, $\mathfrak{v}(\phi)(g; \overline{f}) = u$, and the hypothesis that $\phi \in \Gamma$ entails $\boldsymbol{M}(u) \models \phi[g; \overline{f}]$. Since g is arbitrary in $\boldsymbol{M}(u)$, we obtain $\boldsymbol{M}(u) \models \forall z\phi[\overline{f}]$, as desired.
d) It is immediate from (b) and (c) that $\sigma \in \Gamma$. Since the hypothesis is equivalent to $\mathfrak{v}(\sigma) = u$, we conclude that $\boldsymbol{M}(u) \models \sigma$, as claimed. □

As a consequence of 24.54.(d), the L-structure of locally constant functions on any topological space with values in a model of a theory T whose axioms are of the form $\forall z(\phi \to \exists y\psi)$, with ϕ, ψ a conjunction of atomic formulas, are also models of T. This applies to many interesting mathematical structures, such as groups, rings, lattices, Boolean or Heyting algebras, etc.

The satisfaction of general formulas in the global sections of $\boldsymbol{M_b}$ is characterized by the following fundamental result :

THEOREM 24.55. (Feferman-Vaught) *Let M be a L-structure and X a Boolean space. To each formula $\phi(v_1, \ldots, v_n)$ in L we can recursively associate a sequence*

$$\langle\, \Phi(x_1, \ldots, x_m);\; \psi_1(v_1, \ldots, v_n), \ldots, \psi_m(v_1, \ldots, v_n)\,\rangle$$

such that

* $\Phi(x_1, \ldots, x_m)$ *is a formula of the language of Boolean algebras;*
* *For* $1 \leq k \leq m$, $\psi_k(v_1, \ldots, v_n)$ *are L-formulas in the same free variables as* ϕ*;*
* *For all* $\overline{f} = \langle\, f_1, \ldots, f_n\,\rangle \in \boldsymbol{M_b}(X)^n$,

$$\boldsymbol{M_b}(X) \models \phi[f_1, \ldots, f_n] \quad \textit{iff} \quad B(X) \models \Phi[\mathfrak{v}(\psi_1)(\overline{f}), \ldots, \mathfrak{v}(\psi_m)(\overline{f})].$$

Proof. By induction on the complexity of formulas. The proof given in [**14**] or [**7**] is easily adapted to this case, via 24.46, once it is remarked that the passage through the existential quantifier requires only finite completeness. The reader might also consult [**76**]. $\square$

As an illustration of the techniques used to treat structures of locally constant functions on a topological space, we shall prove the following result, complementing 9.33 and 9.35.

Theorem 24.56. *Let X be a compact space and R a commutative ring with identity. If $\mathcal{M}_\sigma(R)$ is a frame, then $\mathcal{M}_\sigma(\boldsymbol{R}(X))$ is an algebraic frame.*

Proof. We shall use the notation and results in section 9.2. Recall that $B(X)$ is the Boolean algebra of clopens in X.

By 24.54.(d), $\boldsymbol{R}(X) = \mathbb{C}(X, R)$ is a commutative ring with identity $\widehat{1}$ (the constant function with value $1 \in R$) and additive neutral $\widehat{0}$ (the constant function with value $0 \in R$). Addition and multiplication in $\boldsymbol{R}(X)$ are defined pointwise, i.e., for f, $g \in \boldsymbol{R}(X)$ and $x \in X$,

$$[f + g](x) = f(x) + g(x) \quad \text{and} \quad [f \cdot g](x) = f(x)g(x).$$

For $S \subseteq \boldsymbol{R}(X)$ and $x \in X$, set

$$S(x) = \{f(x) \in R : f \in S\}.$$

If f, $g \in \boldsymbol{R}(X)$ and $u \in B(X)$, write

$$f_{|u} \vee g_{|u^c}$$

for the element of $\boldsymbol{R}(X)$ that coincides with f on u and with g on u^c (24.46.(f)).

The proof develops through a series of Facts, that describe the relationship between global and local aspects of the pertinent concepts.

Fact 24.57. *Let S, T be a multiplicative subsets of of $\boldsymbol{R}(X)$. Then*
a) For all $x \in S$, $S(x) \in \mathcal{M}(R)$ and $S \subseteq T \Rightarrow S(x) \subseteq T(x)$.
b) If $S \in \mathcal{M}_\sigma(\boldsymbol{R}(X))$, $f \in S$ and $u \in B(X)$, then $\left(f_{|u} \vee \widehat{1}_{|u^c}\right) \in S$.
c) $S \in \mathcal{M}_\sigma(\boldsymbol{R}(X)) \Rightarrow S(x) \in \mathcal{M}_\sigma(R)$, for all $x \in X$.
d) For all $x \in X$, $\sigma(S)(x) = \sigma(S(x))$ [19].

Proof. Item (a) is straightforward. For (b), consider

$$g = f_{|u} \vee \widehat{1}_{|u^c} \quad \text{and} \quad h = \widehat{1}_{|u} \vee f_{|u^c}.$$

Then, $f = gh$ and saturation implies g, $h \in S$, as desired.
c) Fix $x \in X$ and suppose that $ab = c \in S(x)$. Then, there is $s \in S$ and a clopen u in X, such that s is constantly equal to c in u. By (b), $f = s_{|u} \vee \widehat{1}_{|u^c}$ is in S. Now define g, $h \in \boldsymbol{R}(X)$ as follows :

$$g = \widehat{a}_{|u} \vee \widehat{1}_{|u^c} \quad \text{and} \quad h = \widehat{b}_{|u} \vee \widehat{1}_{|u^c},$$

[19] σ is *saturation*, as in 9.23. Its basic property is described in 9.23.(a).

where $\widehat{a}$ is the constant a-valued map on X. It is clear that $gh = f$. Since S is saturated, we get g, $h \in S$, and so $a = g(x)$ and $b = h(x)$ are both in $S(x)$, as needed.
d) Fix $x \in X$; since $S(x) \subseteq \sigma(S)(x)$ and the latter is saturated, we have $\sigma(S(x)) \subseteq \sigma(S)(x)$. Now suppose that $a \in \sigma(S(x))$; then, there are $b \in R$ and $c \in S(x)$, such that $ab = c$. Select $f \in S$ and $u \in B(X)$ such that f is constantly equal to c on u. Define g, $h \in \boldsymbol{R}(X)$ by

$$g = \widehat{a}_{|u} \vee \widehat{1}_{|u^c} \quad \text{and} \quad h = \widehat{b}_{|u} \vee f_{|u^c}.$$

Then, $gh = f \in S$ and so g, $h \in \sigma(S)$. Hence, $a = g(x) \in \sigma(S)(x)$, as needed. □

FACT 24.58. *If $S, T \in \mathcal{M}_\sigma(\boldsymbol{R}(X))$, then*

$$S = T \quad \Leftrightarrow \quad \text{For all } x \in X, \quad S(x) = T(x).$$

PROOF. It is enough to verify ($\Leftarrow$). This is the only step of the proof in which we shall use the compactness of X. Let $f \in S$. For each $x \in X$, there is $g_x \in T$, such that $g_x(x) = f(x)$. Since all elements of $\boldsymbol{R}(X)$ are locally constant, we conclude that for each $x \in X$, there is a pair $\langle g_x, u_x \rangle \in T \times B(X)$, such that $x \in u_x$ and f, g_x are constant and equal in u_x. Since $\{u_x : x \in X\}$ is a clopen covering of X, compactness entails that it has a finite subcovering. Hence, we may assume that there is a finite collection

$$\{\langle g_k, u_k \rangle \in T \times B(X) : 1 \leq k \leq n\}$$

such that $\bigcup_{k=1}^n u_k = X$ and f, g_k coincide in u_k. Define, by induction on $k \leq n$,

$$\begin{cases} v_1 = u_1 & \text{and} \\ v_k = u_k - \bigcup_{j \leq (k-1)} v_j & \text{if } k \geq 2. \end{cases}$$

Then, $v_k \subseteq u_k$, the v_k are pairwise disjoint and still a covering of X. Hence, the collection

$$\{\langle g_k, v_k \rangle : 1 \leq k \leq n\}$$

still has the property of the original one, i.e., g_k and f are constant and equal in each v_k and the v_k are a (pairwise disjoint) covering of X. Define, for $1 \leq k \leq n$

$$h_k = g_{k|v_k} \vee \widehat{1}_{|v_k^c}.$$

Since T is saturated, (b) guarantees that $h_k \in T$, $1 \leq k \leq n$. Now observe that for each v_k we have

$$(h_1 h_2 \ldots h_n)_{|v_k} = g_{k|v_k} = f_{|v_k},$$

and so $f = h_1 h_2 \ldots h_n \in T$. We have shown that $S \subseteq T$. The argument being symmetrical in S and T, we conclude that $S = T$, as desired. □

FACT 24.59. *If $S, T \in \mathcal{M}_\sigma(\boldsymbol{M}(X))$, then, for all $x \in X$,*
a) $(S \cap T)(x) = S(x) \cap T(x)$; *b) $(S \cdot T)(x) = S(x) \cdot T(x)$;*
c) $(S \vee T)(x) = S(x) \vee T(x)$.

PROOF. a) Clearly, $(S \cap T)(x) \subseteq S(x) \cap T(x)$. If $a \in S(x) \cap T(x)$, select $f \in S$, $g \in T$ and $u \in B(X)$ such that f and g are constantly equal to a on u. By Fact 24.57.(b)

$$h \;=\; f_{|u} \vee \widehat{1}_{|u^c} \;=\; g_{|u} \vee \widehat{1}_{|u^c}$$

are in S and T, establishing that $a \in (S \cap T)(x)$.
b) It suffices to check that $S(x) \cdot T(x) \subseteq (S \cdot T)(x)$. If $c = ab$, with $a \in S(x)$ and $b \in T(x)$, there are $f \in S$, $g \in T$ and $u \in B(X)$ such that f and g are constant on u, with values a and b, respectively. Then, $g = f_{|u} \vee \widehat{1}_{|u^c}$ belongs to S and $h = g_{|u} \vee \widehat{1}_{|u^c}$ is in T, with $gh(x) = c$, as needed.
c) By 9.23.(b), $S \vee T = \sigma(S \cdot T)$; the conclusion follows immediately from (b) and 24.57.(d). $\square$

Since $\mathcal{M}_\sigma(\boldsymbol{R}(X))$ is a complete algebraic lattice (9.25.(c)), by 8.14 it will be a frame iff it is distributive. Hence, to complete the proof, let S, T_1, T_2 be saturated multiplicative sets in $\boldsymbol{R}(X)$. The fact that $\mathcal{M}_\sigma(R)$ is a frame, together with 24.57.(c), implies that that for all $x \in X$

$$S(x) \cap (T_1(x) \vee T_2(x)) \;=\; (S(x) \cap T_1(x)) \vee (S(x) \cap T_2(x)).$$

But then, items (a) and (c) in 24.59 yield

$$S \cap (T_1 \vee T_2) \;=\; (S \cap T_1) \vee (S \cap T_2)$$

as needed. $\square$

Proposition 9.33, together with Theorems 9.35 and 24.56, produce a wide a variety of commutative rings with identity whose lattice of saturated multiplicative sets constitute an algebraic frame.

To end this section we present a further example of sheaf, that are important in applications ([**43**], [**1**], [**36**], [**49**], [**10**]).

Definition 24.60. *Let X be a topological space and M a L-structure. A* **filtration** Σ *in $\langle X, M \rangle$ is a collection of pairs*

$$\Sigma = \{\langle F_i, M_i \rangle : i = 1, \ldots, n\}$$

such that
* $F_1, \ldots, F_n$ *is an increasing sequence of closed set in X;*
* $M_1, \ldots, M_n$ *is an increasing sequence of substructures of M.*

If $u \in \Omega(X)$ write $\Sigma_{|u}$ for the induced filtration on u, that is, [20]

$$\Sigma_{|u} \;=\; \{\langle u \cap F_i, M_i \rangle : i \in \underline{n}\}.$$

Define, for $u \in \Omega(X)$

$$\boldsymbol{M}(u, \Sigma) \;=\; \{s \in \boldsymbol{M}(u) : \forall\, i \in \underline{n}\ (x \in F_i \Rightarrow f(x) \in M_i)\}.$$

With relations, operations and constants induced by $\boldsymbol{M}$, $\boldsymbol{M}(u,\Sigma)$ is a L-structure.

For $u \leq v$ in X, the canonical restrictions of $\boldsymbol{M}$ are L-morphisms, taking $\boldsymbol{M}(v,\Sigma)$ into $\boldsymbol{M}(u,\Sigma)$. Hence, we have an extensional presheaf over X, the **filtered power** *of M over X by Σ, written* $\boldsymbol{M(X, \Sigma)}$.

Write $\boldsymbol{M_b(X, \Sigma)}$ for the subpresheaf of $\boldsymbol{M(X, \Sigma)}$ whose domain is constituted by all sections with finite image in M. $\boldsymbol{M_b(X, \Sigma)}$ is the **bounded filtered**

[20]Recall that $\underline{n} = \{1, \ldots, n\}$ as in page 1.

power *of* M *over* X *by* Σ. *If* $u \in \Omega(X)$, *write* $\boldsymbol{M_b}(u, \Sigma)$ *for the* L*-structure of sections of* $\boldsymbol{M_b(X, \Sigma)}$ *over* u.

REMARK 24.61. Note that when $\Sigma = \{\langle X, M \rangle\}$,

$$\boldsymbol{M(X, \Sigma)} = \boldsymbol{M} \quad \text{and} \quad \boldsymbol{M_b(X, \Sigma)} = \boldsymbol{M_b}.$$

Moreover, neither the filtered power nor the bounded filtered power are changed if the pair $\langle X, M \rangle$ is added to the filtration. Whenever convenient we shall assume that $\langle X, M \rangle$ belongs to the filtration in consideration. □

The following is straightforward :

LEMMA 24.62. *If* Σ *is a filtration on* $\langle X, M \rangle$ *then*
a) $\boldsymbol{M(X, \Sigma)}$ *is a sheaf over* X;
b) $\boldsymbol{M_b(X, \Sigma)}$ *is a finitely complete presheaf over* X. □

LEMMA 24.63. *Let* X, Y *be topological spaces and* M *a* L*-structure. Let*

$$\Sigma_X = \{\langle F_i, M_i \rangle : i \in \underline{n}\} \quad \text{and} \quad \Sigma_Y = \{\langle K_i, M_i \rangle : i \in \underline{n}\}$$

be filtrations on X *and* Y, *respectively. Suppose that* $f : X \longrightarrow Y$ *is a continuous map such that for all* $k \in \underline{n}$, $F_i \subseteq f^{-1}(K_i)$. *If* $\widehat{f}$ *is the map* (23.18),

$$\widehat{f} : \boldsymbol{M}(Y, \Sigma_Y) \longrightarrow \boldsymbol{M}(X, \Sigma_X), \quad \text{given by } s \mapsto s \circ f$$

then, [21]
a) $\widehat{f}$ *is a* L*-morphism.*
b) $Im\, f$ *is dense in* Y, $\widehat{f}$ *is a* L*-monomorphism.*
c) The restriction of $\widehat{f}$ *to* $\boldsymbol{M_b}(Y, \Sigma)$ *takes values in* $\boldsymbol{M_b}(X, \Sigma)$ *and statements (a) and (b) above hold for this restriction.*

PROOF. It is clear that $\widehat{f}$ takes $\boldsymbol{M}(Y, \Sigma_Y)$ to $\boldsymbol{M}(X, \Sigma_X)$ and restricts to a map from the bounded filtered power of M over Y to the bounded filtered power of M over X.
a) Let $R \in rel(n, L)$ be a n-ary relation symbol in L and $\overline{s} = \langle s_1, \ldots, s_n \rangle$ be sections in $\boldsymbol{M}(Y, \Sigma_Y)$. Then,

$$\boldsymbol{M}(Y, \Sigma_Y) \models R[\overline{s}] \quad \text{iff} \quad \forall\, y \in Y, \;\; M \models R[s_1(y), \ldots, s_n(y)].$$

Since $\widehat{f}(s_k) = s_k \circ f$, for each $x \in X$ we have

$$M \models R[s_1(f(x)), \ldots, s_n(f(x))],$$

that is, $\boldsymbol{M}(X, \Sigma_X) \models R[\widehat{f}(s_1), \ldots, \widehat{f}(s_n)]$. Similarly, one shows that $\widehat{f}$ preserves operations in L. It is clear that $\widehat{f}$ takes constant maps to constant maps and so is a L-morphism. Since the bounded filtered power is a L-substructure of the filtered power, the restriction of $\widehat{f}$ to the bounded filtered power is also a L-morphism.
b) In view of (a), to show that $\widehat{f}$ is a L-monomorphism it is enough to check that if $R \in rel(n, L)$ and $s_1, \ldots, s_n \in \boldsymbol{M}(Y, \Sigma_Y)$, then

$$\boldsymbol{M}(X, \Sigma_X) \models R[\widehat{f}(s_1), \ldots, \widehat{f}(s_n)] \quad \Rightarrow \quad \boldsymbol{M}(Y, \Sigma_Y) \models R[\overline{s}].$$

By 24.46.(e), there is a partition $\{v_i : i \in i\}$ of Y into non-empty clopens such that each s_k is constant in v_i, $i \in I$. For $y \in Y$, there is a *unique* $i \in I$ such that

[21] For the definition of L-morphism and L-monomorphism see 17.7.

$y \in v_i$. Since $Im\, f$ is dense in Y, there is $x \in X$ such that $f(x) \in v_i$. Hence, our hypothesis entails

$$M \models R[s_1(f(x)), \ldots, s_n(f(x))],$$

and so $M \models R[s_1(y), \ldots, s_n(y)]$, because each s_k is constant in v_i. Since y is arbitrary in Y, we obtain $\boldsymbol{M}(Y, \Sigma_Y) \models R[\overline{s}]$, as needed. The bounded filtered power being a L-substructure of the filtered power, (c) is also verified. $\square$

Proposition 24.64. *Let M be an L structure, X a topological space and γX the Booleanization of X* (20.7). *Let*

$$\Sigma = \{\langle F_i, M_i \rangle : i \in \underline{n}\}$$

be a filtration on $\langle X, M \rangle$. Then,
a) Σ induces a filtration $\gamma\Sigma = \{\langle \gamma F_i,\ M_i \rangle : i \in \underline{n}\}$, such that for all $i \in \underline{n}$, $F_i \subseteq \gamma^{-1}(\gamma F_i)$.
b) For each $v \in \Omega(\gamma X)$, the map

$$\alpha_v : \boldsymbol{M_b}(v, \gamma\Sigma) \longrightarrow \boldsymbol{M_b}(\gamma^{-1}(v), \Sigma), \text{ given by } \alpha_v(s) = s \circ \gamma$$

is a L-isomorphism, such that for all opens $w \leq v$ in γX, the following diagram commutes :

$$\begin{tikzcd}
\boldsymbol{M_b}(v, \gamma\Sigma) \arrow[r, "\alpha_v"] \arrow[d, "\cdot|_w"'] & \boldsymbol{M_b}(\gamma^{-1}(v), \Sigma) \arrow[d, "\cdot|_{\gamma^{-1}(w)}"] \\
\boldsymbol{M_b}(w, \gamma\Sigma) \arrow[r, "\alpha_w"'] & \boldsymbol{M_b}(\gamma^{-1}(w), \gamma\Sigma)
\end{tikzcd}$$

In particular, for all $u \in B(X)$, $\boldsymbol{M_b}(u, \Sigma)$ is isomorphic to $\boldsymbol{M_b}(S_u, \gamma\Sigma)$.

Proof. a) Since $F \mapsto \gamma F$ is an increasing map (20.9.(c)), it is clear that $\gamma\Sigma$ is a filtration on $\langle \gamma X, M \rangle$. Recalling that

$$\gamma F = \bigcap \{v \in B(\gamma X) : F \subseteq \gamma^{-1}(v)\}$$

we have $F \subseteq \gamma^{-1}(\gamma F)$, for any closed set F in X.
b) Taking 20.9 and 24.63 into account, there only remains to verify that α_v is a *surjection*. The reader can check that the proof used to show that α_v is onto in 20.9 also applies to the situation at hand. $\square$

Remark 24.65. As was the case in 20.9, it follows from 24.64 that $\boldsymbol{M_b}(X, \Sigma)$ is isomorphic to $\boldsymbol{M}(\gamma X, \gamma\Sigma)$, with γX a Boolean space. Hence, if we are interested only in *global sections*, we may as well assume that X is Boolean. However, sheaf-theoretically, the situation is more complex : the structure of sections over each $v \in \Omega(\gamma X)$ is isomorphic to the structure of sections over its inverse image by γ. But X may very well possess opens that are not in the image of γ^*. In fact, the proof of 20.9 indicates what one must do to produce such an example. It is shown that the image of γ^* corresponds to the set of opens in X that are unions of elements of $B(X)$. Hence, if X is not totally disconnected, we must have $\Omega(X) \neq Im\, \gamma^*$. In this case, the presheaves $\boldsymbol{M_b(X,\ \Sigma)}$ and $\boldsymbol{M_b(\gamma X, \gamma\Sigma)}$ *are not* isomorphic.

Since the image of a compact set by a continuous map is compact, if X is not compact and M is infinite, it is clear that there is no compact space that can play the role γX for $\boldsymbol{M(X,\ \Sigma)}$. $\square$

6. Natural Numbers

We now turn to the existence of a "natural number object" in $\boldsymbol{Sh(X)}$.

Perhaps one of the most fundamental properties of $\mathbb{N}$ is the possibility of defining functions by recursion. More precisely, let A be a set, let $A \xrightarrow{f} A$ be a map and let a_0 be an element of A. Then, the following principle holds :

[ind] There is a **unique** map, $g : \mathbb{N} \longrightarrow A$, such that
$$g(0) = a_0 \quad \text{and} \quad g(n+1) = f(g(n)).$$

Consider the class of all triples (A, f, a_0), where A is a set, $f \in A^A$ and a_0 is a distinguished element of A. If σ is the successor function on $\mathbb{N}$, $n \mapsto (n+1)$, then $(\mathbb{N}, \sigma, 0)$ is such a triple. Given $\mathcal{A} = (A, f, a_0)$ and $\mathcal{B} = (B, g, b_0)$, define a morphism $h : \mathcal{A} \longrightarrow \mathcal{B}$, to be a function $h\ :\ A \longrightarrow B$, such that $h(a_0) = b_0$ and $h \circ f = g \circ h$:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] & A \arrow[d, "h"] \\ B \arrow[r, "g"'] & B \end{tikzcd}$$

This collection of objects and morphisms form a category. The principle [ind] means that $(\mathbb{N}, \sigma, 0)$ is **an initial object** in this category. Phrased in this way, it is possible to define the concept of natural number object in any category, **once we know how to interpret the notion of "distinguished element"**.

If A is a set, there is a bijective correspondence between A and the set of maps from $\{*\}$ to A. Actually, any one element set could take the place of $\{*\}$, since all of them represent the **final object**, **1**, in **Set**. We therefore define, in a category with final object, **1**, a **distinguished element** of an object A as a **morphism from 1 to** $\boldsymbol{A}$. This is the notion we shall adopt. It has shortcomings : the usual notion of Robinson diagram of a structure can not be rendered sheaf theoretic with this definition of interpretation of constants.

In $\boldsymbol{Sh(X)}$, the final object is the constant $\{*\}$ sheaf, written **1**. Hence, by 24.50, the "elements" of a presheaf A are **the global sections** of A.

Let $\mathcal{I}nd$ be the category of triples (S, f, c), with S a sheaf over X, f is a morphism from S to S and c is a global section of S, with morphisms as above. A **natural number object in** $\boldsymbol{Sh(X)}$ is an initial object of $\mathcal{I}nd$, i.e., a triple (N, σ, ω) in $\mathcal{I}nd$, satisfying :

[ind] For all (S, f, c) in $\mathcal{I}nd$, there is a unique $g : N \longrightarrow S$, such that $f_X(\omega) = c$ and $f \circ g = g \circ \sigma$.

$$\begin{tikzcd} N \arrow[r, "\sigma"] \arrow[d, "g"'] & N \arrow[d, "g"] \\ S \arrow[r, "f"'] & S \end{tikzcd}$$

Clearly, if it exists, such an object is unique, up to isomorphism.

Theorem 24.66. *The triple* $(\mathbf{N}, \sigma, \widehat{0})$*, consisting of*

∗ *The constant* $\mathbb{N}$ *sheaf,* $\mathbf{N}$*;*

∗ *The 'successor' morphism* σ*, defined, for each* $u \in \Omega(X)$*, by*

$$s \in \mathbf{N}(u) \quad \mapsto \quad \sigma_u(s) = s + 1;$$

∗ *The global section constantly equal to* 0*, denoted by* $\widehat{0}$*,*

is the natural number object in $\boldsymbol{Sh(X)}$*. Furthermore, the following induction principle holds in* $\mathbf{N}$ *: if* S *is a subsheaf of* $\mathbf{N}$*, the following are equivalent :*

(1) $S = \mathbf{N}$.

(2) (i) $\widehat{0} \in S(X)$ $\qquad$ (ii) $\sigma(S) \subseteq S$.

Proof. Let $Q \in \boldsymbol{Sh(X)}$, let $q_0 \in Q(X)$ be a global section of Q and let $\mu : Q \longrightarrow Q$ be a morphism. By the classical principle of definition by recursion, there is a **unique** $g : \mathbb{N} \longrightarrow Q(X)$, such that $g(0) = q_0$ and $g(n+1) = \mu_X(g(n))$, $\forall\, n \in \mathbb{N}$. By Proposition 24.49.(b), there is a unique morphism $\lambda : \mathbf{N} \longrightarrow Q$, such that $\lambda_X(\widehat{n}) = g(n)$. It is straightforward that λ is the unique morphism from $\mathbf{N}$ to Q, such that $\lambda_X(\widehat{0}) = q_0$ and $\lambda \circ \sigma = \mu \circ \lambda$, as needed.

For the induction principle, clearly (1) $\Rightarrow$ (2); for the converse, the conditions in (2) and the usual induction principle entail that $\widehat{n} \in |S|$, for all $n \in \mathbb{N}$. Since $\widehat{\mathbb{N}}$ is dense in $\mathbf{N}$ (24.46.(a).(2)) and S is a *subsheaf* of $\mathbf{N}$, we obtain $S = \mathbf{N}$, as needed. $\square$

We now introduce the sheaf-theoretic counterpart of the familiar structure on $\mathbf{N}$. For s, $t \in |\mathbf{N}|$, define

$$\begin{cases} s + t : Es \cap Et \longrightarrow \mathbb{N} & \text{by} \quad [s+t](x) = s(x) + t(x); \\ s \cdot t : Es \cap Et \longrightarrow \mathbb{N} & \text{by} \quad [s \cdot t](x) = s(x)t(x), \end{cases}$$

where the operations in the right-hand side are the usual ones in $\mathbb{N}$. Note that for all $a \in \mathbb{N}$,

$$\begin{cases} [(s+t) = \widehat{a}] = \bigcup_{b+c=a} [\![s = \widehat{b}]\!] \cap [\![t = \widehat{c}]\!] \\ [(s \cdot t) = \widehat{a}] = \bigcup_{bc=a} [\![s = \widehat{b}]\!] \cap [\![t = \widehat{c}]\!], \end{cases}$$

and so $s + t$, st are, by 24.46.(a), in $|\mathbf{N}|$. It is readily verified that addition and product are *sheaf morphisms* from $\mathbf{N} \times \mathbf{N}$ to $\mathbf{N}$.

We also define a binary relation $\leq$ on $\mathbb{N}$ by the following prescription, where $u \in \Omega(X)$:

$$\leq(u) = \{\langle s, t \rangle \in \mathbb{N}(u) \times \mathbb{N}(u) : \forall\, x \in u, \;\; s(x) \leq t(x)\},$$

with restrictions induced by $\mathbb{N} \times \mathbb{N}$. The reader can check that $\leq$ is a subsheaf of $\mathbb{N}^2$. We can consider $\mathbb{N}$ as a structure

$$\mathbb{N} = \langle \mathbb{N}, \leq, +, \cdot, \widehat{0}, \widehat{1} \rangle,$$

that has basic properties that are analogous to those found in elementary number theory. Although we have not defined the value of formulas in presheaves, we shall discuss the interpretation of very simple formulas in $\mathbb{N}$, in particular because it is a good introduction to the general case. For instance

(i) $$\forall s\, \forall t\; (s + t = t + s)$$

expressing the commutativity of addition is true because addition is defined pointwise and it is commutative in $\mathbb{N}$. Similarly, the sheaf operations $+$ and $\cdot$ are commutative, associative and have $\widehat{0}$ and $\widehat{1}$ as neutrals, respectively. Of a slightly different sort are the properties

(ii) $$\forall s (s \neq \widehat{0} \;\; \rightarrow \;\; \exists t (s = t + \widehat{1}));$$

(iii) $$\forall s\; (s = \widehat{0} \;\; \vee \;\; \exists t\; (s = t + \widehat{1}).$$

Classically, (ii) and (iii) are equivalent, but not intuitionistically (see 6.8.(h)); in general, we only have (iii) $\Rightarrow$ (ii). To see that $\mathbb{N} \models$ (iii), note that, if $s \in |\mathbb{N}|$, its extent Es can be partitioned into two disjoint *opens* :

$$Es = [\![s \, = \, \widehat{0}]\!] \cup u, \;\;\text{where } u = \textstyle\bigcup_{n \geq 1} [\![s \, = \, \widehat{n}]\!]. \qquad (*)$$

In u we may certainly write s as $t + \widehat{1}$. Since the union of where the two alternatives in (iii) hold is Es, we consider that $\mathbb{N} \models$ (iii). For (ii), let s be a section in $\mathbb{N}$; the idea is that *in* Es (where s exists), if it is distinct from $\widehat{0}$, then it must *locally* be the successor of a section of $\mathbb{N}$, that is

$$Es \cap int\; \{x \in Es : s(x) \neq 0\} \;\; \subseteq \;\; \textstyle\bigcup_{t \in |\mathbb{N}|} [\![s \, = \, (t+1)]\!]. \qquad (**)$$

But this is exactly the content of (*), verifying that $\mathbb{N} \models$ (ii) [22]. In a similar vein, one can prove that $\leq$ is a *linear order* on $\mathbb{N}$, that is a partial order satisfying

(iv) $$\forall s\, \forall t\; (s \leq t \;\; \vee \;\; t \leq s).$$

For, consider

$$u = \{x \in Es \cap Et : s(x) \leq t(x)\};$$

since s and t are locally constant, for $x \in u$ select a clopen v in ν_x, such that $v \subseteq (Es \cap Et)$ and s, t are constant in v. Then $v \subseteq u$, and so $u \in \Omega(X)$. A similar reasoning shows that

$$v = \{x \in Es \cap Et : t(x) \leq s(x)\} \in \Omega(X).$$

Since $u \cup v = Es \cap Et$, it follows that $\mathbb{N}$ models (iv).

Once we have $\mathbb{N}$, there are well known constructions to produce the integers, $\mathbb{Z}$, and the rationals, $\mathbb{Q}$. The proof of this left to the reader as Exercise 24.83

It is also possible to construct the reals in $\boldsymbol{Sh(X)}$. In general however, "completion by Cauchy sequences" gives a different result than "completion by Dedekind

[22] This can be obtained on general grounds, since, as it shall be seen, sheaves are models for the intuitionistic predicate calculus.

cuts". **The completion by Dedekind cuts** of the sheaf $\mathbb{Q}$ is the sheaf of continuous real valued functions on X, $\mathbb{C}(X)$. We refer the interested reader to [**55**].

If A is a set, a **sequence** in A is a map $f : \mathbb{N} \longrightarrow A$. For presheaves we have

Definition 24.67. *If P is a presheaf over X, a* **sequence in *P*** *is a morphism from* $\mathbf{N}$ *to* P.

From 24.49 we get

Corollary 24.68. *Let P be a presheaf over X. There is a natural bijective correspondence between sequences in P and (set-theoretical) sequences in $P(X)$.*

Exercises

24.69. Determine domain, equality and extent for the initial and final objects $\mathbf{0}$ and $\mathbf{1}$ (as in 24.1). □

24.70. Let $P \underset{\lambda}{\overset{\beta}{\rightrightarrows}} Q$ be morphisms of presheaves.

a) $P \times_\lambda P$ is the equalizer of $P \underset{\lambda}{\overset{\lambda}{\rightrightarrows}} Q$.

b) Determine the domain, equality and extent of $Eq(\lambda, \beta)$, establishing it is an extensional presheaf of sets over X. □

24.71. Let P_α, $\alpha \in A$, be a family of subpresheaves of P. For $u \in \Omega(X)$, set

$$B(u) = \bigcup_{\alpha \in A} P_\alpha(u),$$

with restrictions induced by P. Then, B is a presheaf and the join of the P_α in the complete lattice $\mathcal{P}P(X)$. □

24.72. a) The constant presheaf A over X (23.16) is dense in the constant sheaf $\mathbf{A}$ over X.

b) If P_α, $\alpha \in A$, are closed subpresheaves of P, then so is their intersection. Conclude that $\mathfrak{P}P(X)$ is a complete lattice.

c) Show that $\mathcal{P}P(X)$ and $\mathfrak{P}P(X)$ are closed under the restriction operation defined in Example 23.12. □

24.73. a) Completion is a functor, $c : \boldsymbol{pSh(X)} \longrightarrow \boldsymbol{Sh(X)}$.

b) The functor c preserves monics and epics. □

24.74. Let P be a presheaf over X and S, $T \subseteq |P|$.

a) $S \subseteq T \;\Rightarrow\; \overline{S} \subseteq \overline{T}$ and $\overline{S \cap T} = \overline{S} \cap \overline{T}$.

b) The map $\overline{(\cdot)} : \mathcal{P}P(X) \longrightarrow \mathcal{P}P(X)$, $S \mapsto \overline{S}$, is an increasing nucleus on $\mathcal{P}P(X)$ (13.1), whose set of fixed points is $\mathfrak{P}P(X)$. □

24.75. a) Generalize Lemma 18.17 to $\boldsymbol{pSh(X)}$.

b) Let $\{P_i : i \in I\}$ be presheaves over X and let $P = \prod_{i \in I} P_i$. For $J \subseteq I$, set

$$P_{|J} = \prod_{j \in J} P_j$$

and let $\widehat{\pi}_J : P \longrightarrow P_{|J}$ be the projection that forgets the coordinates outside J. Then, the map

$$S \in \mathfrak{P}P_{|J}(X) \;\;\mapsto\;\; \widehat{\pi}_J{}^{-1}(S) \in \mathfrak{P}P(X),$$

is a natural bijective correspondence between closed subpresheaves of $P_{|J}$ and closed subpresheaves of P that depend only on J. A similar statement holds for $\mathcal{P}P(X)$ and $\mathcal{P}P_{|J}(X)$. □

24.76. If P is a presheaf over X, the diagonal of P^n,
$\Delta_n \;=\; \{(\langle s_1, \ldots, s_n \rangle \in |P|^n : s_i = s_j, \forall\, i, j \leq n\}$,
is a n-ary relation on P. □

24.77. If R is a relation from P to Q, define *domain* and *image* of R. □

24.78. Let Q be a sheaf over X.
a) The map $u \in \Omega(X) \mapsto Q_{|u} \in \mathfrak{P}Q(X)$ is a frame isomorphism of $\Omega(X)$ onto its image.
b) Is every subsheaf of Q of the form $Q_{|u}$, for some $u \in \Omega(X)$? □

24.79. Extend 24.49 to any finite product of constant sheaves. □

24.80. Every finitely complete $\mathcal{C}$-presheaf is extensional. □

24.81. Let $R_1, \ldots, R_n$ be a commutative rings with identity such that $\mathcal{M}_\sigma(R_i)$ is an algebraic frame. If $R = \prod_{i=1}^{n} R_i$, then $\mathcal{M}_\sigma(R)$ is a frame [23]. □

24.82. Let $\mathbb{N}$ be the natural numbers and $\mathbb{Z}$ the ring of integers. Let $R = Z^{\mathbb{N}}$ be the ring of sequences of integers, with its natural coordinate-wise structure.
a) If $S \in \mathcal{M}(\mathbb{Z})$ and F is a filter on $\mathbb{N}$, then

$$[F, S] = \{s \in R : \{n \in \mathbb{N} : s(n) \in S\} \in F\}$$

is a multiplicative subset of R, which is saturated if the same is true of S. Moreover

$$F \subseteq G \;\text{ and }\; S \subseteq T \quad\Rightarrow\quad [F, S] \subseteq [G, T].$$

b) Let F be the filter of cofinite subsets of $\mathbb{N}$ and $S = U(\mathbb{Z}) = \{\pm 1\}$. Then, for all $n \in \mathbb{N}$, [24]

$$[F, S](n) \;=\; \{s(n) : s \in [F, S]\} \;=\; \mathbb{Z}.$$

c) Show that the equivalence in 24.58 is false if X is not compact. □

24.83. Show that the usual way of introducing the additive inverse in $\mathbb{N}$ to construct the integers $\mathbb{Z}$, produces, in $\boldsymbol{Sh(X)}$, the constant sheaf $\boldsymbol{\mathbb{Z}}$. Show that the "field of fractions" construction, applied to $\boldsymbol{\mathbb{Z}}$, yields the constant sheaf $\boldsymbol{\mathbb{Q}}$. □

[23] If $R_i = R$, $1 \leq i \leq n$, this is an immediate consequence of 24.56.
[24] Notation as in the proof of 24.56.

Part 5

L-sets

In this Part we begin the study of an abstract version of sheaves and presheaves that originated with [**26**] but developed essentially in [**15**] and [**50**]. The symbol L denotes a semilattice (3.1), H stands for a Heyting algebra (6.1), while Ω will be reserved for a frame (8.1).

Many of the results are generalizations of the ones proven for sheaves and presheaves over topological spaces. However, since in general HAs do not have points, many of the constructions have to be obtained in an intrinsic way. The proofs of a fair amount of the properties of L-sets are straightforward transcriptions of those for presheaves of sets over topological spaces.

CHAPTER 25

L-sets and L-presheaves

We start with a general concept of a set with values in a semilattice, obtained by transcribing to the semilattice context the first two properties in 23.7.(b).

DEFINITION 25.1. *Let L be a semilattice. A* **L-set**, *A, consists of a set, $|A|$, (the domain of A), together with a map*

$$[\![\cdot\, = \,\cdot]\!] : |A| \times |A| \longrightarrow L,$$

satisfying, for all $x, y, z \in |A|$,

[= 1] : $[\![x = y]\!] \;=\; [\![y = x]\!]$;

[= 2] : $[\![x = y]\!] \wedge [\![y = z]\!] \;\leq\; [\![x = z]\!]$.

The map $[\![\cdot = \cdot]\!]$ is the **equality relation on A**. *For $x \in |A|$,*

$$Ex \;\;=_{def}\;\; [\![x = x]\!]$$

is the **extent of x**. *Clearly, $[\![x = y]\!] \leq Ex \wedge Ey$. For $S \subseteq |A|$, the* **support of S** *is*

$$ES = \bigvee\nolimits_{a \in S} Ea,$$

whenever this sup exists in L. When there is need to register explicitly that equality, extent or support refer to A, write $[\![\cdot = \cdot]\!]_A$ and E_A.

We refer to an element in $|A|$ as a **section of A**. *For $p \in L$,*

$$A(p) = \{x \in |A| : Ex = p\},$$

is the set of **sections of A over p**. *A* **global section** *of A is an element of $A(\top)$. We shall assume that whenever L has $\perp$, any L-set has a* **unique** *section over $\perp$, indicated by $*$.*

A L-set A is **extensional** *iff it satisfies*

[ext] *For all $x, y \in |A|$,* $\;\; Ex = Ey = [\![x = y]\!] \;\Rightarrow\; x = y.$

Unless explicit mention to the contrary, all L-sets are assumed to be extensional.

If A is an L-set and $|B| \subseteq |A|$, then $|B|$ is the domain of a L-set, obtained by restricting $[\![\cdot = \cdot]\!]$ to $|B| \times |B|$. Moreover, if A is extensional, the same will be true of B; write $B \subseteq A$ to indicate the L-set structure induced by A on $|B|$.

EXAMPLE 25.2. Presheaves, P, over a topological space X give rise to a $\Omega(X)$-set, also denoted by P, with $|P| = \coprod_{u \in \Omega(X)} P(u)$ and for $s \in P(u)$, $t \in P(v)$,

$$[\![s = t]\!] \;=\; \bigcup \{w \in \Omega(u \cap v) : s_{|w} = t_{|w}\}.$$

By 23.7.(b), this equality relation makes P into a $\Omega(X)$-set. It will be extensional iff P is an extensional presheaf over X. □

EXAMPLE 25.3. If A is a L-set and $p \in L$,

$$|A_{|p}| \;=\; \{x \in |A| : Ex \leq p\}.$$

is the domain of a L-set $A_{|p}$, the **restriction of $\boldsymbol{A}$ to $\boldsymbol{p}$**. Equality in $A_{|p}$ is induced by A, i.e., for $x,\, y \in |A_{|p}|$, $[\![x = y]\!]_{A_{|p}} \;=\; [\![x = y]\!]_A$. Clearly, $A_{|p}$ is extensional whenever the same is true of A. $\square$

EXAMPLE 25.4. Let H be a HA. Define an H-set, $\widetilde{H}$, as follows :

$*\ |\widetilde{H}| = \{\langle a, p\rangle \in H \times H : a \leq p\} = \bigcup_{p \in H} p^{\leftarrow} \times \{p\}$;

$*\ [\![\langle a, p\rangle = \langle b, q\rangle]\!] = p \wedge q \wedge (a \leftrightarrow b)$,

where $\leftrightarrow$ is *equivalence* in H (6.9). By 6.23, this equality satisfies [= 1] and [= 2]. Clearly, $E\langle a, p\rangle = p$, for all $\langle a, p\rangle \in |\widetilde{H}|$. $\widetilde{H}$ corresponds to the sheaf of opens of a topological space X (23.13). To establish that $\widetilde{H}$ is extensional, note that $E\langle a, p\rangle \;=\; E\langle b, q\rangle \;=\; [\![\langle a, p\rangle = \langle b, q\rangle]\!]$ entails $p = q = p \wedge (a \leftrightarrow b)$. Hence, $p \leq (a \leftrightarrow b)$, and 6.10.(a) yields $a = a \wedge p = b \wedge p = b$, as needed. $\square$

EXAMPLE 25.5. Let L be a semilattice. Define a L-set, $\mathbf{1}$, by

$*\ |\mathbf{1}| = L$; $\qquad$ $*\ [\![x = y]\!] = x \wedge y$.

Clearly, $\mathbf{1}$ is an extensional L-set. $\square$

EXAMPLE 25.6. Let R be a commutative ring with identity and let Ω be the frame $Rad(R)$, of radical ideals in R, of Chapter 9, the results of which we shall use freely. Recall that (x) is the principal ideal generated by $x \in R$. Define a Ω-set, $\widetilde{R}$, as follows :

$*\ |\widetilde{R}| \;=\; R$; $\qquad$ $*\ [\![a = b]\!] \;=\; (\sqrt{a} \leftrightarrow \sqrt{b}) \;=\; (\sqrt{a} : \sqrt{b}) \cap (\sqrt{b} : \sqrt{a})$.

Recalling that $\leftrightarrow$ is the equivalence operation in the frame Ω, it follows from 6.23 that $\widetilde{R}$ is a $Rad(R)$-set, in which every section is global.

$\widetilde{R}$ will be extensional iff for all $a,\, b \in R$

$$\sqrt{a} = \sqrt{b} \;\;\Rightarrow\;\; a = b.$$

Since $\sqrt{(a^2)} = \sqrt{a}$ (the set of primes containing a and a^2 are the same), we must have that $a^2 = a$, for all $a \in R$, that is, R *is a Boolean ring*. Conversely, if R is a Boolean ring, then $\sqrt{a} = \sqrt{b} \;\;\Rightarrow\;\; a = b$. Indeed, by 9.12.(b), the hypothesis means that there are $n,\, m \geq 1$ such that

$$a^n = a \in (b) \quad \text{and} \quad b^m = b \in (a).$$

These relations entail the existence of $x,\, y \in R$ such that $a = xb$ and $b = ya$; the first of these equations yields $ab = xb^2 = xb = a$, while the second implies $ab = ya^2 = ya = b$, showing that $a = b$. Hence,

$\widetilde{R}$ is an extensional $Rad(R)$-set **iff** R is a Boolean ring. $\square$

EXAMPLE 25.7. Let R be a commutative regular ring with identity and let $B(R)$ the BA of idempotents in R (19.19). For $a,\, b \in R$, there is a unique $e_{ab} \in B(R)$, such that the principal ideal $(a - b)$ is equal to the principal ideal (e_{ab}). Hence,

$$(a - b)e_{ab} \;=\; (a - b) \;\text{ and }\; e_{ab} \;=\; \alpha_{ab}(a - b), \tag{I}$$

for some $\alpha_{ab} \in R$. Define a $B(R)$-set, $\mathcal{R}$, as follows :

$* \ |\mathcal{R}| \ = \ \{\langle a, e \rangle \in R \times B(R) : ae = a\};$

$* \ [\![\langle a, e \rangle = \langle b, f \rangle]\!] \ = \ (1 - e_{ab})ef.$

Clearly, $E\langle a, e \rangle = e$. One should keep in mind that $e{\cdot}f$ is $e \wedge f$ in $B(R)$. This construction corresponds to the structure sheaf of R over the Boolean space $Spec(R)$ (Example 22.19 and Corollary 22.20). $\mathcal{R}$ is extensional, because

$$E\langle a, e \rangle \ = \ E\langle b, f \rangle \ = \ [\![\langle a, e \rangle = \langle b, f \rangle]\!],$$

implies $e = f = (1 - e_{ab})ef$; hence, $e_{ab}e = 0$, and so

$$a - b \ = \ ae - be \ = \ e(a - b) \ = \ ee_{ab}(a - b) \ = \ 0,$$

verifying that $a = b$, as desired. □

EXAMPLE 25.8. This example is an abstract version of the presheaf $\boldsymbol{fA}$ in 23.16. Let A be a set and let L be a *distributive lattice*, with $\bot$ and $\top$. If $A \xrightarrow{s} L$ is a map, the *support of* s is

$$spt(s) \ = \ \{a \in A : s(a) \neq \bot\}.$$

Now define [1]

$$|\boldsymbol{fA}| \ = \ \left\{ s \in L^A : \begin{array}{c} spt(s) \subseteq_f A \quad \text{and} \quad \forall\, a, b \in A \\ a \neq b \ \Rightarrow \ s(a) \wedge s(b) = \bot. \end{array} \right\}$$

For s, $t \in |\boldsymbol{fA}|$, set

$$[\![s = t]\!] \ = \ \textstyle\bigvee_{a \in A} s(a) \wedge t(a).$$

Note that the join in the definition of $[\![\cdot = \cdot]\!]$ is *finite* because the support of s and t are finite. Clearly, $[\![\cdot = \cdot]\!]$ satisfies [= 1]; for [= 2], we have, using distributivity and the fact that $t(a)$ is disjoint from $t(b)$ for $a \neq b$,

$$\begin{aligned} [\![s = t]\!] \wedge [\![t = z]\!] \ &= \ \textstyle\bigvee_{a \in A} s(a) \wedge t(a) \ \wedge \ \bigvee_{a \in A} t(a) \wedge z(a) \\ &= \ \textstyle\bigvee_{a, b \in A} s(a) \wedge t(a) \wedge t(b) \wedge z(b) \\ &= \ \textstyle\bigvee_{a \in A} s(a) \wedge t(a) \wedge z(a) \\ &\leq \ \textstyle\bigvee_{a \in A} s(a) \wedge z(a) \ = \ [\![s = z]\!]. \end{aligned}$$

Note that $Es = \bigvee_{a \in A} s(a)$. Hence, $\boldsymbol{fA}$ is a L-set, the **bounded constant *A* *L*-set**. To see it is extensional, suppose $Es = Et = [\![s = t]\!]$; for $a \in A$, we have

$$\begin{aligned} s(a) \ &= \ s(a) \wedge Es \ = \ s(a) \wedge [\![s = t]\!] \ = \ s(a) \wedge \textstyle\bigvee_{b \in A} s(b) \wedge t(b) \\ &= \ \textstyle\bigvee_{b \in A} s(a) \wedge s(b) \wedge t(b) \ = \ s(a) \wedge t\,(a), \end{aligned}$$

and so $s(a) \leq t(a)$. Since the argument is symmetrical in s and t, we conclude that $s(a) = t(a)$, as needed.

To describe the sections in $\boldsymbol{fA}$ that correspond to the constant functions, for each $a \in A$, define $\breve{a} : A \longrightarrow L$ by

$$\breve{a}(b) \ = \ \begin{cases} \bot & \text{if } a \neq b; \\ \top & \text{if } b = a. \end{cases}$$

Then, $\breve{a}$ is a global section in $\boldsymbol{fA}$; write $\breve{A} = \{\breve{a} : a \in A\}$. The map

$$a \in A \longmapsto \breve{a} \in \boldsymbol{fA}(\top)$$

is an injection of A into the global sections of $\boldsymbol{fA}$, with image $\breve{A}$.

[1]Recall (page 15) that $\subseteq_f$ means “finite subset of”.

If L is $\Omega(X)$, X a topological space, the construction above is **isomorphic** to that in Example 23.16. We leave details to the reader, but make the following observations :

$*$ If $f : X \longrightarrow A$ is a continuous map with finite image, define

$$\alpha(f) : A \longrightarrow \Omega(X), \text{ given by } \alpha(f)(a) \;=\; [\![f = \widehat{a}]\!].$$

Note that $\alpha(f) \in \boldsymbol{fA}$;

$*$ If $s \in |\boldsymbol{fA}|$, define $\beta(s) : X \longrightarrow A$ by

$$\beta(s)(x) = a \quad \text{iff} \quad x \in s(a).$$

Then, $\beta(s)$ is a continuous map, with finite image in A;

$*$ $\alpha(\beta(s)) = s$ and $\beta(\alpha(f)) = f$.

$*$ α and β are morphisms of $\Omega(X)$-sets [2]. □

EXAMPLE 25.9. This example is the abstract counterpart of the constant sheaf presented in 23.17 and discussed in section 24.5.

Let Ω be a frame and A be a set. Let

$$|\boldsymbol{A}| \;=\; \{s \in \Omega^A : \forall\, a, b \in A,\;\; a \neq b \;\Rightarrow\; s(a) \wedge s(b) = \bot\}.$$

Thus, $|\boldsymbol{A}|$ is analogous to $|\boldsymbol{fA}|$, but without the finiteness assumption on support. For s, $t \in |\boldsymbol{A}|$

$$[\![s = t]\!] \;=\; \bigvee_{a \in A} s(a) \wedge t(a).$$

As in 25.8

$*$ $\boldsymbol{A}$ is an extensional Ω-set, the **constant $\boldsymbol{A}$ $\boldsymbol{\Omega}$-set**;

$*$ The map $\check{\cdot} : A \longrightarrow \check{A} \subseteq \boldsymbol{A}(\top)$ is an injection of A into the set of global sections of $\boldsymbol{A}$.

$*$ As was the case in 25.8, if Ω is the frame of opens of a topological space, then the definition of $\boldsymbol{A}$ given here and that in 23.17 are isomorphic presentations of the same object. □

We now introduce the notion of morphism of L-sets.

DEFINITION 25.10. *Let A, B be L-sets. A morphism, $A \xrightarrow{f} B$, is a set-theoretical map, $f : |A| \longrightarrow |B|$, such that for all x, $y \in |A|$*

[mor 1] : $E_B f x = E_A x$;

[mor 2] : $[\![x = y]\!]_A \leq [\![fx = fy]\!]_B$.

*L-sets and their morphisms form a category, written **Lset**.*

Note that there is no extensionality assumption in 25.10. Moreover, 25.10 is just a rewriting, with L in place of $\Omega(X)$, of item (3) in 23.19 and hence a direct descendant of the corresponding notion for presheaves over a topological space.

25.11. **Final and Initial Object.** The final object in ***Lset*** is the L-set $\mathbf{1}$ of Example 25.5. If A is a L-set, the unique morphism from A to $\mathbf{1}$ is given by $a \mapsto Ea$. The final object in ***Lset*** is the empty L-set, written $\emptyset$. Recall that if L has $\bot$, then $|\emptyset| = \{*\}$, with $E* = \bot$. □

[2]Defined in 25.10.

25.12. **Products.** Let $A_1, \ldots, A_n$ be L-sets. Define a L-set, $\prod_{i=1}^n A_i$, by the following rules :

$* \ |\prod_{i=1}^n A_i| \ = \ \{\langle a_1, \ldots, a_n \rangle \in \prod_{i=1}^n |A_i| : Ea_1 = Ea_2 = \ldots = Ea_n\}$;

$* \ [\![\overline{x} = \overline{y}]\!] \ = \ \bigwedge_{i=1}^n [\![x_i = y_i]\!]$,

where $\overline{x}$, $\overline{y} \in |\prod_{i=1}^n A_i|$. Note the distinction between the domain of the product and the product of the domains. Moreover, $\prod_{i=1}^n A_i$ is extensional if the same is true of each coordinate. There are natural morphisms of L-sets

$$\pi_i : \prod_{i=1}^n A_i \longrightarrow A_i, \ \ \pi_i(\overline{a}) = a_i,$$

the *projections* to the i^{th} coordinate. The family

$$\langle \prod_{i=1}^n A_i, \ \{\pi_i \ : \ 1 \le i \le n\} \rangle$$

is the product of the A_i in the category ***Lset***. If L is *complete* lattice, an analogous construction shows that ***Lset*** has all products. As usual in any category, the **empty product** is the final object **1**. □

Before going on to describe some of the other standard constructions in the category ***Lset***, we set down a vector notation for L-sets, complementing the conventions set down in 1.4.

25.13. **Notation.** Let L be a semilattice and let $A_1, \ldots, A_n$ be L-sets. Write $\overline{x} = \langle x_1, \ldots, x_n \rangle$ for an element of $\prod_{i=1}^n |A_i|$. For $\overline{x}, \overline{y} \in \prod_{i=1}^n |A_i|$, define

$$* \ E\overline{x} \ = \ \bigwedge_{i=1}^n Ex_i; \qquad\qquad * \ [\![\overline{x} = \overline{y}]\!] \ = \ \bigwedge_{i=1}^n [\![x_i = y_i]\!].$$

If *L* is complete, this notation applies to any family of L-sets. □

Lemma 25.14. *a) If $A_1, \ldots, A_n$ are L-sets and $\overline{a}$, $\overline{b}$, $\overline{c} \in \prod_{i=1}^n A_i$, then*

(1) $[\![\overline{a} = \overline{b}]\!] \ \le \ E\overline{a} \wedge E\overline{b}$; (2) $[\![\overline{a} = \overline{b}]\!] \wedge [\![\overline{b} = \overline{c}]\!] \ \le \ [\![\overline{a} = \overline{c}]\!]$.

If L is a complete lattice, the above laws hold for any family of L-sets.

b) Let $A \xrightarrow{f} B \xrightarrow{g} C$ be morphisms of L-sets with $h = g \circ f$. If L is a complete lattice, then for all sets I, $\overline{a} \in |A|^I$ and $\overline{c} \in |C|^I$,

$$[\![h\overline{a} = \overline{c}]\!] \ = \ \bigvee_{\overline{b} \in |B|^I} [\![f\overline{a} = \overline{b}]\!] \wedge [\![g\overline{b} = \overline{c}]\!].$$

Proof. Item (a) is a straightforward consequence that meets – even infinitary –, are associative (Lemma 7.7).

b) Fix $\overline{a} \in |A|^I$ and $\overline{c} \in |C|^I$; since g is a morphism, for each $\overline{b} \in |B|^I$ and $i \in I$, we have

$$[\![fa_i = b_i]\!] \ \le \ [\![gfa_i = gb_i]\!] \ = \ [\![ha_i = gb_i]\!],$$

and so, taking meets over $i \in I$ yields $[\![f\overline{a} = \overline{b}]\!] \ \le \ [\![h\overline{a} = g\overline{b}]\!]$. Thus, if $\overline{b} \in |B|^I$

$$[\![f\overline{a} = \overline{b}]\!] \wedge [\![g\overline{b} = \overline{c}]\!] \ \le \ [\![h\overline{a} = g\overline{b}]\!] \wedge [\![g\overline{b} = \overline{c}]\!] \ \le \ [\![h\overline{a} = \overline{c}]\!],$$

wherefrom it follows that

$$\bigvee_{\overline{b} \in |B|^I} [\![f\overline{a} = \overline{b}]\!] \wedge [\![g\overline{b} = \overline{c}]\!] \ \le \ [\![h\overline{a} = \overline{c}]\!].$$

On the other hand, if we take $\overline{b} = f(\overline{a})$ then

$$\bigvee_{\overline{b} \in |B|^I} [\![f\overline{a} = \overline{b}]\!] \wedge [\![g\overline{b} = \overline{c}]\!] \ \ge \ [\![f\overline{a} = f\overline{a}]\!] \wedge [\![gf\overline{a} = \overline{c}]\!]. \qquad (1)$$

Since g is a morphism, one has

$$[\![f\overline{a} = f\overline{a}]\!] \;=\; Ef\overline{a} \;=\; Egf\overline{a} \;=\; Eh\overline{a}.$$

Thus, item (a) yields

$$[\![f\overline{a} = f\overline{a}]\!] \wedge [\![gf\overline{a} = \overline{c}]\!] \;=\; Eh\overline{a} \wedge [\![h\overline{a} = \overline{c}]\!] \;=\; [\![h\overline{a} = \overline{c}]\!],$$

and (1) implies the desired equality. □

Recall (25.1) that the *support* of a L-set A is $EA \;=\; \bigvee_{a\in|A|} Ea$, whenever this join exists in L. The next result discusses the support of non-empty powers and products [3].

LEMMA 25.15. *Let L be a complete lattice.*

a) A is a L-set and $I \neq \emptyset$ is a set, then

$$EA \;=\; EA^I \;=\bigvee_{\overline{a}\in|A|^I} E\overline{a}.$$

b) If A_i, $i \in I$, are L-sets, then

(1) $E(\Pi_{i\in I}\, A_i) \;\leq\; \bigvee_{\overline{c}\in\Pi|A_i|} E\overline{c} \;\leq\; \bigwedge_{i\in I} EA_i$.

(2) *If L is a frame and I is* finite*, then*

$$\bigvee_{\overline{c}\in\Pi|A_i|} E\overline{c} \;=\; \bigwedge_{i\in I} EA_i.$$

PROOF. a) The diagonal morphism, $\Delta : A \longrightarrow A^I$, $\Delta(a) = \widehat{a}$ [4] and 25.36 entail $EA \leq EA^I$. Since $|A^I| \subseteq |A|^I$, it suffices to check that $\bigvee_{\overline{a}\in|A|^I} E\overline{a} \;\leq\; EA$. But this is clear, because for $\overline{a} \in \prod_{i\in I} |A_i|$ we have, for some fixed $k \in I$,

$$E\overline{a} \;=\; \bigwedge_{i\in I} Ea_i \;\leq\; Ea_k \;\leq\; EA,$$

as needed.

b) The inequality in (1) is clear, recalling that $|\prod_{i\in I} A_i| \subseteq \prod_{i\in I} |A_i|$. If I is finite and L is a frame, distributivity of joins over finite meets (8.4) yields, with

$$\begin{aligned}\bigwedge_{i\in I} EA_i \;&=\; \bigwedge_{i\in I} \bigvee_{a_i\in|A_i|} Ea_i \;=\; \bigvee_{\overline{a}\in\Pi|A_i|} \bigwedge_{i\in I} Ea_i\\ &=\; \bigvee_{\overline{a}\in\Pi|A_i|} E\overline{a}.\end{aligned}$$

ending the proof. □

REMARK 25.16. The reader will easily find examples of Ω-sets A, B such that $|A \times B| = \{*\}$ [5], but $EA \wedge EB \neq \bot$. Hence, the equality between the support of a product and the meet of the supports of its components is false for Ω-sets, even for finite products. For Ω-presheaves, the relation holds for finite products (see 26.13); however, even for presheaves, it is false for arbitrary products, as shown in 26.14. The same example shows that the equality (2) in 25.15.(b) is also false for infinite products. □

25.17. **Fibered product over a map.** Let $f : A \longrightarrow B$ be a morphism in ***Lset***. Define a L-set $A \times_f A$ by the rules :

$*$ $|A \times_f A| = \{\langle x, y\rangle \in A \times A : fx = fy\}$
$\qquad = \{\langle t, z\rangle \in |A| \times |A| : Ex = Et \text{ and } ft = fz\}$;

$*$ Equality is that induced by $A \times A$.

[3] The support of the empty product, $\mathbf{1}$, is $\top$.

[4] The constant I-sequence with entries a.

[5] $*$ is the unique section over $\bot$ as in 25.1.

$A \times_f A$ is extensional, whenever A is extensional. There are natural morphisms, $\rho_1,\ \rho_2 : A \times_f A \longrightarrow A$, given by the restrictions of the coordinate projections of $A \times A$. Clearly, $f \circ \rho_1 = f \circ \rho_2$.

The triple $\langle\, A \times_f A,\ \rho_1,\ \rho_2 \,\rangle$ is the fibered product of A over the morphism f; it is characterized by the fact that the following diagram is a pull-back :

$$\begin{tikzcd} A \times_f A \arrow[r, "\rho_1"] \arrow[d, "\rho_2"'] & A \arrow[d, "f"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

□

25.18. **Equalizers.** Let $f,\ g : A \longrightarrow B$ be a pair of morphisms in L-sets. Define a L-set $Eq(f,g)$, by the rules :

$*$ $|Eq(f,g)| = \{a \in A : fa = ga\}$; $*$ Equality is that induced by A.

There is a natural morphism, $Eq(f,g) \xrightarrow{\iota} A$, whose carrier is the inclusion of $|Eq(f,g)|$ into $|A|$. The pair $\langle\, Eq(f,g),\ \iota \,\rangle$ is the equalizer of $\langle\, f, g \,\rangle$ in ***Lset***. □

25.19. **Coproducts.** Let A_i, $i \in I$, be a family of L-sets and assume that L has $\perp$. Define a L-set, $\coprod_{i\in I} A_i$, by

$*$ $|\coprod_{i\in I} A_i| = \bigcup_{i\in I} |A_i| \times \{i\}$;

$*$ $[\![\langle\, x, i \,\rangle = \langle\, y, j \,\rangle]\!] = \begin{cases} [\![x = y]\!] & \text{if } i = j \\ \perp & \text{if } i \neq j. \end{cases}$

For each $i \in I$, the map $A_i \xrightarrow{\alpha_i} \coprod_{i\in I} A_i$, whose carrier is $a \mapsto \langle\, a, i \,\rangle$, is a L-set morphism. The family $\langle \coprod_{i\in I} A_i,\ \{\alpha_i : i \in I\} \rangle$ is the coproduct of the A_i in ***Lset***. It is straightforward that the **empty coproduct** is the initial object $\emptyset$. □

From 25.12, 25.18, 25.11 and Theorem 16.31, we get

Corollary 25.20. *The category* ***Lset*** *is finitely complete. If L is a complete lattice, then* ***Lset*** *is complete.*

We now to the description of monics and epics n ***Lset***.

Lemma 25.21. *For a morphism in* ***Lset***, $A \xrightarrow{f} B$, *consider the conditions :*

(1) f *is monic in* ***Lset***; (2) f *is an injection of* $|A|$ *into* $|B|$;

(3) *For all* $x,\ y \in |A|$, $[\![x = y]\!]_A = [\![fx = fy]\!]_B$.

Then, (1) $\Leftrightarrow$ (2). *If A is extensional, then* (3) $\Rightarrow$ (2) (*and* (1)).

Proof. Clearly, (2) $\Rightarrow$ (1). For the converse, we use 25.17; if f is monic, then $\rho_1 = \rho_2$. Hence, if $x,\ y \in |A|$ satisfy $fx = fy$, we have $Ex = Ey$ and $\langle\, x, y \,\rangle \in |A \times_f A|$. Consequently,

$$x = \rho_1(\langle x, y \rangle) = \rho_2(\langle x, y \rangle) = y,$$

showing that f is an injection. To complete the proof, it is enough to verify that (3) implies (2), when A is extensional. If x, $y \in |A|$ are such that $fx = fy$, then,

$$[\![x = y]\!] = [\![fx = fy]\!] = Efx = Efy = Ex = Ey,$$

and extensionality yields $x = y$, as needed. □

By Lemma 25.21, the notion of subobject in ***Lset*** is just **subset of the domain, with the induced equality**. Hence, the construction described right after 25.1, written, $A \subseteq B$, yields *all* sub L-sets of B.

The L-set morphism verifying (3) in 25.21 deserve a special name :

Definition 25.22. *A L-set morphism $A \xrightarrow{f} B$ is a* **regular monic** *if for all x, $y \in |A|$ $[\![fx = fy]\!]_B = [\![x = y]\!]_A$.*

Lemma 25.23. *If $f : A \longrightarrow B$ is a morphism of L-sets, the following are equivalent :*

(1) *f is an isomorphism, i.e., there is a morphism $g : B \longrightarrow A$, such that $g \circ f = Id_A$ and $f \circ g = Id_B$.*

(2) *f is bijective and a regular monic* (25.22)*.*

If A is extensional, these conditions are equivalent to

(3) *f is surjective and a regular monic.*

Proof. <u>(1) ⇒ (2) :</u> It is enough to check that $[\![fx = fy]\!]_B \leq [\![x = y]\!]_A$. Since $g \circ f = Id_A$, the needed inequality amounts to

$$[\![gfx = gfy]\!]_A \geq [\![fx = fy]\!]_B,$$

that is a consequence of g being a morphism.
<u>(2) ⇒ (1) :</u> Let g be the (set-theoretical) inverse of f; thus, for all $\langle a, b \rangle \in |A| \times |B|$,

$$f\,a = b \quad \text{iff} \quad gb = a.$$

It is immediate that $Egb = Eb$, for all $b \in |B|$. If $b = fa$ and $b' = fa'$, then

$$[\![b = b']\!]_B = [\![fa = fa']\!]_B = [\![a = a']\!]_A = [\![gb = gb']\!]_A,$$

verifying that g is a morphism, as needed. To end the proof, just note that, in case A is extensional, 25.21 guarantees that a regular monic is injective. □

The description of epics appears in

Lemma 25.24. *Let $f : A \longrightarrow B$ be a morphism of L-sets. Assume that L is a* frame *and consider the following conditions :*

(1) *f is epic in **Lset**;* (2) $\forall\, b \in |B|$, $Eb = \bigvee_{a \in |A|} [\![b = fa]\!]$.

Then, (1) ⇒ (2)*; if B is extensional, these conditions are equivalent.*

Proof. <u>(1) ⇒ (2) :</u> Let $\widetilde{L}$ be the L-set of 25.4. We construct two morphisms h, $k : B \longrightarrow \widetilde{L}$, as follows : for $b \in |B|$

$$\begin{cases} h(b) = \langle Eb, Eb \rangle; \\ k(b) = \langle \bigvee_{a \in |A|} [\![b = fa]\!], Eb \rangle. \end{cases}$$

Note that $Ehb = Ekb = Eb$; to see that k is indeed a morphism, it is enough to show that

$$[\![b = b']\!] \leq [\![kb = kb']\!] = Eb \wedge Eb' \wedge \left(\bigvee_{a\in|A|} [\![b = fa]\!] \leftrightarrow \bigvee_{a\in|A|} [\![b' = fa]\!]\right),$$

or equivalently (6.10.(a)) that

$$[\![b = b']\!] \wedge \bigvee_{a\in|A|} [\![b = fa]\!] = [\![b = b']\!] \wedge \bigvee_{a\in|A|} [\![b = fa]\!]. \tag{I}$$

Since L is a frame, we have

$$\begin{aligned} [\![b = b']\!] \wedge \bigvee_{a\in|A|} [\![b = fa]\!] &= \bigvee_{a\in|A|} [\![b = b']\!] \wedge [\![b = fa]\!] \\ &\leq \bigvee_{a\in|A|} [\![b' = fa]\!]. \end{aligned}$$

By symmetry, we obtain the equality in (I), as needed. The verification that h is also a morphism is straightforward. Now observe that for all $a \in |A|$

$$k(fa) = \langle \bigvee_{a'\in|A|} [\![fa = fa']\!], Efa \rangle = \langle Efa, Efa \rangle = h(fa),$$

and so $h \circ f = k \circ f$. Since f is epic, we conclude that $h = k$, establishing the equality in (2) for all $b \in B$.

(2) $\Rightarrow$ (1) : Assume that B is extensional and let $B \underset{h}{\overset{k}{\rightrightarrows}} C$ be morphisms, such that $f \circ h = f \circ k$. For $b \in |B|$, the transitivity of equality ([= 2] in 25.1) yields, with $a \in |A|$

$$\begin{aligned} [\![hb = kb]\!] &\geq [\![hb = hfa]\!] \wedge [\![hfa = kb]\!] = [\![hb = hfa]\!] \wedge [\![kfa = kb]\!] \\ &\geq [\![b = fa]\!] \wedge [\![fa = b]\!] = [\![b = fa]\!]. \end{aligned} \tag{II}$$

Hence, taking joins over $a \in |A|$ in (II) yields

$$[\![hb = kb]\!] \geq \bigvee_{a\in|A|} [\![b = fa]\!] = Eb,$$

and extensionality entails $hb = kb$, concluding the proof. $\square$

The notions of compatibility and of "gluing" compatible families is at the heart of sheaf theory. We shall discuss these notions in the context of L-sets, indicating the need for L to satisfy a $[\wedge, \bigvee]$- law (8.6) in order to have a smooth theory of sheaves and presheaves.

Let A be L-set, with L a semilattice. How would we define compatibility of sections in $|A|$? Given $x \in |A|$, we think of Ex as the largest element of L where x exists. Equality, is thought of as the largest element of L over which x and y coincide. Thus, two sections x, y of A should be compatible if and only if the value of their equality is the intersection of their extents. Hence,

Definition 25.25. *A set of sections $S \subseteq |A|$ in a L-set A is* **compatible** *iff for all $s, t \in S$ $\quad Es \wedge Et = [\![s = t]\!]$.*

Next, what would be the right concept of "gluing" ? We wish to find, given a compatible $S \subseteq |A|$, a section $t \in |A|$ that "extends" all $s \in S$. Consequently, t must have extent, at least $\bigvee_{s\in S} Es$; in particular, we must assume that this sup exists in L. Secondly, it is expected that in each Es, t and s coincide. Since there is no information about t outside $\bigvee_{s\in S} Es$, we are led to require that t satisfy

$$[\text{gl } \top] \qquad \begin{cases} (i)\ Et = \bigvee_{s \in S} Es \\ (ii) \text{ For all } s \in S, \quad Es = [\![t = s]\!]. \end{cases}$$

Note that if t and t' satisfy (i) and (ii) in [gl $\top$], then, for $s \in S$,

$$[\![t = t']\!] \;\geq\; [\![t = s]\!] \wedge [\![t' = s]\!] \;=\; Es \wedge Es \;=\; Es,$$

and so $[\![t = t']\!] \;\geq\; \bigvee_{s \in S} Es \;=\; Et \;=\; Et'$. Hence, extensionality implies *uniqueness* of a section satisfying [gl $\top$]. All seems to be working fine, even in the general context of semilattices. It is a different matter when we try to localize the above reasoning.

DEFINITION 25.26. *Let L be a semilattice and $p \in L$. A set of sections S in a L-set A is* **compatible over p** *iff for all s, $t \in S$,*

$$p \wedge Es \wedge Es \;=\; p \wedge [\![s = t]\!].\ ^{6}$$

If S is compatible over $p \in L$, one expects to be able to glue, uniquely, the pieces of elements of S that lie over p. To each $s \in S$, there corresponds a piece of s over p − which may not be in $|A|$ −, but whose extent ought to be $p \wedge Es$. Hence, by analogy with [gl $\top$], if $\bigvee_{s \in S} p \wedge Es$ exists in L, we are looking for $t \in |A|$, such that

$$[\text{gl } p]_* \qquad \begin{cases} (i)\ Et = \bigvee_{s \in S} p \wedge Es; \\ (ii) \text{ For all } s \in S, \quad p \wedge Es = p \wedge [\![t = s]\!]. \end{cases}$$

If t and t' satisfy [gl $p]_*$, then, for $s \in S$,

$$[\![t = t']\!] \;\geq\; [\![t = s]\!] \wedge [\![s = t']\!] \;=\; p \wedge Es,$$

and so, $[\![t = 't']\!] \;\geq\; \bigvee_{s \in S} p \wedge Es \;=\; Et \;=\; Et'$, and extensionality again entails the desired uniqueness.

But there is another plausible alternative for the extent condition (i) in [gl $p]_*$: $Et \;=\; p \wedge \bigvee_{s \in S} Es$. After all, we have information to determine t in the intersection of p and $\bigvee_{s \in S} Es$, assuming, of course, that this join exists in L. Since L does not necessarily satisfy the $[\wedge, \bigvee]$-law of 8.6, we may have

$$p \wedge \bigvee_{s \in S} Es \;>\; \bigvee_{s \in S} p \wedge Es,$$

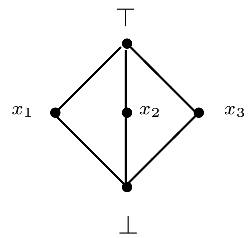
with a (possibly) significant difference between these terms. Requiring that the extent of the gluing be the largest it can be is coded by

$$[\text{gl } p]^* \qquad \begin{cases} (i)\ Et = p \wedge \bigvee_{s \in S} Es; \\ (ii) \text{ For all } s \in S, \quad p \wedge Es = p \wedge [\![t = s]\!]. \end{cases}$$

Perhaps a fruitful analogy is with integration : condition [gl $p]_*$ yields a "lower gluing", while [gl $p]^*$ defines an "upper gluing" of S over p. In general, upper gluings *are not* unique :

EXAMPLE 25.27. Let L be the complete (non-distributive) lattice of Remark 5.2 :

[6] Thus, compatibility (25.25) corresponds to the case $p = \top$ in the present definition.



Let $|C| = \{\langle x, x \rangle : x \in L \text{ and } x \neq x_3\} \cup \{\langle a, x_3 \rangle, \langle b, x_3 \rangle\}$, where a and b are distinct. For $\langle \alpha, p \rangle, \langle \beta, q \rangle \in |C|$,

$$[\![\langle \alpha, p \rangle = \langle \beta, q \rangle]\!] = \begin{cases} \bot & \text{if } p = q = x_3 \text{ and } \alpha \neq \beta \\ p \wedge q & \text{otherwise} \end{cases}$$

A straightforward (and tedious) computation shows that this is an extensional equality on $|C|$, with which it becomes a L-set, C (compare with 25.5). If $x \neq x_3$ in L, write $\widehat{x}$ for $\langle x, x \rangle$. Note that $E\widehat{x} = x$, while $E\langle a, x_3 \rangle = E\langle b, x_3 \rangle = x_3$.

Let $S = \{\widehat{x}_1, \widehat{x}_2\}$; S is compatible in C because

$$E\widehat{x}_1 \wedge E\widehat{x}_2 = \bot = [\![\widehat{x}_1 = \widehat{x}_2]\!].$$

We have $\top = (E\widehat{x}_1 \vee E\widehat{x}_2)$, and $\widehat{\top}$ is the unique section in C that satisfies [gl $\top$] for S. Now, S is compatible over x_3, because

$$x_3 \wedge E\widehat{x}_1 \wedge E\widehat{x}_2 = \bot = x_3 \wedge [\![\widehat{x}_1 = \widehat{x}_2]\!].$$

Observe that $(x_3 \wedge x_1) \vee (x_3 \wedge x_2) = \bot$ and that $\widehat{\bot}$ is the unique element of $|C|$ that satisfies [gl $x_3]_*$ for S. On the other hand,

$$x_3 \wedge (x_2 \vee x_1) = x_3,$$

and there are *two* distinct sections of extent x_3, verifying [gl $x_3]^*$ for S, namely $\langle a, x_3 \rangle$ and $\langle b, x_3 \rangle$. □

If we wish to have gluings that satisfy maximum extent and uniqueness, we are led to work in a context where [gl $p]_*$ and [gl $p]^*$ are equivalent. This happens for $p = \top$, because $\top$ is distributive in L, that is, an element q, verifying, for all $K \subseteq L$,

$$q \wedge \bigvee K = \bigvee_{k \in K} q \wedge k,$$

with the convention that if $\bigvee K$ exists in L, so does $\bigvee_{k \in K} q \wedge k$, and they are equal. Moreover, if we are interested in gluing only *finite* subsets of a L-set, then it is enough that L be a distributive lattice. The route we shall take here is to define gluings when the base algebra is a frame. Nevertheless, it is useful to develop properties for general L-sets, even if sometimes it will be assumed that L is a $[\wedge, \bigvee]$-semilattice, that is,

Definition 25.28. *A semilattice L is a $[\wedge, \bigvee]$-semilattice if for all $S \subseteq L$ and $x \in L$,*

$$x \wedge \bigvee S = \bigvee_{s \in S} x \wedge s,$$

with the usual convention that if one term of this equality exists in L, then so does the other, and they are the same.

The following result gives a characterization of $[\wedge, \bigvee]$-semilattices.

Proposition 25.29. *Let L be a semilattice with $\bot$ and $\top$. The following are equivalent :*

(1) *L is a $[\wedge, \bigvee]$-semilattice.* (2) *L can be regularly embedded in a frame.*

Proof. The definition of regular embedding appears in 7.3. By 8.7, (2) implies (1). For the converse, the reader can verify that Definition 14.1, as well as the proofs of Lemma 14.2 and Theorem 14.4 apply, *verbatim*, to a $[\wedge, \bigvee]$-semilattice. $\square$

The hypothesis of regular in 25.29.(2) cannot be omitted, for the complete lattice L in 25.27 can be embedded, as a semilattice, into a complete Boolean algebra.

By 8.7, HAs and BAs are $[\wedge, \bigvee]$-semilattices; moreover, a complete $[\wedge, \bigvee]$-semilattice is a frame. Thus, there is enough variety in structure to sample important constructions. We formally set down [7]

Definition 25.30. *Let A be an Ω-set, $S \cup \{t\} \subseteq |A|$ and $p \in \Omega$. The section t is a* **gluing of *S* over *p*** *if the following conditions are satisfied*

$$[\text{glu } p] \qquad \begin{cases} (i)\ Et = p \wedge \bigvee_{s \in S} Es; \\ (ii)\ \text{For all } s \in S, \quad p \wedge Es \;=\; [\![t = s]\!], \end{cases}$$

When $p = \top$, t is a **gluing of *S* in *A***.

The next result shows that equality of gluings is determined by the equality of the pieces being glued.

Lemma 25.31. *Let A be a Ω-set and x, y be sections in A, that are gluings over $p \in \Omega$ of S, $T \subseteq |A|$, respectively. Then,*

$$p \wedge [\![x = y]\!] \;=\; p \wedge \bigvee_{\langle s,t\rangle \in S \times T} [\![s = t]\!].$$

Proof. We shall treat the case in which $p = \top$, leaving the straightforward generalization to the reader. By 25.30, we have

$$\begin{array}{ll} (i)\ Ex = \bigvee_{s \in S} Es; & (iii)\ Ey = \bigvee_{t \in t} Et; \\ (ii)\ \forall\, s \in S, \quad Es = [\![x = s]\!]; & (iv)\ \forall\, t \in t, \quad Et = [\![y = t]\!]; \end{array} \qquad (*)$$

The equations in (*), the distributive law in 8.4 and Exercise 25.35 yield

$$\begin{aligned} [\![x = y]\!] &= [\![x = y]\!] \wedge Ex \wedge Ey \;=\; [\![x = y]\!] \wedge \bigvee_{s \in S} Es \wedge \bigvee_{t \in T} Et \\ &= \bigvee_{\langle s,t\rangle \in S \times T} [\![x = y]\!] \wedge Es \wedge Et \\ &= \bigvee_{\langle s,t\rangle \in S \times T} [\![x = y]\!] \wedge [\![x = s]\!] \wedge [\![y = t]\!] \\ &= \bigvee_{\langle s,t\rangle \in S \times T} [\![s = t]\!] \wedge [\![x = s]\!] \wedge [\![y = t]\!] \\ &= \bigvee_{\langle s,t\rangle \in S \times T} [\![s = t]\!] \wedge Es \wedge Et \\ &= \bigvee_{\langle s,t\rangle \in S \times T} [\![s = t]\!], \end{aligned}$$

as desired. $\square$

Another concept that can be conveniently treated when the base is a frame is *denseness*.

[7] Recall our standing convention that L is a semilattice, H is a HA and Ω a frame.

Definition 25.32. *Let A be a Ω-set and S, $T \subseteq |A|$.*

a) S is **dense** *in T if for all $t \in T$,* $Et \;=\; \bigvee_{s\in S} [\![s = t]\!]$.

b) The **density** *of A, $d(A)$, is the least cardinal γ such that there is dense subset of A of cardinal γ. A is* **separable** *if $d(A)$ is at most countable.*

Lemma 25.33. *Let A be a Ω-set and D be a dense set of sections in A. Let B be an* extensional *Ω-set.*
a) The relation of being dense is transitive.

b) If a, $b \in |A|$, then $[\![a = b]\!] \;=\; \bigvee_{d,d'\in D} [\![d = d']\!] \wedge [\![a = d]\!] \wedge [\![b = d']\!]$.

c) If f, $g : A \longrightarrow B$ are morphisms of Ω-sets, then $f_{|D} = g_{|D} \;\Rightarrow\; f = g$.

c) A morphism $A \xrightarrow{f} B$ is epic iff $f(A)$ is dense in B.

Proof. a) If S is dense in T, T is dense in X and $x \in X$, then

$$\begin{aligned} Ex &= \bigvee_{t\in T} [\![x = t]\!] \;=\; \bigvee_{t\in T} [\![x = t]\!] \wedge Et \\ &= \bigvee_{t\in T} [\![x = t]\!] \wedge \big(\bigvee_{s\in S} [\![t = s]\!]\big) \\ &= \bigvee_{s\in S} \bigvee_{t\in T} [\![x = t]\!] \wedge [\![t = s]\!] \\ &\leq \bigvee_{s\in S} [\![x = s]\!] \;\leq\; Ex, \end{aligned}$$

establishing that S is dense in X.

b) As in the proof of 25.31, if a, $b \in |A|$, then 25.35 yields

$$\begin{aligned} [\![a = b]\!] &= [\![a = b]\!] \wedge Ea \wedge Eb \\ &= [\![a = b]\!] \wedge \bigvee_{d\in D} [\![a = d]\!] \wedge \bigvee_{d'\in D} [\![b = d']\!] \\ &= \bigvee_{d,d'\in D} [\![a = b]\!] \wedge [\![a = d]\!] \wedge [\![b = d']\!] \\ &= \bigvee_{d,d'\in D} [\![d = d']\!] \wedge [\![a = d]\!] \wedge [\![b = d']\!], \end{aligned}$$

as needed.

c) For $a \in A$ and $d \in D$, we have

$$\begin{aligned} [\![fa = ga]\!] &\geq [\![fa = fd]\!] \wedge [\![fd = ga]\!] \;=\; [\![fa = fd]\!] \wedge [\![gd = ga]\!] \\ &\geq [\![a = d]\!], \end{aligned}$$

and so taking joins over $d \in D$, we arrive at $[\![fa = ga]\!] \geq Ea$. Since B is extensional, we conclude that $fa = ga$, as desired. Item (c) is just a rephrasing of the equivalence in 25.24. □

We end this section with the concept of *finite completeness.*

Definition 25.34. *Let L be a distributive lattice. A L-set A is* **finitely complete (fc)** *if for all $p \in L$ and $S \subseteq_f |A|$, if S is compatible over p, then there is* a unique *$t \in |A|$ such that*

$$Et = p \wedge \bigvee_{s\in S} E\,s \quad \text{and} \quad p \wedge Es = [\![t = s]\!],\ \forall\ s \in S.$$

Exercises

25.35. If A is a L-set and $a, b, x, y \in |A|$, then the following **exchange rules** hold true :

a) $[\![a = b]\!] \wedge [\![a = x]\!] \wedge [\![b = y]\!] \;=\; [\![x = y]\!] \wedge [\![a = x]\!] \wedge [\![b = y]\!]$.

b) $[\![x = b]\!] \wedge [\![x = a]\!] \;=\; [\![a = b]\!] \wedge [\![x = a]\!]$. $\square$

25.36. If A, B are L-sets such that $EA, EB \in L$ (25.1) and there is a morphism from A to B, then $EA \leq EB$. $\square$

25.37. If L is a semilattice, $p \in L$ and I is a finite set,

$$(\mathbf{1}_{|p})^I \approx \mathbf{1}_{|p}.$$

The same is true for any set I whenever L is complete. $\square$

25.38. Let A, B be Ω-sets, $S \subseteq |A|$ and $p \in \Omega$. Let $f \;:\; A \longrightarrow B$ be a morphism of Ω-sets.

a) If S has a gluing over p, then S is compatible over p.

b) If A is extensional, then gluings are unique *(whenever they exist)*.

c) If S is compatible over p in A, then $f(S) = \{fs : s \in S\}$ is compatible over p in B.

d) If t is a gluing of S over p in A, then ft is a gluing of $f(S)$ over p in B. $\square$

25.39. a) $\widetilde{H}(\top)$ is dense in $\widetilde{H}$ (25.4).

b) $\check{A}$ is dense in $\boldsymbol{fA}$ (25.8) and in $\boldsymbol{A}$ (25.9).

c) There are separable Ω-sets with domain of arbitrarily large cardinality. $\square$

25.40. a) $\boldsymbol{fA}$ is finitely complete.

b) If H is a HA, $\widetilde{H}$ (25.4) is finitely complete.

c) Show that the categorical constructions discussed in this Chapter lead from fc-sets to fc-sets.

d) Determine which of the Examples of this Chapter are finitely complete. $\square$

CHAPTER 26

Presheaves over a Semilattice

As was the case with L-sets, the definition of a presheaf over a semilattice is obtained by specifying [rest 1] through [rest 3] in 23.7.(a) as axioms :

DEFINITION 26.1. *Let L be a semilattice.*
a) A **presheaf** *P over L* [1] *is a set $|P|$* (the domain of P)*, together with maps*

$$E : |P| \longrightarrow L \text{ (extent)} \qquad \textit{and} \qquad \begin{cases} {}_| : |P| \times L \longrightarrow |P| \\ \langle a, p \rangle \mapsto a_{|p} \end{cases} \text{(restriction)}$$

satisfying, for all $a \in |P|$, p, $q \in L$

$$\text{[rest 1]} : a_{|Ea} \;=\; a; \qquad \text{[rest 2]} : Ea_{|p} \;=\; Ea \wedge p;$$

$$\text{[rest 3]} : (a_{|p})_{|q} \;=\; a_{|p\wedge q}.$$

b) For $q \in L$, $P(q) = \{a \in |P| : Ea = q\}$ is the **set of sections of *P* over *q*;** *an element of $P(\top)$ is called a* **global section** *of P.*

Whenever L has $\perp$, we assume that any presheaf over L has a **unique** *section over $\perp$, written $*$* (as for L-sets).

c) For $p \in L$ and $S \subseteq |P|$,

$*$ $ES = \bigvee_{a\in A} Ea$ is the **support of *S* in *P***, *whenever this join exists in L;*

$*$ S is **compatible over *p*** *iff for all x, $y \in S$, $x_{|p\wedge Ex\wedge Ey} \;=\; y_{|p\wedge Ex\wedge Ey}$.*

$*$ S is **compatible** *if $x_{|Ex\wedge Ey} \;=\; y_{|Ex\wedge Ey}$, for all x, $y \in S$.*

d) If $S \subseteq |P|$ and $t \in |P|$, t is a **gluing of *S* in *P*** *if*

$$Et \;=\; \bigvee_{s\in S} Es \quad \textit{and} \quad t_{|Es} \;=\; s, \forall\, s \in S.$$

e) A presheaf P over L is **extensional** *if for all x, $y \in |P|$ and $D \subseteq L$*

[ext] $\qquad \forall\, p \in D,\; x_{|p} = y_{|p}$ *and* $Ex = Ey = \bigvee D \qquad \Rightarrow \qquad x = y.$

f) If P and Q are L-presheaves, a map $f : |P| \longrightarrow |Q|$ is a **morphism** *iff for all $x \in |P|$ and $p \in L$*

$$\text{[pmor 1]} : Efx = Ex \qquad\qquad \text{[pmor 2]} : f(x_{|p}) \;=\; (fx)_{|p}.$$

Write ***pSh(L)*** *for the category of L-presheaves and their morphisms.*

---

[1] Or L-presheaf.

As for L-sets (25.36), the support of the source of a morphism is less than or equal to the support of its target.

By Proposition 23.7, every presheaf over a topological space X gives rise to a $\Omega(X)$-presheaf, with same domain and restriction, which is extensional iff the original presheaf is extensional. Moreover, by 23.19, a morphism of presheaves over X gives rise to a morphism of $\Omega(X)$-presheaves. In fact, we have

LEMMA 26.2. *The categories* $\boldsymbol{pSh(X)}$ *and* $\boldsymbol{pSh(\Omega(X))}$ *are* **isomorphic***, the same being true of the corresponding extensional subcategories.*

PROOF. We only comment in how a $\Omega(X)$-presheaf P gives rise to a presheaf over X. If $u \leq v$, the restriction map of P yields a map

$$p_{vu} : P(u) \longrightarrow P(v), \quad p_{vu}(s) = s_{|v},$$

which satisfies $p_{uu} = Id_{P(u)}$ (by [rest 1]) and $p_{vu} \circ p_{wv} = p_{wu}$, if $u \leq v \leq w$ (by [rest 3]); that morphisms correspond bijectively follows from 23.19.(c). $\square$

Because of 26.2*, we shall not distinguish between the categories of* $\Omega(X)$*-presheaves and presheaves over* X*.*

Exercise 26.23 collects some of basic properties of L-presheaves and their morphisms, entirely analogous to the ones for presheaves over topological spaces. The concept of $[\wedge, \bigvee]$-semilattice is defined in 25.28.

EXAMPLE 26.3. Let $2 = \{\bot, \top\}$; a 2-presheaf can be naturally identified with its set of global sections; and morphisms identified with the induced map on global sections. Hence, the categories $\boldsymbol{pSh(2)}$ and $\boldsymbol{Set}$ are naturally isomorphic and Sheaf Theory can be considered as a generalization of Set Theory. $\square$

EXAMPLE 26.4. Let A be a set and L a semilattice. We construct a L-presheaf, also written A, as follows :

(i) $|A| = \bigcup_{\bot \neq p \in L} A \times \{p\} \cup \{\langle *, \bot \rangle\}$; *(ii)* $E\langle x, p \rangle = p$;

(iii) $\langle x, p \rangle_{|q} = \langle x, p \wedge q \rangle$.

A is the **constant presheaf** on L. If $f : A \longrightarrow B$ is a map, f induces a morphism of L-presheaves,

$$\langle x, p \rangle \longmapsto \begin{cases} \langle fx, p \rangle & \text{if } p \neq \bot \\ \langle *, \bot \rangle & \text{if } p = \bot, \end{cases}$$

still indicated by f. It is straightforward that these definitions yield a functor from $\boldsymbol{Set}$ to $\boldsymbol{pSh(L)}$, the *constant presheaf functor*. $\square$

EXAMPLE 26.5. Let $L \subseteq R$ be semilattices and let A be a R-presheaf. We define a L-presheaf, $A_{|L}$, the **restriction of $\boldsymbol{A}$ to $\boldsymbol{L}$**, by the following rules :

i) $|A_{|L}| = \coprod_{p \in R} A(p)$;

ii) Extent and restriction are those induced by A.

Note that for p, $q \in L$ and $x \in A(p)$, we have

$$x_{|q} = x_{|p \wedge q} \in A(p \wedge q) \subseteq |A_{|L}|,$$

and $A_{|L}$ is indeed a L-presheaf. Clearly, $A_{|L}$ is extensional whenever the same is true of A. If $L = p^{\leftarrow}$, $p \in R$, write $A_{|p}$ for the restriction of A to $p^{\leftarrow}$. Note that

$$|A_{|p}| = \{x \in |A| : Ex \leq p\},$$

as was the case for L-sets in 25.3. There is a natural **injection**

$$\mathfrak{r}_A : A_{|L} \longrightarrow A, \text{ given by } x \in |A_{|L}| \longmapsto x \in |A|,$$

such that for all $p \in L$ and $x \in |A_{|L}|$, $\mathfrak{r}_A(x_{|p}) = (\mathfrak{r}_A x)_{|p}$. □

Before giving more examples, we establish a connection between L-presheaves and L-sets, starting with the following

DEFINITION 26.6. *Let L be a semilattice and let A be a L-set which is also a L-presheaf. The equality and restriction in A are* **compatible** *if for all x, $y \in |A|$ and all p, $q \in L$*

$$(i)\ Ex = [\![x = x]\!]; \qquad (ii)\ [\![x_{|p} = y_{|q}]\!] = p \wedge q \wedge [\![x = y]\!],$$

where in (i), Ex is the extent map that comes with the L-presheaf structure of A.

LEMMA 26.7. *Let L be a semilattice and let A be a L-presheaf with a compatible structure of L-set.*
a) For $x \in |A|$ and $p \in L$, $Ex_{|p} = [\![x_{|p} = x]\!] = p \wedge Ex$. If A is an extensional L-set, then $x_{|p}$ is the **unique** *element $t \in |A|$ satisfying $Et = [\![t = x]\!] = p \wedge Ex$.*
b) If A is an extensional L-set, then A is an extensional presheaf.
c) Let B be an extensional L-set with a compatible structure of L-presheaf. Then every L-set morphism from A to B is a presheaf morphism.

PROOF. a) The definition of L-presheaf entails $Ex_{|p} = p \wedge Ex$; on the other hand, (i) and (ii) in 26.6 yield $[\![x_{|p} = x]\!] = p \wedge Ex$, as desired. The remaining statement is clear.
b) Suppose $x, y \in |A|$ and there is $D \subseteq L$ such that $\bigvee D = Ex = Ey$ and $x_{|p} = y_{|p}$, for all $p \in D$. Transitivity of equality and (a) yield, for $p \in D$,

$$\begin{aligned}[\![x = y]\!] &\geq [\![x = x_{|p}]\!] \wedge [\![x_{|p} = y]\!] = p \wedge Ex \wedge [\![y_{|p} = y]\!] \\ &= p \wedge Ex \wedge Ey = p,\end{aligned}$$

and so $[\![x = y]\!] \geq \bigvee D = Ex = Ey$. Hence, extensionality entails $x = y$, as needed.
c) It must be shown that if $x \in |A|$ and $p \in L$, then $f(x_{|p}) = (fx)_{|p}$. Since f is a morphism, item (a) yields

$$\begin{cases} *\ \ Ef(x_{|p}) = Ex_{|p} = p \wedge Ex = E(fx)_{|p}; \\ *\ p \wedge Ex = [\![x_{|p} = x]\!] \leq [\![f(x_{|p}) = fx]\!] \leq Ef(x_{|p}) = p \wedge Ex, \end{cases}$$

and the extensionality of B implies $f(x_{|p}) = (fx)_{|p}$, as desired. □

The next result guarantees a large supply of presheaves with a compatible equality.

Theorem 26.8. *Let P be a presheaf over a subsemilattice L of a frame Ω. For $x,\, y \in |P|$, set*

[equu] $$[\![x\,=\,y]\!] \;=\; \bigvee\,\{p \in L : p \le Ex \wedge Ey \quad and \quad x_{|p} = y_{|p}\},$$

where this join is computed **in $\mathbf{\Omega}$**. *Then, for $x,\, y \in |A|$ and $p,\, q \in L$*

a) $[\![x\,=\,y]\!]$ is an equality in $|P|$, with which it becomes a Ω-set, called the **$\mathbf{\Omega}$-set associated to the presheaf $\boldsymbol{P}$**, *still indicated by P. Furthermore, P is an extensional Ω-set iff P is an extensional presheaf over L.*

b) $[\![x_{|p}\,=\,y_{|q}]\!] \;=\; p \wedge q \wedge [\![x\,=\,y]\!]$.

c) $Ex_{|p} \;=\; [\![x_{|p}\,=\,x]\!] \;=\; p \wedge Ex$. *If P is extensional, $x_{|p}$ is the* **unique** *t in $|P|$ such that $Et = [\![t\,=\,x]\!] = p \wedge Ex$.*

d) If P is extensional, then

(1) $x_{|p} = y_{|p}$ *iff* $p \;\le\; (Ex \vee Ey) \to [\![x\,=\,y]\!]$ [2].

(2) *If $p \le Ex \wedge Ey$, then* $x_{|p} = y_{|p}$ *iff* $p \le [\![x\,=\,y]\!]$.

(3) *If $[\![x\,=\,y]\!] \in L$, then $x_{|[\![x=y]\!]} = y_{|[\![x=y]\!]}$.*

e) Let P and Q be presheaves over L and let $f : |P| \longrightarrow |Q|$ be a map. Consider the following conditions :

(1) *f is a presheaf morphism.* (2) *f is a Ω-set morphism.*

Then, (1) $\Rightarrow$ (2) *and they are equivalent if Q is extensional.*

Proof. Throughout the proof, write

$$E_{xy} = \{p \in L : p \le Ex \wedge Ey \quad \text{and} \quad x_{|p} = y_{|p}\},$$

where $x,\, y \in |P|$. Hence, $[\![x\,=\,y]\!] = \bigvee E_{xy}$. For $q \in L$, let

$$q \wedge E_{xy} = \{q \wedge p : p \in E_{xy}\}.$$

a) It is clear that $[\![x\,=\,x]\!] = Ex$ and that $[\![x\,=\,y]\!] = [\![y\,=\,x]\!]$. For transitivity, if $p \in E_{xy}$ and $q \in E_{yz}$, note that $(p \wedge q) \le Ex \wedge Ez$ and

$$x_{|p\wedge q} \;=\; (x_{|p})_{|q} \;=\; (y_{|p})_{|q} \;=\; y_{|p\wedge q},$$

that is, $p \wedge q \in E_{xz}$. Hence,

$$\{p \wedge q : p \in E_{xy} \text{ and } q \in E_{yz}\} \subseteq E_{xz}. \tag{*}$$

Now, (*) and 8.4 immediately imply $[\![x\,=\,y]\!] \wedge [\![y\,=\,z]\!] \le [\![x\,=\,z]\!]$, as desired. It is straightforward that P is an extensional Ω-set iff it is an extensional L-presheaf.

b) Write $s = x_{|p}$ and $t = y_{|q}$; since $Es = p \wedge Ex$ and $Et = q \wedge Ey$, if $r \in E_{st}$, then

$$\begin{cases} Er \le p \wedge q \wedge Ex \wedge Ey \quad \text{and} \\ x_{|r} = x_{|p\wedge r} = (x_{|p})_{|r} = s_{|r} = t_{|r} = (y_{|q})_{|r} = y_{|q\wedge r} = y_{|r}, \end{cases}$$

wherefrom we conclude that $r \in E_{xy}$. Consequently,

$$[\![x_{|p}\,=\,y_{|q}]\!] \;\le\; p \wedge q \wedge [\![x\,=\,y]\!].$$

[2] $\vee$ and $\to$ are join and implication in Ω.

To prove the reverse inequality, it is enough to verify, because Ω is a frame, that if $u \in E_{xy}$, then $p \wedge q \wedge u \in E_{st}$. We have,

$$\begin{aligned} s_{|p\wedge q\wedge u} &= (x_{|p})_{|p\wedge q\wedge u} = x_{|p\wedge q\wedge u} = (x_{|u})_{|p\wedge q} = (y_{|u})_{|p\wedge q} = y_{|p\wedge q\wedge u} \\ &= (y_{|q})_{|p\wedge q\wedge u} = t_{|p\wedge q\wedge u}, \end{aligned}$$

completing the proof of (b); item (c) follows from (b), as in 26.7.(a).

d) Suppose that $p \leq (Ex \vee Ey) \to [\![x = y]\!]$, that is (by $[\to]$ in 6.1)

$$p \wedge (Ex \vee Ey) \;\leq\; [\![x = y]\!] \;\leq\; Ex \wedge Ey.$$

The above inequalities entail $p \wedge Ex = p \wedge Ey = p \wedge Ex \wedge Ey$ and hence,

$$p \wedge Ex \;=\; Ex_{|p} \;=\; p \wedge Ey \;=\; Ey_{|p} \;=\; p \wedge [\![x = y]\!].$$

Consider the set $D \;=\; p \wedge E_{xy}$; Ω being a frame, we have

$$\bigvee D \;=\; p \wedge \bigvee E_{xy} \;=\; p \wedge [\![x = y]\!] \;=\; p \wedge Ex \;=\; p \wedge Ey.$$

Since L is a subsemilattice of Ω, $p \wedge E_{xy} \subseteq L$ and $p \wedge Ex \in L$, it follows that $\bigvee D$ exists in L and is equal to $p \wedge Ex = p \wedge Ey$. Because $x_{|q} = y_{|q}$, for all $q \in E_{xy}$, it follows that for all $r = (p \wedge q)$ in D, we have

$$x_{|r} \;=\; x_{|p\wedge q} \;=\; (x_{|q})_{|p} \;=\; (y_{|q})_{|p} \;=\; y_{|p\wedge q} \;=\; y_{|r},$$

and extensionality implies $x_{|p} = y_{|p}$. Conversely, suppose that $x_{|p} = y_{|p}$. By item (b),

$$Ex_{|p} \;=\; p \wedge Ex \;=\; p \wedge Ey \;=\; Ey_{|p} \;=\; [\![x_{|p} = y_{|p}]\!] \;=\; p \wedge [\![x = y]\!].$$

Hence, $p \wedge (Ex \vee Ey) \;=\; p \wedge Ex \;=\; p \wedge [\![x = y]\!] \;\leq\; [\![x = y]\!]$, as needed. Items (2) and (3) in (d) follow straightforwardly from what has already been proven.

e) Let $f : |P| \longrightarrow |Q|$ be a map and assume that f is the carrier of a presheaf morphism. For $x,\, y \in |P|$, if $p \in E_{xy}$, [pmor 2] implies that $p \in E_{fx,fy}$; hence, $[\![x = y]\!] \;\leq\; [\![fx = fy]\!]$. Since extent is the same for a presheaf over L and for the associated Ω-set, we conclude that f is the carrier of a morphism of Ω-sets. The proof of the converse is analogous to that of 26.7.(c). $\square$

Remark 26.9. If L is a subsemilattice of a frame and A is a L-presheaf, A will always be considered as an Ω-set with the equality in 26.8 (the Ω-set associated to A). In particular, every Ω-presheaf gives rise to a Ω-set, indicated by the same symbol. Without explicit mention to the contrary, all L-set concepts used for a L-presheaf A, refer to the equality in 26.8. $\square$

Example 26.10. All the examples in Chapter 25, except 25.3 and 25.6, can be given a compatible structure of extensional presheaf, as follows (numbering refers to the original example) :

25.4. For $\langle a, p \rangle \in |\widetilde{H}|$ and $q \in H$, the rules

$$E\langle a, p \rangle = p \quad\text{and}\quad \langle a, p \rangle_{|q} = \langle a \wedge q,\; p \wedge q \rangle,$$

make $|\widetilde{H}|$ a presheaf over H. Now suppose that $\langle a, p \rangle$ and $\langle b, p \rangle$ are sections in $\widetilde{H}$, such that there is $p_i \in H$ satisfying $p = \bigvee_{i\in I} p_i$ and

$$\langle a, p \rangle_{|p_i} = \langle a \wedge p_i,\ p \wedge p_i \rangle = \langle b \wedge p_i,\ p \wedge p_i \rangle.$$

Then, for all $i \in I$, $a \wedge p_i = b \wedge p_i$, that is, $p_i \leq (a \leftrightarrow b)$. Hence, $p \leq (a \leftrightarrow b)$, and so, $a = a \wedge p = b \wedge p = b$, showing that, as a presheaf, $\widetilde{H}$ satisfies [ext].

25.5. For $p, q \in L$, set $Ep = p$ and $p_{|q} = p \wedge q$; extensionality is clear.

25.7. Let R be a commutative regular ring and let $B(R)$ the BA of idempotents in R. For $\langle a, e \rangle \in |cR|$ and $f \in B(R)$, set

$$E\langle a, e \rangle = e \quad \text{and} \quad \langle a, e \rangle_{|f} = \langle af, ef \rangle.$$

Then, $\mathcal{R}$ is a $B(R)$-presheaf. To see that it is extensional, assume that $\langle a, e \rangle$, $\langle b, e \rangle \in |\mathcal{R}|$ are such that there are $f_i \in B(R)$, with

$$\bigvee_{i \in I} f_i = e \quad \text{and} \quad f_i(a - b) = 0, \quad \text{for all } i \in I.$$

It follows from (I) in 25.7, that $e_{ab} f_i = 0$, for all $i \in I$. Since $B(R)$ is a Boolean algebra, it is a $[\wedge, \bigvee]$-lattice (8.7) and so,

$$e_{ab} e \;=\; e_{ab} \wedge \bigvee_{i \in I} f_i \;=\; \bigvee_{i \in I} e_{ab} f_i \;=\; 0.$$

But then, again by (I) in 25.7, we get

$$a - b \;=\; e_{ab}(a - b) \;=\; e_{ab}(ae - be) \;=\; e_{ab} e(a - b) \;=\; 0,$$

as needed to show that $\mathcal{R}$ is an extensional presheaf.

25.8. For $s \in |\boldsymbol{fA}|$ and $p \in L$, set $Es = \bigvee_{a \in A} s(a)$. Now define

$$s_{|p} : A \longrightarrow L \quad \text{by} \quad s_{|p}(a) = p \wedge s(a).$$

It is clear that $spt(s_{|p}) \subseteq spt(s)$, being therefore finite in A. These definitions make $\boldsymbol{fA}$ into an extensional L-presheaf. The proof of extensionality is essentially the same as that given in 25.8. The same technique will show that the constant Ω-set $\boldsymbol{A}$ is a Ω-presheaf.

25.27. Recall that $|A| = \{\widehat{x}_1, \widehat{x}_2, \widehat{\top}, \widehat{\bot}, \langle a, x_3 \rangle, \langle b, x_3 \rangle\}$. Define

$*$ $E\widehat{x} = x$, if $x \neq x_3$, and $E\langle a, x_3 \rangle = E\langle b, x_3 \rangle = x_3$;

$$* \ \langle \alpha, p \rangle_{|q} = \begin{cases} \langle \alpha, p \rangle & \text{if } p \leq q \\ \widehat{\bot} & \text{if } p \wedge q = \bot \\ \widehat{x} & \text{if } \alpha = p = \top \text{ and } q \neq x_3 \\ \langle a, x_3 \rangle & \text{if } \alpha = p = \top \text{ and } q = x_3, \end{cases}$$

yielding extent and restriction that verify the conditions in 26.1. Since $\top$ is the only element of L with a non-trivial cover, and $A(\top)$ is a singleton, A is extensional. $\square$

To describe some of the usual categorical constructions in $\boldsymbol{pSh(L)}$, we introduce a vector notation, complementing 25.13 :

26.11. **Notation.** Let $A_1, \ldots, A_n$ be L-presheaves. If $\overline{x} \in \prod_{i=1}^{n} |A_i|$ and $p \in L$, set

$$* \ E\overline{x} = \bigwedge_{i=1}^{n} Ex_i; \qquad\qquad * \ \overline{x}_{|p} = \langle x_{i_{|p}} \rangle.$$

If L is complete, this notation applies to any family of presheaves over L. $\square$

26.12. **Products.** Let A_i, $i \in I$, be L-presheaves. Let

$$|\prod_{i\in I} A_i| \;=\; \{\overline{x} \in \prod_{i\in I} |A_i| : \forall\, i, j \in I, \;\; Ex_i = Ex_j\}.$$

Define extent and restriction in $\prod_{i\in I} A_i$ by

$$\overline{x} \longmapsto E\overline{x} \quad \text{and} \quad \langle\, \overline{x}, p \,\rangle \longmapsto \overline{x}_{|p},$$

where $p \in L$. Then, $\prod_{i\in I} A_i$ is a L-presheaf, which is extensional iff the same is true of each coordinate. There are maps

$$\pi_i : \prod_{i\in I} A_i \longrightarrow A_i, \quad \pi_i(\overline{x}) = x_i,$$

which are morphisms of presheaves, called the projections on the i^{th} coordinate. The family $\langle\, \prod_{i\in I} A_i,\ \{\pi_i \,:\, i \,\in\, I\}\,\rangle$ is the product of the A_i in $\boldsymbol{pSh(L)}$. $\square$

Lemma 26.13. *If $A_1, \ldots, A_n$ are Ω-presheaves then*

$$E(\Pi_{i=1}^n A_n) \;=\; \bigvee_{\overline{a}\in\Pi|A_i|} E\overline{a} \;=\; \bigwedge_{i=1}^n EA_i.$$

Proof. The second equality follows from 25.15.(b).(2). Set $P = \prod_{i\in I} A_i$. Note that for $\overline{a} \in \prod_{i\in I} |A_i|$, 26.8.(c) yields

$$E\overline{a} \;=\; E\overline{a}_{|E\overline{a}} \quad \text{and} \quad \overline{a}_{|E\overline{a}} \in |P|.$$

Thus, $EP \geq \bigvee_{\overline{a}\in\Pi|A_i|} E\overline{a}$, and so equality follows from (1) and (2) in 25.15.(b). $\square$

Example 26.14. We shall construct a sequence A_n of presheaves, such that $EA_n = \top$ but $E\left(\prod_{n\geq 1} A_n\right) = \bot$. Let B be the BA of clopens in the Cantor space $2^{\mathbb{N}}$. As in Chapter 18, for $n \geq 1$ and $s \in 2^n$, $V_s \;=\; \{t \in 2^{\mathbb{N}} : t_{|n} = s\}$. Moreover, for each $n \geq 1$, $\bigcup_{s\in 2^n} V_s \;=\; 2^{\mathbb{N}}$, the top of the BA B. Let A be any B-presheaf with $EA = \top = 2^{\mathbb{N}}$. Define, for $n \geq 1$

$$A_n = \coprod_{s\in 2^n} A_{|V_s},$$

with restriction as in 26.5. It is straightforward that $EA_n = 2^{\mathbb{N}}$, for all $n \geq 1$. On the other hand, the domain of the product of the A_n contains only the section over $\bot = \emptyset$, because the intersection of any infinite collection of the V_s is the bottom element of B. Thus, in this case

$$\bigwedge_{n\geq 1} EA_n = \top = 2^{\mathbb{N}} \quad \text{and} \quad E\left(\prod_{n\geq 1} A_n\right) = \bot = \emptyset,$$

as asserted. If $\overline{a} \in \prod_{n\geq 1} |A_n|$, $E\overline{a} \;=\; \bot$, and so $\bigvee_{\overline{a}\in\Pi|A_n|} E\overline{a} \;=\; \bot$, showing that the equality in 25.15.(b).(2) may fail. $\square$

26.15. **Fibered product over a map.** Let $f : A \longrightarrow B$ be a morphism in $\boldsymbol{pSh(L)}$. Define a L-presheaf $A \times_f A$ by the rules :

$*$ $|A \times_f A| = \{\langle\, x, y \,\rangle \in A \times A : fx = fy\}$
$\qquad = \{\langle\, t, z \,\rangle \in |A| \times |A| : Ex = Et \text{ and } ft = fz\}$;

$*$ Extent and restriction are induced by $A \times A$.

$A \times_f A$ inherits extensionality from A. There are natural morphisms, $\rho_1, \rho_2 : A \times_f A \longrightarrow A$, the restrictions of the coordinate projections of $A \times A$ to $A \times_f A$. The triple $\langle\, A \times_f A,\ \rho_1,\ \rho_2 \,\rangle$ is the **fibered product of $\boldsymbol{A}$ over $\boldsymbol{f}$**; as noted in 25.17, it is a pull-back in $\boldsymbol{pSh(L)}$ and $f \circ \rho_1 = f \circ \rho_2$. $\square$

The next result should be compared with 25.23.

LEMMA 26.16. *If $f : A \longrightarrow B$ is a morphism of L-presheaves, the following are equivalent :*

(1) *f is an isomorphism, i.e., there is a morphism, $g : B \longrightarrow A$, such that $f \circ g = Id_B$ and $g \circ f = Id_A$;*

(2) *f is bijective.*

PROOF. It is enough to check that (2) $\Rightarrow$ (1). Let g be the (set-theoretical) inverse of f. Since

$$\forall \langle a, b \rangle \in |A| \times |B|, \quad fa = b \quad \text{iff} \quad gb = a,$$

it is clear that $Egb = Eb$, $\forall\, b \in |B|$. To show that g is a morphism of presheaves, let $b \in |B|$ and $p \in L$; then $f((gx)_{|p}) = fg(x)_{|p} = x_{|p}$, and so $(gx)_{|p} = g(x_{|p})$, as needed. $\square$

LEMMA 26.17. *Let $A \xrightarrow{f} B$ be a morphism in $\boldsymbol{pSh(L)}$.*

a) The following are equivalent :

(1) *f is monic in $\boldsymbol{pSh(L)}$;* (2) *f is injective from $|A|$ into $|B|$.*

b) If A has a compatible structure of L-set (26.6)*, with which it is extensional, then the conditions in (a) are equivalent to*

(3) *f is a regular monic* (25.22)*.*

PROOF. The proof that (1) and (2) in (a) are equivalent is the same as in 25.21. The latter result and 26.8.(f) also yield (3) $\Rightarrow$ (1) and (2). For the converse, note that if $x, y \in |A|$, then 26.8.(d).(3) entails

$$f(x_{|[\![fx=fy]\!]}) \;=\; fx_{|[\![fx=fy]\!]} \;=\; fy_{|[\![fx=fy]\!]} \;=\; f(y_{|[\![fx=fy]\!]}),$$

and so $x_{|[\![fx=fy]\!]} = y_{|[\![fx=fy]\!]}$. Hence, $[\![fx = fy]\!] = [\![x = y]\!]$, as needed. $\square$

Lemma 26.17 allows the following conclusions :

∗ For presheaves over Ω, a morphism is monic iff it is a regular monic;

∗ The notion of subobject in $\boldsymbol{pSh(L)}$ coincides with **subset of the domain, with induced restriction and extent**. Write $A \subseteq B$ if A is a subpresheaf of B.

26.18. **Initial and Final object.** The final object in $\boldsymbol{pSh(L)}$ is the presheaf version of $\mathbf{1}$ in 25.5, described in 26.10. The initial object $\boldsymbol{pSh(L)}$ is the presheaf corresponding to the empty L-set. $\square$

26.19. **Equalizers.** Let $f, g : A \longrightarrow B$ be morphisms of presheaves. Define a presheaf $Eq(f, g)$, by setting

∗ $|Eq(f, g)| = \{x \in |A| : fx = gx\}$;

∗ Extent and restriction are those induced by A.

There is a natural morphism, $Eq(f, g) \xrightarrow{\iota} A$, whose carrier is the canonical inclusion. The pair $\langle Eq(f, g), \iota \rangle$ is the equalizer of $\langle f, g \rangle$ in $\boldsymbol{pSh(L)}$. Observe that if A is extensional, so is $Eq(f, g)$. $\square$

26.20. **Coproducts.** Let A_i, $i \in I$, be a family of presheaves over L. Define a L-presheaf $\coprod_{i\in I} A_i$, by

$*$ $|\coprod_{i\in I} A_i| = \bigcup_{i\in I} |A_i| \times \{i\}$; $*$ $E\langle a, i \rangle = Ea$;

$*$ ${\langle a, i \rangle}_{|p} = \langle a_{|p}, i \rangle$.

There are presheaf morphisms, $a \in A_i \overset{\alpha_i}{\longmapsto} \langle a, i \rangle$, making $\langle \coprod_{i\in I} A_i,\ \{\alpha_i : i \in I\} \rangle$ the coproduct of the A_i in $\boldsymbol{pSh(L)}$. Note that the coproduct preserves extensionality. It is also clear that if L is complete, then

$$E\left(\coprod_{i\in I} A_i\right) \;=\; \bigvee_{i\in I} EA_i,$$

that is the support (26.1) of the coproduct is the join of the supports of the component presheaves. $\square$

The notion of *restriction density* in Exercise 26.28.(d) allows extension of morphisms, as follows :

LEMMA 26.21. *Let A be a L-presheaf and B a restriction dense subset of $|A|$. Let C be a L-presheaf and $B \xrightarrow{f} |C|$ be a map such that for x, $y \in B$ and $p \in L$*

$$Efx \;=\; Ex \quad \textit{and} \quad x_{|p} \;=\; y_{|p} \;\Rightarrow\; (fx)_{|p} \;=\; (fy)_{|p}.$$

Then, f has a unique *extension to a presheaf morphism, $\widetilde{f} : A \longrightarrow C$.*

PROOF. If $a = b_{|Ea}$, set $\widetilde{f}(a) = (fb)_{|Ea}$. $\square$

An important distinction between presheaves and general Ω-sets is that a dense subset in each coordinate generates a dense subset of a finite product. The statement and proof closely resemble that of 24.17.

PROPOSITION 26.22. *Let $A_1, \ldots, A_n$ be Ω-presheaves and D_i be dense subsets of A_i, $1 \leq i \leq n$. Then,*

$$D \;=\; \{\overline{d}_{|E\overline{d}} : \overline{d} \in \textstyle\prod_{i=1}^n D_i\}$$

is dense in $\prod_{i=1}^n A_i$ and for all $\overline{x} \in \prod_{i=1}^n |A_i|$, $E\overline{x} \;=\; \bigvee_{\overline{d}\in\prod D_i} [\![\overline{x} = \overline{d}]\!]$.

PROOF. We prove only the last assertion; let $A = \prod_{i=1}^n A_i$. Since $\overline{x}_{|E\overline{x}} \in |A|$, and $E\overline{x}_{|E\overline{x}} = E\overline{x}$, the first part of the statement yields

$$\begin{aligned} E\overline{x} \;&=\; \textstyle\bigvee_{\overline{c}\in D} [\![\overline{x}_{|E\overline{x}} = \overline{c}]\!] \;=\; \bigvee_{\overline{d}\in\prod D_i} [\![\overline{x}_{|E\overline{x}} = \overline{d}_{|E\overline{d}}]\!] \\ &=\; \textstyle\bigvee_{\overline{d}\in\prod D_i} E\overline{x} \wedge E\overline{d} \wedge [\![\overline{x} = \overline{d}]\!] \;=\; \bigvee_{\overline{d}\in\prod D_i} [\![\overline{x} = \overline{d}]\!], \end{aligned}$$

as needed. $\square$

Exercises

26.23. Let $f : P \longrightarrow Q$ be a morphism of presheaves over a semilattice L.

a) If $S \subseteq |P|$ and $p \in L$, let $S_{|p} = \{s_{|p} : s \in S\}$. Then, S is compatible over p iff $S_{|p}$ is compatible [3].

b) If $S \subseteq |P|$ has a gluing in P, then S is compatible.

c) If P is extensional and L is a $[\wedge, \bigvee]$-semilattice, then gluings are unique (whenever they exist).

d) If t is a gluing of S in P and L is a $[\wedge, \bigvee]$-semilattice, then $t_{|p}$ is a gluing of $S_{|p}$ for all $p \in L$.

e) If $S \subseteq |P|$ is compatible over $p \in L$, then $f(S) = \{fs : s \in S\}$ is compatible over p in Q.

f) If t is a gluing of $S \subseteq |P|$ in P, then ft is a gluing of $f(S)$ in Q. $\square$

26.24. A semilattice L may be considered as a category, whose objects are its elements and whose arrows are given by

$$[x, y] = \begin{cases} \emptyset & \text{if } x \notin y^{\leftarrow} \\ \{\langle x, y \rangle\} & \text{if } x \leq y. \end{cases}$$

The collection of contravariant functors from L to $\boldsymbol{Set}$, with natural transformations as morphisms, is a category, $\mathfrak{p}\mathfrak{S}(L)$. Then, there is a natural equivalence between $\mathfrak{p}\mathfrak{S}(L)$ and $\boldsymbol{pSh(L)}$. $\square$

26.25. With notation as in 26.5,

a) If C is a L-presheaf and $|C| \xrightarrow{g} |A|$ is a map such that for $\langle x, p \rangle \in |C| \times L$,

$$Egx = Ex \quad \text{and} \quad g(x_{|p}) = (gx)_{|p},$$

then there is a *unique* morphism of L-presheaves, $f : C \longrightarrow A_{|L}$, such that the triangle below is commutative :

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[dr, "g"'] & A_{|L} \arrow[d, "\mathfrak{r}_A"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\mathfrak{r}_A"'] & B \arrow[d, "\mathfrak{r}_B"] \\ A_{|L} \arrow[r, "\mathfrak{r}f"'] & B_{|L} \end{tikzcd}$$

b) If B is a R-presheaf and $f : A \longrightarrow B$ is a morphism of R-presheaves, there is a *unique* morphism of L-presheaves, $\mathfrak{r}f : A_{|L} \longrightarrow B_{|L}$ such that the square above is commutative.

c) Restriction is a covariant functor $\mathfrak{r} : \boldsymbol{pSh(R)} \longrightarrow \boldsymbol{pSh(L)}$. $\square$

[3] Hence, for presheaves, local compatibility can be reduced to compatibility. This **is not true** for L-sets (25.25, 25.26).

The exercise that follows describes a way to extensionalize a presheaf, under reasonable hypotheses on the base.

26.26. Let L be a $[\wedge, \bigvee]$-semilattice and let A be a L-presheaf. Define a binary relation Σ on $|A|$ as follows :

$$s \;\Sigma\; t \qquad \text{iff} \qquad \begin{array}{l} Es = Et \text{ and } \exists\ \alpha \subseteq L \text{ such that } \ \bigvee \alpha = Es \ \text{ and} \\ s_{|p} = t_{|p},\ \forall\ p \in \alpha. \end{array}$$

a) Σ is an equivalence relation in $|A|$ and a congruence with respect to restriction, i.e., for all s, $t \in |A|$ and $p \in L$

$$s \;\Sigma\; t \ \Rightarrow\ s_{|p} \;\Sigma\; t_{|p}.$$

b) Let $|A_e| = |A|/\Sigma = \{s_e : s \in |A|\}$ be the set of equivalence classes of $|A|$ by Σ. For $s \in |A|$ and $p \in L$, define

$$Es_e = Es \quad \text{and} \quad (s_e)_{|p} = (s_{|p})_e.$$

With this structure, A_e is an extensional L-presheaf, and $s \in |A| \longmapsto s_e \in |A_e|$ is a morphism of L-presheaves, $e : A \longrightarrow A_e$, with the following universal property :

If $A \xrightarrow{f} B$ is a morphism of L-presheaves and B is extensional, there is a *unique* morphism of L-presheaves, $\widehat{f} : A_e \longrightarrow B$ making the following diagram commutative :

$$\begin{tikzcd} A \arrow[rr, "e"] \arrow[dr, "f"'] & & A_e \arrow[dl, "\widehat{f}"] \\ & B & \end{tikzcd}$$

c) Extensionalization is a covariant functor from $\boldsymbol{pSh(L)}$ to the full subcategory of extensional presheaves over L, left adjoint to the associated forgetful functor. $\square$

26.27. Let A be a presheaf over Ω, $S \subseteq |A|$, $p \in \Omega$ and $t \in |A|$. Then :

a) S is compatible over p in the presheaf sense (26.1.(c) iff it is compatible in the Ω-set sense (25.26).

b) t is a gluing of S in the presheaf sense (26.1.(d)) iff it is a gluing of S in the Ω-set sense (25.30). $\square$

26.28. Let A be a L-presheaf and $D \subseteq |A|$. D is **$\boldsymbol{\rho}$-dense in $\boldsymbol{A}$** iff for all $x \in |A|$, there is $\{\langle d_i, p_i \rangle \in D \times L : i \in I\}$ such that

$$(i)\ Ex \ = \ \bigvee p_i \qquad \text{and} \qquad (ii)\ \forall\ i \in I,\ \ x_{|p_i} \ = \ (d_i)_{|p_i}.$$

Thus, D is ρ-dense in A if all sections in A are locally equal to restrictions of sections in D.

a) If L is a frame, then D is ρ-dense in A iff D is dense in A [4].

[4]In the sense of 25.32.

b) If L is a $[\wedge, \bigvee]$-semilattice, ρ-denseness is transitive.

c) The constant L-presheaf C is ρ-dense in $\boldsymbol{fC}$ (25.8 in 26.10).

d) $B \subseteq |A|$ is **restriction dense** if for all $x \in |A|$, there is $b \in B$ with $x = b_{|Ex}$. Show that a restriction dense subset is ρ-dense in A.

e) A is **flabby** [5] if its set of global sections is restriction dense. If H is a HA, $\widetilde{H}$ (25.4, 26.10) is a flabby presheaf over H. $\square$

26.29. Let L be a distributive lattice. A finitely complete L-set (25.34) has a compatible structure of extensional presheaf over L (26.6). How are L-set and presheaf morphisms related ? $\square$

[5]This type of presheaf is important in characterizing injectivity in categories of sheaves of modules; see Prop. 2.4.6, p. 98 in [**37**].

CHAPTER 27

Sheaves and Complete Ω-sets

This chapter introduces the abstract version of *sheaves*. The discussion at the end of Chapter 25 and Definition 23.8 lead to

DEFINITION 27.1. *Let Ω be a frame.*
a) A Ω-set A is **complete** *iff for all $p \in \Omega$ and $S \subseteq |A|$, if S is compatible over p, then S has a* unique *gluing over p* [1]*. A morphism of complete Ω-sets is a morphism of the underlying Ω-sets.*

b) A Ω-presheaf P is a **sheaf** *if all compatible $S \subseteq |P|$ can be uniquely glued in P*[2]*. A morphism of Ω-sheaves is a morphism of the underlying Ω-presheaves.*

Clearly, complete Ω-sets and their morphism, as well as Ω-sheaves and their morphisms, are categories.

LEMMA 27.2. *Complete Ω-sets and Ω-sheaves are extensional.*

PROOF. Let A be a complete Ω-set and $s, t \in |A|$ satisfy $Es = Et = [\![s = t]\!]$. Note that $\{s\}$ is compatible over Es and that s satisfies (i) and (ii) in [glu Es] (27.1). It is straightforward that t also satisfies [glu Es] for $\{s\}$ and so uniqueness entails $s = t$, as needed. The reasoning for Ω-sheaves is analogous. $\square$

The concepts in 27.1 are equivalent :

PROPOSITION 27.3. *The categories of complete Ω-sets and of Ω-sheaves are* **isomorphic**.

PROOF. Let A be a complete Ω-set. We must show that there are natural extent and restriction maps in A, compatible with its equality (26.6), with which it becomes a sheaf over Ω. For $s \in |A|$, set
$$Es = [\![s = s]\!].$$
Now fix $s \in |A|$ and $p \in \Omega$. Since $\{s\}$ is compatible over p, there is a *unique* $t \in |A|$ such that
$$Et \;=\; p \wedge Es \;=\; [\![s = t]\!]. \qquad (*)$$
Define $s_{|p} = t$; clearly, this restriction map satisfies [rest 2] in 26.1. Moreover,

$*$ From (*) we get $Es_{|Es} \;=\; Es \;=\; [\![s = s_{|Es}]\!]$, and so extensionality (27.2) entails $s_{|Es} = s$, verifying [rest 1] in 26.1;

[1] Here compatibility is as in 25.26 and gluing as in 25.30.
[2] Here compatibility and gluing are as in 26.1, although 25.38 implies that it does not matter.

$*$ If $p,\, q \in \Omega$, then
$$p \wedge q \wedge Es \;=\; Es_{|p\wedge q} \;=\; q \wedge (p \wedge Es) \;=\; q \wedge Es_{|p} \;=\; E(s_{|p})_{|q}. \qquad (+)$$
The transitivity of equality and (*) yield
$$\begin{aligned} [\![s_{|p\wedge q} = (s_{|p})_{|q}]\!] &\geq [\![s_{|p\wedge q} = s]\!] \wedge [\![s = s_{|p}]\!] \wedge [\![s_{|p} = (s_{|p})_{|q}]\!] \\ &= (p \wedge q \wedge Es) \wedge (p \wedge Es) \wedge (q \wedge Es_{|p}) \\ &= p \wedge q \wedge Es, \end{aligned}$$
which, together with (+) and extensionality, entails $s_{|p\wedge q} = (s_{|p})_{|q}$, establishing [rest 3] and showing that A is a Ω-presheaf.

Fact. *For all $s,\, t \in |A|$ and $p,\, q \in \Omega$*
a) $[\![s_{|p} = t_{|q}]\!]_A \;=\; p \wedge q \wedge [\![s = t]\!]_A$.
b) $[\![s = t]\!]_A \;=\; \bigvee \{p \leq Es \wedge Et : s_{|p} = t_{|p}\}$.

Proof. Here, $[\![* = *]\!]_A$ is the *original* equality in A and *not* that defined in 26.8 [3]. With that in mind, we omit A from the notation.

a) The transitive law of equality in A and (*) yield
$$\begin{aligned} [\![s_{|p} = t_{|q}]\!] &\geq [\![s_{|p} = s]\!] \wedge [\![s = t]\!] \wedge [\![t = t_{|q}]\!] \\ &= p \wedge Es \wedge [\![s = t]\!] \wedge q \wedge Et \;=\; p \wedge q \wedge [\![s = t]\!]. \end{aligned} \qquad \text{(I)}$$
On the other hand, we also have
$$\begin{aligned} [\![s = t]\!] &\geq [\![s = s_{|p}]\!] \wedge [\![s_{|p} = t_{|q}]\!] \wedge [\![t = t_{|q}]\!] \\ &= Es_{|p} \wedge [\![s_{|p} = t_{|q}]\!] \wedge Et_{|q} \;=\; [\![s_{|p} = t_{|q}]\!]. \end{aligned} \qquad \text{(II)}$$
It is clear that (I) and (II) imply the desired equality [4].
b) First note that extensionality and (*) yield $s_{|[\![s=t]\!]} = t_{|[\![s=t]\!]}$. Next, if $s_{|p} = t_{|p}$, then
$$\begin{aligned} p \wedge Es &= Es_{|p} \;=\; Et_{|p} \;=\; p \wedge Et \;=\; [\![s_{|p} = t_{|p}]\!] \\ &= p \wedge [\![s = t]\!], \end{aligned} \qquad (++)$$
where the last equality comes from (a). If $p \leq Es \wedge Et$, it follows from (++) that $p \leq [\![s = t]\!]$, completing the proof of the Fact.

By the Fact, the equality generated by the restriction on A, as in 26.8, is its original equality. Hence,
$*$ By 26.27, $S \subseteq |A|$ is presheaf compatible iff it is Ω-set compatible; and presheaf gluings correspond to Ω-set gluings. Hence, with the extent and restriction defined above, A is a Ω-sheaf;
$*$ By 26.8.(f), Ω-set morphisms correspond to presheaf morphism.

It is left to the reader to check that if A is a Ω-sheaf, then with the equality defined in 26.8, A is a complete Ω-set. $\square$

In view of Proposition 27.3, we set down

[3] That is exactly the crux of the matter.
[4] We have shown that (*) and the laws of equality entail 26.8.(b).

Definition 27.4. *If Ω is a frame, write $\boldsymbol{Sh(\Omega)}$ for the isomorphic categories of sheaves over Ω and complete Ω-sets.*

Example 27.5. By 26.2 and 27.3, all sheaves over a topological space X are examples of complete $\Omega(X)$-sets. □

Example 27.6. Let Ω be a frame and let A be a set. The constant Ω-set $\boldsymbol{A}$ (25.9) is complete. To see this, let $p \in \Omega$ and suppose that $S \subseteq |\boldsymbol{A}|$ is compatible over p. Thus, for $s,\, t \in S$

$$p \wedge Es \wedge Et = p \wedge [\![s = t]\!], \tag{I}$$

that is,

$$p \wedge \bigvee_{a,b \in A} s(a) \wedge t(b) \;\; = \;\; p \wedge \bigvee_{a \in A} s(a) \wedge t(a). \tag{II}$$

Fact 1. *For $a \neq b$ in A, $\;p \wedge s(a) \wedge t(b) = \bot$.*

Proof. Relation (I) and the fact that the values of s and t in distinct points of A are disjoint yields,

$$\begin{aligned} p \wedge s(a) \wedge t(b) \;&=\; p \wedge s(a) \wedge t(b) \wedge Es \wedge Et \\ &=\; p \wedge s(a) \wedge t(b) \wedge [\![s = t]\!] \\ &=\; p \wedge s(a) \wedge t(b) \wedge \textstyle\bigvee_{c \in A} s(c) \wedge t(c) \\ &=\; p \wedge s(a) \wedge t(a) \wedge t(b) \;=\; \bot, \end{aligned}$$

as claimed. Define $x : A \longrightarrow \Omega$ as follows :

$$x(a) = p \wedge \bigvee_{s \in S} s(a).$$

If $a \neq b$ in A, then, Fact 1 entails

$$x(a) \wedge x(b) \;\; = \;\; p \wedge \bigvee_{s,t \in S} s(a) \wedge t(b) \;\; = \;\; \bot,$$

verifying that $x \in |\boldsymbol{A}|$. Note that

$$Ex \;\; = \;\; \bigvee_{a \in A} x(a) \;\; = \;\; \bigvee_{a \in A} p \wedge \bigvee_{s \in S} s(a) \;\; = \;\; p \wedge \bigvee_{s \in S} Es.$$

Moreover, for $s \in S$,

$$\begin{aligned} p \wedge [\![x = s]\!] \;&=\; p \wedge \textstyle\bigvee_{a \in A} x(a) \wedge s(a) \\ &=\; p \wedge \textstyle\bigvee_{a \in A} \big(\bigvee_{t \in S} t(a)\big) \wedge s(a) \\ &=\; p \wedge \textstyle\bigvee_{a \in A} s(a) \;=\; p \wedge Es, \end{aligned}$$

verifying [glu p] in Definition 25.30. Since we already know that $\boldsymbol{A}$ is extensional, the gluing of S over p is unique and $\boldsymbol{A}$ is a complete Ω-set. In particular, the final object $\mathbf{1}$ and the initial object $\mathbf{0}$ (25.11) in $\boldsymbol{\Omega set}$ are complete Ω-sets. □

Example 27.7. If Ω is a frame, the Ω-set $\widetilde{\Omega}$ of 25.4 is complete. To see this, let $u \in \Omega$ and suppose that $S \subseteq |\widetilde{\Omega}|$ is a set of sections compatible over u. Hence, for all $\langle a, p \rangle$, $\langle b, q \rangle \in S$

$$u \wedge p \wedge q \;\; = \;\; u \wedge (a \leftrightarrow b) \wedge p \wedge q,$$

or equivalently, $u \wedge p \wedge q \;\; \leq \;\; (a \leftrightarrow b)$. Since $a \leq p$ and $b \leq q$, 6.10.(a) yields

$$\text{For all } \langle a, p \rangle, \langle b, q \rangle \in S, \;\; u \wedge q \wedge a = u \wedge p \wedge b. \tag{I}$$

Define

$$x \;\; = \;\; u \wedge \bigvee_{\langle a,p \rangle \in S} a \quad \text{ and } w \;\; = \;\; u \wedge \bigvee_{\langle a,p \rangle \in S} p.$$

Clearly, $x \leq q$. We shall verify that $\langle x, w \rangle$ is the gluing of S over u, which will be unique, because $\widetilde{\Omega}$ is extensional. Since the extent of $\langle a, p \rangle \in S$ is p, we have

$$E\langle x, w\rangle = w = u \wedge \bigvee_{s\in S} Es.$$

verifying condition (i) for [glu u] in 25.30. Next, we must check that if $\langle a, p\rangle \in S$,

$$u \wedge E\langle a, p\rangle \;=\; u \wedge [\![\langle a, p\rangle = \langle x, w\rangle]\!],$$

that is, keeping in mind that $u \wedge p \le w$,

$$u \wedge p = u \wedge p \wedge w \wedge (a \leftrightarrow x) = u \wedge p \wedge (a \leftrightarrow x).$$

Since this relation is equivalent to $u \wedge p \le (a \leftrightarrow x)$, what must be shown reduces, via 6.10.(a), to

$$u \wedge p \wedge a = u \wedge p \wedge x. \tag{II}$$

From the definition of x and (I) comes

$$\begin{aligned} u \wedge p \wedge x &= u \wedge p \wedge \textstyle\bigvee_{\langle b,q\rangle\in S} b = \bigvee_{\langle b,q\rangle\in S} u \wedge p \wedge b \\ &= \textstyle\bigvee_{\langle b,q\rangle\in S} u \wedge q \wedge a \;\le a, \end{aligned}$$

establishing that $u \wedge p \wedge x \le u \wedge p \wedge a$. Since $u \wedge a \le x$, the reverse inequality is obvious and the proof of (II) and that $\widetilde{\Omega}$ is a sheaf over Ω is complete. □

LEMMA 27.8. *If* $A \xrightarrow{f} B$ *is a morphism of sheaves, then* f *is an isomorphism* $\Leftrightarrow$ f *is monic and epic.*

PROOF. It suffices to establish ($\Leftarrow$). Since A, B are sheaves, they are extensional and have compatible restriction and equality. Hence, 26.17.(b) guarantees that f is injective and

$$\forall\, x, y \in |A|,\;\; [\![x = y]\!] = [\![fx = fy]\!]. \tag{I}$$

By 25.23, it remains to check that f is onto B. If $s \in |B|$, the fact that f is epic entails, by 25.24,

$$Es = \textstyle\bigvee_{a\in|A|} [\![s = fa]\!].$$

Consider the set $S = \{a_{|[\![s=fa]\!]} \in |A| : a \in |A|\}$; note that

$$Ea_{|[\![s=fa]\!]} = Ea \wedge [\![s = fa]\!] = Efa \wedge [\![s = fa]\!] = [\![s = fa]\!].$$

Because $[\![s = fa]\!] \wedge [\![s = fa']\!] \le [\![fa = fa']\!] = [\![a = a']\!]$, (I) implies

$$\begin{aligned} [\![s = fa]\!] \wedge [\![s = fa']\!] &= [\![s = fa]\!]\, [\![s = fa']\!] \wedge [\![a = a']\!] \\ &= [\![a_{|[\![s=fa]\!]} = a'_{|[\![s=fa']\!]}]\!], \end{aligned}$$

wherefrom we conclude that S is compatible in A. Since A is a sheaf, S has a unique gluing, x, in A, verifying, for all $a \in |A|$,

$$\text{(II)} \qquad \begin{cases} (i)\; Ex = Efx = \bigvee_{a\in|A|} [\![s = fa]\!] = Es; \\ (ii)\; [\![s = fa]\!] = [\![x = a_{|[\![s=fa]\!]}]\!] = [\![s = fa]\!] \wedge [\![x = a]\!]. \end{cases}$$

Since $[\![fx = s]\!] \ge [\![s = fa]\!] \wedge [\![fx = fa]\!]$, taking joins with respect to $a \in |A|$, yields, in view of (i) and (ii) in (II) above,

$$\begin{aligned} Ex = Es \ge [\![fx = s]\!] &\ge \textstyle\bigvee_{a\in|A|} [\![s = fa]\!] \wedge [\![fx = fa]\!] \\ &= \textstyle\bigvee_{a\in|A|} [\![s = fa]\!] \wedge [\![x = a]\!] \\ &= \textstyle\bigvee_{a\in|A|} [\![s = fa]\!] \;=\; Es, \end{aligned}$$

and $fx = s$, completing the proof. □

The main result of this section is the following

THEOREM 27.9. *Let A be a Ω-set. Then, there is a sheaf cA over Ω and a morphism $c : A \longrightarrow cA$, satisfying the following properties :*

(1) *For $a,\, b \in |A|$,* $[\![a = b]\!] = [\![ca = cb]\!]$.

(2) *For all $s \in |cA|$,* $Es = \bigvee_{a \in A} [\![s = ca]\!]$.

(3) *If B is a sheaf over Ω and $g : A \longrightarrow B$ is a morphism, there is a* unique *morphism $\widehat{g} : cA \longrightarrow B$ that makes the following diagram commutative :*

$$\begin{tikzcd} A \arrow[rr, "c"] \arrow[dr, "g"'] & & cA \arrow[dl, "\widehat{g}"] \\ & B & \end{tikzcd}$$

The sheaf cA of 27.9 is the **completion** of A. This construction shows that the forgetful functor from $\boldsymbol{Sh(\Omega)}$ to $\boldsymbol{\Omega set}$ has a left adjoint. The proof of 27.9 uses ideas due to D. Scott and first published in [**15**]. We start with

DEFINITION 27.10. *Let A be a L-set. A map $s : |A| \longrightarrow L$ is a* **singleton** *in A if for all $a,\, b \in |A|$*

[sin 1] : $s(a) \wedge s(b) \;\leq\; [\![a = b]\!]$;

[sin 2] : $s(a) \wedge [\![a = b]\!] \;\leq\; s(b)$.

Write $|cA|$ for the set of singletons in A. If L is a complete lattice, the **extent** *of $s \in |cA|$ is $Es \;=\; \bigvee_{a \in |A|} s(a)$.*

EXAMPLE 27.11. If A is a Ω-set and $x \in |A|$, then

$$cx : |A| \longrightarrow \Omega, \text{ given by } y \longmapsto [\![x = y]\!],$$

is, by 26.8.(a), a singleton in A, such that $Ex = Ecx$. □

LEMMA 27.12. *Let A be L-set. $a,\, b \in |A|$, $s \in |cA|$ and $p \in L$*

a) $s(a) \leq Ea$; $s(a) \wedge [\![a = b]\!] = s(b) \wedge [\![a = b]\!]$.

b) If L is a HA, then $[\![a = b]\!] \leq s(a) \leftrightarrow s(b)$, where $\leftrightarrow$ is equivalence in L (6.9).

c) The map

$$p \wedge s : |A| \longrightarrow L, \text{ defined by } [p \wedge s](a) = p \wedge s(a)$$

is a singleton in A. If L is a frame, then $E(p \wedge s) = p \wedge Es$.

PROOF. The first relation in (a) comes from [sin 1] with $a = b$, while the second follows easily from [sin 2] in 27.10, by symmetry; (b) is immediate from (a). Item (c) is straightforward and left to the reader. □

LEMMA 27.13. *Let L be a frame, $A \xrightarrow{f} B$ a morphism of L-sets and s a singleton in A*

a) If A is a complete *L-set, there is a* unique *$a \in |A|$ such that*

$$\text{For all } x \in |A|, \quad s(x) \; = \; [\![a = x]\!].$$

In particular, $Ea = Es = s(a)$.

b) The map $s_f : |B| \longrightarrow L$ defined by

$$s_f(b) = \bigvee_{a \in |A|} s(a) \wedge [\![fa = b]\!]$$

is a singleton in B, with $Es = Es_f$. Moreover, notation as in 27.11*, if $a \in |A|$, then $(ca)_f = c(fa)$.*

Proof. a) By 27.3, A has a restriction compatible with its equality, that is, the formulas in 26.8 apply to the situation at hand [5].

Given $s \in |cA|$, let $S = \{c_{|s(c)} : c \in |A|\}$. By 26.8.(b) (or (a) of the Fact in the proof of 27.3) and [sin 1]

$$\begin{aligned}[\![c_{|s(c)} = b_{|s(b)}]\!] \; &= \; s(c) \wedge s(b) \wedge [\![c = b]\!] \; = \; s(c) \wedge s(b) \\ &= \; Ec_{|s(c)} \wedge Eb_{|s(b)},\end{aligned}$$

showing that S is compatible. By completeness, there is $a \in |A|$ such that

$$\begin{cases} (i) \; Ea \; = \; \bigvee_{c \in |A|} Ec_{|s(c)} \; = \; \bigvee_{c \in |A|} s(c) \; = \; Es; \\ (ii) \; Ec_{|s(c)} \; = \; s(c) \; = \; [\![a = c_{|s(c)}]\!] \; = \; s(c) \wedge [\![a = c]\!]. \end{cases} \tag{*}$$

It follows from (ii) in (*) that $s(c) \leq [\![a = c]\!]$. Thus, [sin 2] implies

$$s(c) \; = \; s(c) \wedge [\![a = c]\!] \; \leq \; s(a),$$

that, by taking joins over $c \in |A|$, entails, together with (i) in (*),

$$Ea \; = \; \bigvee_{c \in |A|} s(c) \; \leq \; s(a),$$

and so, $Ea = s(a)$. But then, another application of [sin 2] yields

$$[\![a = c]\!] \; = \; Ea \wedge [\![a = c]\!] \; = \; s(a) \wedge [\![a = c]\!] \; \leq \; s(c),$$

which in view of (ii) in (*) entails $s(c) \; = \; [\![a = c]\!]$, as desired.

b) For [sin 1], let $b, b' \in |B|$. Since $[\![a = a']\!] \; \leq \; [\![fa = fa']\!]$, we get

$$\begin{aligned} s_f(b) \wedge s_f(b') \; &= \; \bigvee_{a \in |A|} s(a) \wedge [\![fa = b]\!] \;\; \wedge \;\; \bigvee_{a' \in A'} s(a') \wedge [\![fa' = b']\!] \\ &= \; \bigvee_{a, a' \in |A|} s(a) \wedge s(a') \wedge [\![fa = b]\!] \wedge [\![fa' = b']\!] \\ &\leq \; \bigvee_{a, a' \in |A|} [\![a = a']\!] \wedge [\![fa = b]\!] \wedge [\![fa' = b']\!] \\ &\leq \; \bigvee_{a, a' \in |A|} [\![fa = fa']\!] \wedge [\![fa = b]\!] \wedge [\![fa' = b']\!] \\ &\leq \; [\![b = b']\!], \end{aligned}$$

as desired. For [sin 2], we have

$$\begin{aligned} s(b) \wedge [\![b = b']\!] \; &= \; \bigvee_{a \in |A|} s(a) \wedge [\![fa = b]\!] \wedge [\![b = b']\!] \\ &\leq \; \bigvee_{a \in |A|} s(a) \wedge [\![fa = b']\!] \; = \; s_f(b'), \end{aligned}$$

and s_f is indeed a singleton in B. Moreover, recalling 27.12.(a), we get

$$\begin{aligned} Es_f \; &= \; \bigvee_{b \in |B|} s_f(b) \; = \; \bigvee_{b \in |B|} \bigvee_{a \in |A|} s(a) \wedge [\![fa = b]\!] \\ &= \; \bigvee_{a \in |A|} s(a) \wedge \left(\bigvee_{b \in |B|} [\![fa = b]\!] \right) \; = \; \bigvee_{a \in |A|} s(a) \wedge Efa \end{aligned}$$

[5] In fact, all we need is the Fact in the proof of 27.3.

$$= \bigvee_{a \in |A|} s(a) \wedge Ea \;=\; \bigvee_{a \in |A|} s(a) \;=\; Es,$$

as desired. If $a \in |A|$, then for all $b \in |B|$

$$\begin{aligned}(ca)_f(b) &= \bigvee_{x \in |A|} ca(x) \wedge [\![fx = b]\!] \;=\; \bigvee_{x \in |A|} [\![a = x]\!] \wedge [\![fx = b]\!] \\ &\leq \bigvee_{a \in |A|} [\![fx = fa]\!] \wedge [\![fx = b]\!] \;\leq\; [\![fa = b]\!] \\ &= c(fa)(b),\end{aligned}$$

showing that $(ca)_f(b) \leq c(fa)(b)$. For the reverse inequality, we have

$$\begin{aligned}[\![fa = b]\!] &= Efa \wedge [\![fa = b]\!] \;=\; Ea \wedge [\![fa = b]\!] \\ &= \bigvee_{x \in |A|} [\![a = x]\!] \wedge [\![fa = b]\!] \\ &= \bigvee_{x \in |A|} [\![a = x]\!] \wedge [\![fa = fx]\!] \wedge [\![fa = b]\!] \\ &\leq \bigvee_{x \in |A|} [\![a = x]\!] \wedge [\![fx = b]\!] \;=\; (ca)_f(b),\end{aligned}$$

concluding the proof. □

If the base algebra is a frame, cA is a sheaf :

LEMMA 27.14. *Let A be an Ω-set. For $s,\, t \in |cA|$, the prescription*

$$[\![s = t]\!] \;=\; \bigvee_{a \in |A|} s(a) \wedge t(a).$$

defines an extensional equality in $|cA|$, with which it is a sheaf.

PROOF. Clearly, $[\![=]\!]$ verifies [= 1] in 25.1. For [= 2] we use the fact that t is singleton and 8.4 to get

$$\begin{aligned}[\![s = t]\!] \wedge [\![t = z]\!] &= [\bigvee_{a \in |A|} s(a) \wedge t(a)] \;\wedge\; [\bigvee_{b \in |A|} t(b) \wedge z(b)] \\ &= \bigvee_{a,b \in |A|} s(a) \wedge t(a) \wedge t(b) \wedge z(b) \\ &\leq \bigvee_{a,b \in |A|} s(a) \wedge [\![a = b]\!] \wedge z(a) \\ &\leq \bigvee_{a,b \in |A|} s(a) \wedge z(a) \;=\; [\![s = z]\!],\end{aligned}$$

as needed. If $Es = Et = [\![s = t]\!]$ and $a \in |A|$, then

$$\begin{aligned}s(a) &= s(a) \wedge Es \;=\; s(a) \wedge [\![s = t]\!] \;=\; s(a) \wedge \bigvee_{b \in |A|} s(b) \wedge t(b) \\ &= \bigvee_{b \in |A|} s(a) \wedge s(b) \wedge t(b) \;\leq\; \bigvee_{b \in |B|} [\![a = b]\!] \wedge t(b) \;\leq\; t(a).\end{aligned}$$

The argument being symmetrical in s, t, we get $s = t$, proving extensionality. For completeness, let $S \subseteq |cA|$ be compatible over $p \in \Omega$. By 25.26, this means that for all $s,\, s' \in S$

$$p \wedge Es \wedge Es' \;=\; p \wedge [\![s = s']\!] \;=\; p \wedge \bigvee_{a \in |A|} s(a) \wedge s'(a). \tag{*}$$

Define $t : |A| \longrightarrow \Omega$ by

$$t(a) = p \wedge \bigvee_{s \in S} s(a).$$

We shall prove that t is the (unique) gluing of S over p. First note that 7.7 and the fact that Ω is a frame yields

$$\begin{aligned}Et &= \bigvee_{a \in |A|} \bigvee_{s \in S} p \wedge s(a) \;=\; p \wedge \bigvee_{a \in |A|} \bigvee_{s \in S} s(a) \\ &= p \wedge \bigvee_{s \in S} \bigvee_{a \in A} s(a) \;=\; p \wedge \bigvee_{s \in S} Es,\end{aligned}$$

verifying (i) in condition [glu p] of 27.1. To verify (ii) in [glu p], suppose that $s \in S$. Then, (*) yields

$$p \wedge [\![s = t]\!] \;=\; p \wedge \bigvee_{a \in A} s(a) \wedge t(a)$$

$$\begin{aligned}
&= p \wedge \bigvee_{a\in|A|} s(a) \wedge \bigvee_{s'\in S} p \wedge s'(a) \\
&= \bigvee_{a\in|A|} \bigvee_{s'\in S} p \wedge s(a) \wedge s'(a) \\
&= \bigvee_{s'\in S} p \wedge \bigvee_{a\in|A|} s(a) \wedge s'(a) \\
&= \bigvee_{s'\in S} p \wedge [\![s = s']\!] \ = \ p \wedge Es,
\end{aligned}$$

and cA is a complete Ω-set, as desired. □

LEMMA 27.15. *Notation as in* 27.11*, if A is an Ω-set, the map $c : A \longrightarrow cA$, $a \longmapsto ca$ is a morphism of Ω-sets, verifying* (1) *and* (2) *in* 27.9.

PROOF. It has already been remarked that $Eca = Ea$, for all $a \in |A|$. If a, b are in $|A|$, note that

$$\bigvee_{c\in|A|} [\![a = c]\!] \wedge [\![c = b]\!] \ = \ [\![a = b]\!]. \tag{*}$$

It is clear that $[\![a = b]\!]$ is below the left-hand side of (*); the reverse inequality comes from the transitivity of equality ([= 2]). Hence, (*) and the definition of equality in cA entail

$$[\![ca = cb]\!] \ = \ [\![a = b]\!],$$

proving in one stroke that c is a morphism and that it verifies (1) in 27.9. For s in $|cA|$, items (a) and (c) in 27.12 yield

$$\begin{aligned}
\bigvee_{a\in|A|} [\![s = ca]\!] &= \bigvee_{a\in A} \bigvee_{b\in|A|} s(b) \wedge [\![a = b]\!] \\
&= \bigvee_{a\in A} \bigvee_{b\in|A|} s(a) \wedge [\![a = b]\!] \\
&= \bigvee_{a\in|A|} s(a) \wedge \Big(\bigvee_{b\in|A|} [\![a = b]\!] \Big) \\
&= \bigvee_{a\in|A|} s(a) \wedge Ea \ = \ \bigvee_{a\in|A|} s(a) \ = \ Es,
\end{aligned}$$

establishing that c satisfies (2) in 27.9, as desired. □

Proof of Theorem 27.9 : [6] In view of 27.14 and 27.15, it only remains to check that cA has the extension property (3). Let $g : A \longrightarrow B$ be a morphism of Ω-sets, with B a sheaf over Ω. If $s \in |cA|$, 27.13.(b) implies that s_f is a singleton in B. Hence, by 27.13.(a) there is a *unique* $b \in |B|$ such that for all $c \in |B|$

$$s_f(c) \ = \ [\![b = c]\!]. \tag{I}$$

Set $\widehat{g}(s)$ = the unique $b \in |B|$ satisfying (I). By Lemma 27.13.(b), $\widehat{g}(ca) = fa$, for all $a \in |A|$. It is left to the reader to check that $\widehat{g}$ is a morphism of Ω-sets. Uniqueness follows (2) and 25.33.(b), completing the proof. □

We have constructed a functor

$$c : \boldsymbol{\Omega set} \longrightarrow \boldsymbol{Sh(\Omega)},$$

the **completion functor** that is left adjoint to the forgetful functor from $\boldsymbol{Sh(\Omega)}$ to $\boldsymbol{\Omega set}$. By 26.8, a presheaf over a frame Ω is a Ω-set and there is a natural correspondence between presheaf morphisms and Ω-set morphisms. Hence,

COROLLARY 27.16. *Let A be a presheaf over a frame Ω. Then, there is a sheaf cA over Ω and a presheaf morphism $c : A \longrightarrow cA$ satisfying properties* (1), (2) *and* (3) *of* 27.9. □

[6] For an alternate proof, see 27.23.

Sheaves have important morphism extension properties. Generalizing 24.15.(c), we state

THEOREM 27.17. *Let $D \subseteq A$ be extensional Ω-sets, with D dense in A and let $f : D \longrightarrow B$ be a morphism of Ω-sets. If B is a sheaf over Ω, then f has a* unique *extension, $\widehat{f}$, to A. Furthermore,*

(1) *$\widehat{f}$ is a regular monic iff f is a regular monic* [7]*;*

(2) *$\widehat{f}$ is epic iff f is epic.*

PROOF. Once the existence of $\widehat{f}$ is proven, uniqueness follows from 25.33. Because B is a sheaf, we know that it has compatible equality, extent and restriction. Since D is dense in A, for each $a \in |A|$ we have

$$Efa \;=\; Ea \;=\; \bigvee_{d \in |D|} [\![a = d]\!]. \tag{I}$$

Consider $S_a \;=\; \{(fd)_{|[\![a=d]\!]} \in |B| : d \in |D|\}$; note that for all $d \in |D|$,

$$E(fd)_{|[\![a=d]\!]} \;=\; [\![a = d]\!] \wedge Efd \;=\; [\![a = d]\!] \wedge Ed \;=\; [\![a = d]\!]. \tag{II}$$

Moreover, for d, $d' \in D$, we also have

$$[\![a = d]\!] \wedge [\![a = d']\!] \;\leq\; [\![d = d']\!] \;\leq\; [\![fd = fd']\!]. \tag{III}$$

Hence, (III) entails

$$\begin{aligned} [\![(fd)_{|[\![a=d]\!]} = (fd')_{|[\![a=d']\!]}]\!] &= [\![fd = fd']\!] \wedge [\![a = d]\!] \wedge [\![a = d']\!] \\ &= [\![a = d]\!] \wedge [\![a = d']\!], \end{aligned}$$

which, in view of (II), implies that S is compatible in B. Let $\widehat{f}a$ be the *unique* section in B that is the gluing of S_a. For $d \in |D|$, the element of largest extent in S_d is $(fd)_{|Ed} = fd$ and so, $\widehat{f}d = fd$. Hence, if $\widehat{f}$ is a morphism, it will be the required extension of f. It follows from (I) that $E\widehat{f}a = Ea$, for all $a \in |A|$. For a, $b \in |A|$, we make use of 25.31 and 25.35, to get

$$\begin{aligned} [\![\widehat{f}a = \widehat{f}b]\!] &= \bigvee_{d,d' \in |D|} [\![(fd)_{|[\![a=d]\!]} = (fd')_{|[\![b=d']\!]}]\!] \\ &= \bigvee_{d,d' \in |D|} [\![fd = fd']\!] \wedge [\![a = d]\!] \wedge [\![b = d']\!] \\ &\geq \bigvee_{d,d' \in |D|} [\![d = d']\!] \wedge [\![a = d]\!] \wedge [\![b = d']\!] \\ &= \bigvee_{d,d' \in |D|} [\![a = b]\!] \wedge [\![a = d]\!] \wedge [\![b = d']\!] \\ &= [\![a = b]\!] \wedge \bigvee_{d,d' \in |D|} [\![a = d]\!] \wedge [\![b = d']\!] \\ &= [\![a = b]\!] \wedge \bigvee_{d \in |D|} [\![a = d]\!] \wedge \bigvee_{d' \in |D|} [\![b = d']\!] \\ &= [\![a = b]\!] \wedge Ea \wedge Eb \;=\; [\![a = b]\!], \end{aligned}$$

as needed to establish that $\widehat{f}$ is a morphism. The assertion in (1) is a consequence 25.21 and the fact that, if f is monic, the only inequality in the preceding computation can be replaced by an equality. Item (2) is clear. □

From 27.8 and 27.17 we get

COROLLARY 27.18. *If $A \xrightarrow{f} B$ is a morphism of Ω-sets, then f is epic and a regular monic $\Leftrightarrow$ cf is an isomorphism.*

[7] Regular monics are defined in 25.22.

PROOF. With notation as in 27.29 (including the displayed diagram), note that cf is the the unique extension of $(c \circ f)$ to cA. Hence, if f is epic and a regular monic, cf has, by 27.17, the same properties; the conclusion follows from 27.8, because completions are sheaves. □

It is frequent that in concrete situations we know the data of a sheaf only over certain subsets of the base algebra; it is also important to establish conditions under which the extension to a sheaf preserves the original data. A typical result of this kind is Theorem 27.20, below. First, we introduce notions that are localizations of completeness and finite completeness (27.1, 25.34) :

DEFINITION 27.19. *Let L be a semilattice, $p \in L$ and A be a L-presheaf.*

a) A is **finitely complete (fc) over p** *if all compatible $S \subseteq_f |A|$, such that $\bigvee_{s \in S} Es \;=\; p$, have a* unique *gluing in A, i.e., there is $t \in |A|$ satisfying*

$$(i)\ Et = \bigvee_{s \in S} Es \qquad \text{and} \qquad (ii)\ \text{For all } s \in S,\quad s = t_{|Es}.$$

b) A is **complete over p** *if all compatible $S \subseteq |A|$ such that $p \;=\; \bigvee_{s \in S} Es$, have a* unique *gluing in A* [8].

THEOREM 27.20. *Let Ω be a frame and $L \subseteq \Omega$ be a subsemilattice which is a* **basis** *for Ω (7.1). If A is an extensional L-presheaf and $p \in L$ is such that*

(1) *A is finitely complete over p and p is compact in Ω (2.43),*

or

(2) *A is complete over p,*

then, $c_{|A(p)} : A(p) \longrightarrow cA(p)$ is a bijection.

PROOF. By 26.8, A is a Ω-set, whose equality is compatible with its extent and restriction. By 26.8 and 25.21, condition (1) in 27.9 implies that $c_{|A(p)}$ is an injection of $A(p)$ into $cA(p)$. We prove that $c_{|A(p)}$ is surjective whenever (1) holds, leaving the other alternative to the reader.

Fix $t \in cA(p)$; since L is a basis for Ω, for each $a \in |A|$, there is $\alpha_a \subseteq L$ such that $[\![ca \;=\; t]\!] \;=\; \bigvee_{q \in \alpha_a} q$. Hence, 27.9.(2) entails

$$p = Et = \bigvee_{a \in |A|} [\![ca \;=\; t]\!] = \bigvee_{a \in |A|} \bigvee \alpha_a.$$

By compactness, there are $a_1, \ldots, a_n \in |A|$ and $\beta_k \subseteq_f \alpha_{a_k}$, $1 \le k \le n$, such that

$$Et \;=\; p \;=\; \bigvee_{k=1}^{n} \bigvee \beta_k.$$

Consider the *finite* subset of A given by

$$S \;=\; \bigcup_{k=1}^{n} \{a_{k_{|q}} : q \in \beta_k\}.$$

It is clear that

$$\bigvee_{s \in S} Es \;=\; \bigvee_{k=1}^{n} \bigvee \beta_k \;=\; Et.$$

To show that S is compatible, let $q \in \beta_i$, $r \in \beta_k$, $i, k \in \underline{n}$. Recalling (1) in 27.9, as well as the relations

$$q \;\le\; [\![ca_i \;=\; t]\!] \;\le\; Ea_i \quad \text{and} \quad r \;\le\; [\![ca_k \;=\; t]\!] \;\le\; Ea_k,$$

[8] That is, $\exists\, t \in |A|$, verifying (i) and (ii) in item (a).

we obtain

* $Ea_{k|_{q\wedge r}} = Ea_k \wedge q \wedge r = q \wedge r = q \wedge r \wedge Ea_{|_i} = Ea_{i|_{q\wedge r}}$.

* $[\![a_{i|_{q\wedge r}} = a_{k|_{q\wedge r}}]\!] = q \wedge r \wedge [\![ca_i = ca_k]\!]$
$$\geq q \wedge r \wedge [\![ca_i = t]\!] \wedge [\![ca_k = t]\!] = q \wedge r,$$

and the extensionality of A entails $a_{i|_q} = a_{k|_r}$. The finite completeness of A over p yields $a \in A$ that is the gluing of S in A. Since c is a presheaf morphism (27.16), it is straightforward to verify that $ca = t$, completing the proof. □

As an illustration of Theorem 27.20, we present the structure sheaf of a commutative ring with identity, generalizing Example 22.17 and Corollary 22.18.

Example 27.21. **The structure sheaf of a commutative ring.** We shall employ the notation and results of Chapters 9 and 19, in particular the ring of fractions construction, presented in section 9.3.

Let R be a commutative ring with identity. Write R^* for the complement of the ideal η of nilpotent elements in R. Recall that

* (a) is the ideal generated by a in R;

* $Z_a = \{P \in Spec(R) : a \notin P\}$ is the basic compact open of the Zariski topology corresponding to a.

* Let $S_a =_{def} \{a^n : n \geq 0\}$. Note that S_a is a proper multiplicative set iff $a \notin \eta$.

Since η is the intersection of all prime ideals in R, we have
$$Z_a = \emptyset \quad \text{iff} \quad a \in \eta.$$

For $a \in R^*$, let $R_a = RS_a^{-1}$; recall that the elements of R_a are written x/a^n, $n \geq 0$ and $x \in R$, under the equivalence relation
$$x/a^n \equiv y/a^m \quad \text{iff} \quad \exists\, k \geq 0, \quad a^k(xa^m - ya^n) = 0,$$
with the usual definitions of addition and multiplication of fractions. If a, $b \in R$, Propositions 19.5.(c) and 9.12.(c) yield
$$\begin{aligned} Z_a \subseteq Z_b \quad & \text{iff} \quad \sqrt{a} \subseteq \sqrt{b} \quad \text{iff} \quad a \in \sqrt{b} \\ & \text{iff} \quad \exists\, n \geq 0 \text{ such that } a^n \in (b) \\ & \text{iff} \quad \exists\, n \geq 0 \text{ and } u \in R, \text{ such that } a^n = ub. \end{aligned}$$
Therefore, if a, $b \in R^*$, then
$$(*) \qquad Z_a \subseteq Z_b \;\Rightarrow\; \exists\, n \geq 0 \text{ and } u \in R \text{ such that } \frac{1}{b} = \frac{u}{a^n} \text{ in } R_a,$$
that is, b is invertible in R_a. By Proposition 9.36.(c), there is a *unique* ring homomorphism
$$\rho_{ba} : R_b \longrightarrow R_a,$$
such that for all $x \in R$,
$$(**) \qquad \rho_{ba}\left(\frac{x}{b^m}\right) = x\left(\frac{1}{b}\right)^m = \frac{xu^m}{a^{nm}},$$
with u as in (*). Moreover, if $Z_a = Z_b$, i.e., $\sqrt{a} = \sqrt{b}$, ρ_{ba} is an isomorphism, by which we identify R_b with R_a. With this convention,
$$Z_a \subseteq Z_b \subseteq Z_c \;\Rightarrow\; \rho_{ca} = \rho_{ba} \circ \rho_{cb} \quad \text{and} \quad \rho_{aa} = Id_{R_a}.$$
In particular, for each $a \in R^*$ there is a canonical ring homomorphism

$$\rho_{1a} =_{def} \rho_a : R \longrightarrow R_a, \quad \rho_a(x) = \frac{x}{1}.$$

Let $X = Spec(R)$ be the Zariski spectrum of R, as in Chapter 19 and let

$$L(R) \;=\; L \;=\; \{Z_a : a \in R^*\} \cup \{\emptyset\}.$$

By 19.5.(a), L is a semilattice, a subsemilattice of the lattice $\Lambda(X)$ of compact opens in X (19.8). Let $\mathcal{R}$ be the L-presheaf determined the prescriptions :

(1) $|\mathcal{R}| \;=\; \bigcup_{r\in R^*} R_r \times \{Z_r\} \;\cup\; \{\langle *, \emptyset \rangle\}$ [9];

(2) Define extent and restriction in $\mathcal{R}$ as follows :

$$\begin{cases} E\langle\, \xi,\, Z_r \,\rangle \;=\; Z_r; & E\langle *, \emptyset \,\rangle \;=\; \emptyset; \\ \langle\, \xi,\, Z_r \,\rangle_{|Z_s} \;=\; \langle\, \rho_{r,rs}(\xi),\, Z_{rs} \,\rangle; & \langle\, \xi,\, Z_r \,\rangle_{|\emptyset} \;=\; \langle *, \emptyset \,\rangle_{|\emptyset} \;=\; \langle *, \emptyset \,\rangle. \end{cases}$$

If $r,\, s \in R^*$, r and s are invertible in R_{rs} [10]. Thus, (**) above yields, with $\xi = \dfrac{x}{r^n}$,

$$\rho_{r,rs}\left(\frac{x}{r^n}\right) \;=\; \frac{xs^n}{r^n s^n} \;=\; \frac{x}{r^n}.$$

Since $Z_r \cap Z_s = Z_{rs}$ (19.5.(a)), we may write restriction in $\mathcal{R}$ as

(***) $$\langle\, \xi,\, Z_r \,\rangle_{|Z_s} \;=\; \langle\, \xi,\, Z_{rs} \,\rangle \;=\; \langle\, \xi,\, Z_r \,\cap\, Z_s \,\rangle,$$

while the restriction of any section of $\mathcal{R}$ to $\emptyset$ is equal to $\langle *, \emptyset \,\rangle$. It follows readily from these observations that $\mathcal{R}$ is a presheaf over L.

Fact 1. *$\mathcal{R}$ is an extensional L-presheaf.*

Proof. Suppose that $\langle\, \xi,\, Z_r \,\rangle$, $\langle\, \zeta,\, Z_r \,\rangle$ are sections in $\mathcal{R}$ and $D \subseteq L$ satisfies

$$\bigcup D \;=\; Z_r \quad\text{and}\quad \langle\, \xi,\, Z_r \,\rangle_{|d} \;=\; \langle\, \zeta,\, Z_r \,\rangle_{|d},\ \forall\, d \in D.$$

Since $\xi,\, \zeta \in R_r$, one has

$$\xi = \frac{x}{r^n} \quad\text{and}\quad \zeta = \frac{z}{r^n}.$$

We may assume that the exponent of r in both fractions are the same; otherwise, just multiply one of them by a convenient power of r to arrive at such an expression. Moreover, since Z_r is compact in X, we may suppose that D is finite,

$$D = \{Z_{a_i} : 1 \le i \le p\}.$$

For each $1 \le i \le p$, our hypothesis is that

$$\langle\, \xi,\, Z_r \,\rangle_{|Z_{a_i}} \;=\; \langle\, \zeta,\, Z_r \,\rangle_{|Z_{a_i}},$$

and so, (***) entails

$$\frac{x}{r^n} = \frac{z}{r^n} \quad \textit{in } R_{a_i}.$$

Thus, there is $k_i \ge 0$ such that

$$a_i^{k_i} r^n (x - z) \;=\; 0.$$

Consequently, if $m = \max\,\{k_i : 1 \le i \le p\}$, we obtain

(1) $$\text{For all } 1 \le i \le p, \quad a_i^m r^n (x - z) \;=\; 0.$$

[9] Note that this is a disjoint union.

[10] $\dfrac{1}{r} = \dfrac{s}{rs}$ and $\dfrac{1}{s} = \dfrac{r}{rs}$.

It is clear that for all $c \in R$ and all integers $m \geq 1$, $Z_c = Z_{c^m}$, because for all prime ideals P in R, $c \notin P \Leftrightarrow c^m \notin P$. Hence, the hypothesis that D covers Z_r amounts to

$$Z_r \;=\; \bigcup_{i=1}^{p} Z_{a_i^m},$$

where $m \geq 1$ is as in (1). By Corollary 19.9, the preceding equality means that r is in the radical of the ideal generated by $\{a_i^m : i \leq p\}$, that is, there are $k \geq 1$ and $\{c_i : 1 \leq i \leq p\} \subseteq R$, such that

$$(2) \qquad r^k \;=\; \sum_{i=1}^{p} c_i a_i^m.$$

If each of the equations in (1) is multiplied by c_i and then summed over $i \leq p$, (2) yields

$$\sum_{i=1}^{p} c_i a_i^m r^n (x - z) \;=\; r^k r^n (x - z) \;=\; 0,$$

establishing that $\xi = \zeta$ in R_r, as needed. □

Fact 2. *For all $r \in R^*$, $\mathcal{R}$ is finitely complete over Z_r.*

Proof. Let $\langle \xi_i, Z_{a_i} \rangle$, $1 \leq i \leq p$, be a finite collection of sections in $\mathcal{R}$, that are pairwise compatible and such that $Z_r = \bigcup_{i=1}^{p} Z_{a_i}$. Write

$$\xi_i = \frac{x_i}{a_i^n},$$

where, once again, we may take the exponents of the a_i to be the same. By (***), compatibility of the ξ_i is equivalent to the following ring theoretic condition :

$$\exists\, q \geq 0 \text{ such that } \forall\, i, j \leq p, \quad (a_i a_j)^q \,(a_j^n x_i - a_i^n x_j) = 0.$$

Setting $m = q + n$, the preceding equations can be written as

$$(3) \qquad \forall\, i, j \leq p, \quad a_j^m a_i^q x_i - a_i^m a_j^q x_j = 0,$$

where $m \geq 1$. The fact that $\{Z_{a_i}\}$ is a covering of Z_r is expressed by equation (2) in the proof of Fact 1. Let

$$x = \sum_{i=1}^{p} c_i a_i^q x_i,$$

where the c_i are as in (2) above. Then, (3) yields

$$\begin{aligned} a_j^q r^k x_j \;&=\; a_j^q (\textstyle\sum_{i=1}^{p} c_i a_i^m) x_j \;=\; \sum_{i=1}^{p} c_i a_j^q a_i^m x_j \;=\; \sum_{i=1}^{p} c_i a_j^m a_i^q x_i \\ &=\; a_j^m \textstyle\sum_{i=1}^{p} c_i a_i^q x_i \;=\; a_j^m x, \end{aligned}$$

and so, since $m = q + n$, $a_j^q (r^k x_j - a_j^n x) \;=\; 0$, proving that $\langle x/r^k, Z_r \rangle_{|Z_{a_j}} = \langle \xi_j, Z_{a_j} \rangle$, $1 \leq j \leq p$. Hence, $\langle x/r^k, Z_r \rangle$ is the 'gluing' of the given compatible family, completing the proof of Fact 2.

Since L is a basis of *compact* opens for the Zariski topology on $X = Spec(R)$, Theorem 27.20 applies to show that the completion of the L-presheaf $\mathcal{R}$, $c\mathcal{R}$, is such that

$$\text{For all } r \in R^*, \quad c\mathcal{R}(Z_r) \;=\; \mathcal{R}(Z_r) \;=\; R_r.$$

In particular, since X is compact, this applies to the global sections of $\mathcal{R}$ and $c\mathcal{R}$ and we have

$$c\mathcal{R}(X) \;=\; \mathcal{R}(X) \;=\; R.$$

The sheaf $c\mathcal{R}$ is the **structure sheaf** of R, due to A. Grothendieck. These sheaves, the *affine schemes*, are the basic building blocks of modern Algebraic Geometry

([**67**], [**25**], [**24**]). A general *scheme* is a sheaf of rings that is locally affine. References for the very important theory of *algebraic groups* are [**27**] and [**56**]. $\square$

Example 27.21 yields

THEOREM 27.22. *A commutative ring with identity, R, is isomorphic to the global sections of a sheaf of rings over $Spec(R)$. Moreover, if $a \in (R - \eta)$, the sections of this sheaf over the compact open Z_a is the ring of fractions RS_a^{-1}, where $S_a = \{a^n : n \geq 0\}$.* $\square$

The last theme of this Chapter is the preservation of finite products of presheaves by the functor c. It should be said that c *does not, in general*, preserve products of Ω-sets or infinite products of Ω-presheaves (see Exercise 27.33). However, there are conditions under which finite products are preserved. For instance, we have

PROPOSITION 27.23. *If $A_1, \ldots, A_n$ are presheaves over Ω, then* [11]

$$c(\textstyle\prod_{i=1}^n A_i) \quad = \quad \prod_{i=1}^n cA_i.$$

PROOF. It is enough to verify the statement for $n = 2$. Let A, B be presheaves over Ω and

$$f : A \times B \longrightarrow cA \times cB, \text{ given by } \quad \langle a, b \rangle \mapsto \langle ca, cb \rangle. \qquad [12]$$

We prove that if T is a Ω-sheaf and $g : A \times B \longrightarrow T$ is a morphism, then there is a unique $\widehat{g} : cA \times cB \longrightarrow T$, such that the following diagram commutes :

(D)

$$\begin{tikzcd} A \times B \arrow[rr, "f"] \arrow[dr, "g"'] & & cA \times cB \arrow[dl, "\widehat{g}"] \\ & T & \end{tikzcd}$$

The universal property (3) in 27.9 will then imply that $c(A \times B)$ is isomorphic to $cA \times cB$.

Let $\langle x, y \rangle \in |cA \times cB|$; then $Ex = Ey$ and (2) in 27.9 yields

$$Ex = \textstyle\bigvee_{a \in |A|} [\![x = ca]\!] \quad \text{and} \quad Ey = \bigvee_{b \in |B|} [\![y = cb]\!],$$

and so taking meets we obtain

$$Ex = Ey = \textstyle\bigvee_{a \in A, b \in B} [\![x = ca]\!] \wedge [\![y = cb]\!]. \tag{I}$$

For $a \in |A|$ and $b \in |B|$, set

$$p(a, b) \quad = \quad ([\![x = ca]\!] \wedge [\![y = cb]\!]).$$

Since $Ea = Eca$ and $Eb = Ecb$, we have

$$Ea \wedge p(a, b) = Eca \wedge p(a, b) = p(a, b) = Eb \wedge p(a, b).$$

Hence, for $\langle a, b \rangle \in |A \times B|$,

$$Ea_{|p(a,b)} \quad = \quad p(a, b) \quad = \quad Eb_{|p(a,b)}. \tag{II}$$

[11] Equality, up to isomorphism, of course.

[12] Hence, $f = c \times c$.

Thus, $S_{xy} = \{\langle\, a_{|p(a,b)}, b_{|p(a,b)} \rangle : a \in |A| \text{ and } b \in |B|\} \subseteq |A \times B|$. Moreover, if $a' \in |A|$ and $b' \in |B|$, the relations

$$\begin{cases} [\![x = ca]\!] \wedge [\![x = ca']\!] \;\le\; [\![ca = ca']\!]; & ([=2] \text{ in } 25.1); \\ [\![a = a']\!] \;=\; [\![ca = ca']\!], & ((1) \text{ in } 27.9), \end{cases}$$

entail, together with 26.8.(b),

$$\begin{aligned} [\![a_{|p(a,b)} = a'_{|p(a',b')}]\!] &= [\![a = a']\!] \wedge p(a,b) \wedge p(a',b') = \\ &= [\![ca = ca']\!] \wedge [\![x = ca]\!] \wedge [\![y = cb]\!] \wedge [\![x = ca']\!] \wedge [\![y = cb']\!] \\ &= p(a,b) \wedge p(a',b'). \end{aligned} \tag{III}$$

Since the role of a and a' may be played by b and b', we also get

$$\begin{aligned} [\![b_{|p(a,b)} = b'_{|p(a',b')}]\!] &= [\![b = b']\!] \wedge p(a,b) \wedge p(a',b') \\ &= p(a,b) \wedge p(a',b'). \end{aligned} \tag{IV}$$

(II), (III) and (IV) imply that S_{xy} is compatible in $A \times B$. Indeed, we have

$$\begin{aligned} E\langle\, a_{|p(a,b)}, b_{|p(a,b)} \rangle \wedge E\langle\, a'_{|p(a',b')}, b'_{|p(a',b')} \rangle &= p(a,b) \wedge p(a',b') = \\ &= [\![a_{|p(a,b)} = a'_{|p(a',b')}]\!] \wedge [\![b_{|p(a,b)} = b'_{|p(a',b')}]\!] \\ &= [\![\langle\, a_{|p(a,b)}, b_{|p(a,b)} \rangle = \langle\, a'_{|p(a',b')}, b'_{|p(a',b')} \rangle]\!], \end{aligned}$$

as claimed. Since g is a morphism, $g(S_{xy}) = \{gs : s \in S_{xy}\}$ is compatible in T (26.23.(e)). Thus, there is a *unique* $\xi \in |T|$ that is the gluing of $g(S_{xy})$. Define

$$\widehat{g}(x,y) = \xi.$$

Note that (I) implies that $E\xi = Ex = Ey = E\langle\, x, y \rangle$. It is left to the reader to check that $\widehat{g}$ is a morphism of Ω-sets. For $\langle\, a, b \rangle \in |A \times B|$,

$$a_{|[\![ca=ca]\!]} = a_{|Ea} = a \quad \text{and} \quad b_{|[\![cb=cb]\!]} = b_{|Eb} = b,$$

and so in case $x = ca$ and $y = cb$, $\langle\, a, b \rangle$ is the gluing of $S_{ca,cb}$ in $A \times B$. Hence, $\widehat{g}(ca, cb) = f(a, b)$, and diagram (D) is commutative. Uniqueness of $\widehat{g}$ comes from the denseness of the image of the completion map, the commutativity of diagram (D) and 26.22. □

27.24. **The category $\boldsymbol{Sh(\Omega)}$.** The following categorical constructs lead directly from sheaves to sheaves

$$\begin{cases} * \;\text{ Products (26.12);} & * \;\text{ Fibered product over a map (26.15);} \\ * \;\text{ Equalizers (26.19);} & * \;\text{ Initial and Final object (26.18);} \\ * \;\text{ Coproducts (26.20);} & \end{cases}$$

To see that the coproduct of sheaves is a sheaf, just observe that two sections in a coproduct are compatible iff they belong to the same component and are compatible therein.

From the list above, we conclude that $\boldsymbol{Sh(\Omega)}$ is a complete category. It is also cocomplete, but to get coequalizers is slightly more delicate, as quotients (to be discussed in Chapter 43) do not immediately furnish sheaves. However, the way to correct that is exactly the same as that discussed in section 24.3 : take the completion of the presheaf or Ω-set version of the construction. □

DEFINITION 27.25. *A Ω-presheaf A is* **locally flabby** [13] *if for all $p \in \Omega$, there is $\alpha \subseteq \Omega$ such that*
$*$ *$p = \bigvee \alpha$;*
$*$ *For all $q \in \alpha$, the restriction map from $A(p)$ to $A(q)$ is surjective.*

EXAMPLE 27.26. Let $X = 2^{\omega}$ be the Cantor space, that is, the product of ω copies of $2 = \{0, 1\}$, with the product topology. X is a compact metric space, whose distance function is given by

$$d(P, Q) \;=\; \begin{cases} \dfrac{1}{1 \;+\; \min\{k \in \omega \;:\; P(k) \neq Q(k)\}} & \text{if } P \neq Q; \\ 0 & \text{otherwise.} \end{cases}$$

If P is a point in X, the map $f_P : X \longrightarrow \mathbb{R}$, defined by

$$f_P(Q) = d(P, Q),$$

is continuous, being equal to zero only at P. Consequently, $g_P = \dfrac{1}{f_P}$ is continuous and unbounded in the open set $U = X - \{P\}$. This construction shows that the sheaf of continuous functions on X, $\mathbb{C}(X)$, is locally flabby, although it *is not* flabby. Indeed,
$*$ Since X is Boolean, every open in X is the union of a (countable) family of clopens;
$*$ Any continuous real map defined on a clopen in X can be extended (in many ways) to a continuous map on X. Hence, the restriction map from global sections to any clopen is surjective;
$*$ The above arguments show that $\mathbb{C}(X)$ is locally flabby. On the other hand, since any continuous map on a compact is bounded, it is clear that $g_P \in \mathbb{C}(U)$ cannot be the restriction of a global section of $\mathbb{C}(X)$.

The reader should check that constant sheaves on X are also examples of locally flabby, non-flabby, presheaves.

Exercise 27.34, below, describes an important property of locally flabby presheaves, generalizing Proposition 24.49.(b) □

Exercises

27.27. If Ω is a frame, then $\boldsymbol{Sh(\Omega)}$ is closed under products (25.12), fibered products (25.17), equalizers (25.18) and coproducts (25.19), constituting a complete category. □

27.28. Let $2 = \{\bot, \top\}$ be the two-element BA. Generalizing 23.25, show that the categories $\boldsymbol{pSh(2)}$, $\boldsymbol{2set}$ and $\boldsymbol{Sh(2)}$ are all naturally isomorphic to $\boldsymbol{Set}$. □

27.29. a) With notation as in 27.13.(b), let $A \xrightarrow{f} B$ be a morphism of Ω-sets. If $s, t \in |cA|$, then

$$Es = Es_f \quad \text{and} \quad [\![s \;=\; t]\!] \leq [\![s_f \;=\; t_f]\!].$$

[13] Compare with 26.28.(e).

Conclude that $s \in |cA| \mapsto s_f \in |cB|$ is a morphism of Ω-sets, **indicated by cf**, making the following diagram commutative :

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "c"'] & B \arrow[d, "c"] \\ cA \arrow[r, "cf"'] & cB \end{tikzcd}$$

b) We have $c(Id_A) = Id_{cA}$ and $c(g \circ f) = cf \circ cg$. $\square$

27.30. Let $c\mathcal{R}$ be the structure sheaf of the commutative ring with identity R, as in 27.21. Let $U = \bigcup_{i=1}^{n} Z_{a_i}$ be a compact open in $Spec(R)$ and

$$M_U \quad = \quad \bigcap \{P^c : P \in U\}$$

be the saturated multiplicative set associated to U (9.27). Then,

$$c\mathcal{R}(U) = RM_U^{-1},$$

that is, the ring of sections of $c\mathcal{R}$ over U is the ring of fractions of R relative to the saturated multiplicative set M_U [14]. $\square$

27.31. Notation as in 27.21, let $P \in Spec(R)$ be a prime ideal in R. Let

$$\nu_P \quad = \quad \{Z_a \in L : P \in Z_a\} \quad = \quad \{Z_a : a \notin P\}.$$

a) $(\nu_P)^{op}$ is up-directed.

b) Let $\mathcal{P} = \langle R_a; \{\rho_{ba} : Z_a \subseteq Z_b \text{ in } \nu_P\} \rangle$ be the inductive system associated to the opposite of the inclusion order in ν_P. Then, $R_P \; = \; \varinjlim \mathcal{P}$.

c) Construct a *geometric* sheaf of rings $\mathfrak{R}$ over $Spec(R)$ whose stalk at $P \in Spec(R)$ is R_P and that is *isomorphic* to $c\mathcal{R}$ [15]. $\square$

27.32. Let R be a commutative regular ring. Give a description, in terms of its idempotents, of the structure sheaf of R. $\square$

27.33. a) If A_n are the presheaves of 26.14, then $cA_n = cA$, for all $n \geq 1$.

b) Construct Ω-sets A, B and a sequence of Ω-presheaves A_n, such that

$$c(A \times B) \neq cA \times cB \quad \text{and} \quad c\left(\prod_{n \geq 1} A_n\right) \neq \prod_{n \geq 1} cA_n. \qquad \square$$

27.34. Let A be a locally flabby presheaf over a frame Ω (27.25). If B is a sheaf over Ω, the map

$$\eta \in [A, B] \quad \longmapsto \quad \eta_{|A(\top)} \in B(\top)^{A(\top)}$$

is a natural bijective correspondence between the morphisms from A to B and the set of maps from $A(\top)$ to $B(\top)$. $\square$

[14] By 9.38.(b), this is consistent with $c\mathcal{R}(Z_a) = \mathcal{R}(Z_a) = R_a$.

[15] Example 22.17 gives an indication on how to proceed.

CHAPTER 28

Strict Equality

If H is a Heyting algebra and A is a H-set, there is another type of equality that can be defined in A, called *strict equality*, due to Dana Scott ([**65**], [**15**]).

DEFINITION 28.1. *If A is a H-set, define, for $x, y \in |A|$*

$$[x \equiv y] \;=\; (Ex \vee Ey) \to [\![x = y]\!],$$

called the **strict equality** *between x and y in A.*

The next Lemma rephrases familiar notions in terms of $[\cdot \equiv \cdot]$.

LEMMA 28.2. *If A is a H-set and $x, y, z \in |A|$, then*

a) $[\![x = y]\!] \;\leq\; [x \equiv y] \;=\; [y \equiv x]$.

b) $[x \equiv y] \wedge [y \equiv z] \;\leq\; [x \equiv z]$.

c) $(Ex \vee Ey) \wedge [x \equiv y] \;=\; Ex \wedge Ey \wedge [x \equiv y] \;=\; [\![x = y]\!]$.

d) $\{x, y\}$ *is compatible in A iff* $(Ex \wedge Ey) \;\leq\; [x \equiv y]$.

e) The following are equivalent :

(1) *A is extensional* (2) $[x \equiv y] = \top \;\Rightarrow\; x = y$.

f) If B is a H-set and $f : |A| \longrightarrow |B|$ is a map, the following are equivalent :

(1) *f is a H-set morphism;*

(2) *For all $x, y \in |A|$,* $\begin{cases} (i)\ Efx = Ex; \\ (ii)\ [x \equiv y] \;\leq\; [fx \equiv fy]. \end{cases}$

g) If A is an extensional H-presheaf, then for all $p \in H$,

$$x_{|p} = y_{|p} \quad \textit{iff} \quad p \;\leq\; [x \equiv y].$$

PROOF. Item (a) is clear; for (b), 6.4.(b) (or $[\to]$ in 6.1) yields

$$\begin{aligned} [x \equiv y] \wedge [y \equiv z] \wedge (Ex \vee Ez) \;&\leq \\ &\leq\; (Ex \vee Ey \vee Ez) \wedge [x \equiv y] \wedge [y \equiv z] \\ &\leq\; [\![x = y]\!] \wedge [\![y = z]\!] \;\leq\; [\![x = z]\!], \end{aligned}$$

and the adjunction $[\to]$ in 6.1 entails the desired conclusion.

c) We have

$$Ex \wedge Ey \wedge [x \equiv y] \;\leq\; (Ex \vee Ey) \wedge [x \equiv y] \;\leq\; [\![x = y]\!],$$

and the second equality follows from (a). The remaining statement and (d) are clear.

e) Assume that A is extensional and $[x \equiv y] = \top$. By 6.4.(a), $Ex \vee Ey = [\![x = y]\!]$ and so $x = y$. Now suppose $Ex = Ey = [\![x = y]\!]$. Clearly, $[x \equiv y] = \top$, and (2) entails $x = y$, as needed. Item (f) is left as an exercise. Regarding (g), just observe that the proof of the corresponding result in 26.8.(d) does not use completeness and thus holds true in any HA. $\square$

Note that $\langle A, [\cdot \equiv \cdot] \rangle$ is a H-set in which all sections are global ($[x \equiv x] = \top$).

In the case of the product of presheaves, one must exercise care in defining strict equality for elements in the product of their domains.

28.3. **Strict equality for vectors.** Let A_i, $i \in I$, be a family of Ω-sets. For $\overline{x}, \overline{y} \in \prod_{i \in I} |A_i|$, define

$$[\overline{x} \equiv \overline{y}] \;=\; (E\overline{x} \vee E\overline{y}) \to [\![\overline{x} = \overline{y}]\!].$$

Note that $[\overline{x} \equiv \overline{y}]$ *is not* the meet of the strict equality of the components of $\overline{x}$ and $\overline{y}$. In fact, we have

$$[\overline{x} \equiv \overline{y}] \;=\; [\overline{x}_{|E\overline{x}} \equiv \overline{y}_{|E\overline{y}}],$$

where the strict equality on the right-hand side is that of $\prod_{i \in I} A_i$. $\square$

28.4. **The $\boldsymbol{H}$-set $\mathfrak{s}_k \boldsymbol{A}$.** Let $k : H \longrightarrow L$ be a semilattice morphism, *which will remain fixed until further notice.* If A is a H-set, define a H-set, $\mathfrak{s}_k A$, by the following prescriptions :

a) $|\mathfrak{s}_k A| \;=\; \{\langle x, p \rangle \in |A| \times L : Ex \leq p$ and $k(Ex) = k(p)\}$;

b) $[\![\langle x, p \rangle = \langle y, q \rangle]\!] \;=\; p \wedge q \wedge [x \equiv y]$.

Note that $E\langle x, p \rangle = p \wedge [x \equiv x] = p \wedge \top = p$. It is straightforward from 28.2 that $\mathfrak{s}_k A$ is a H-set. For $x \in |A|$, set $\mathfrak{s}_k x = \langle x, Ex \rangle \in |\mathfrak{s}_k A|$. $\square$

Proposition 28.5. *Let $h : H \longrightarrow L$ be a semilattice morphism and A an extensional H-set. With notation as above,*

a) $\mathfrak{s}_k A$ is extensional and $\mathfrak{s}_k$ is a monic in H-set. If k is injective, then $\mathfrak{s}_k$ is an isomorphism, by which we may identify A and $\mathfrak{s}_k A$.

b) If $f : A \longrightarrow B$ is a morphism of H-sets, the map

$$\mathfrak{s}_k f : |\mathfrak{s}_k A| \longrightarrow |\mathfrak{s}_k B|, \text{ given by } \mathfrak{s}_k f(\langle x, p \rangle) = \langle fx, p \rangle,$$

is the carrier of a morphism of H-sets, $\mathfrak{s}_k f$, making the following diagram commutative :

$$\begin{tikzcd} A \arrow[r, "\mathfrak{s}_k"] \arrow[d, "f"'] & \mathfrak{s}_k A \arrow[d, "\mathfrak{s}_k f"] \\ B \arrow[r, "\mathfrak{s}_k"'] & \mathfrak{s}_k B \end{tikzcd}$$

Moreover, $\mathfrak{s}_k(Id_A) = Id_{\mathfrak{s}_k A}$ *and* $\mathfrak{s}_k(g \circ f) = \mathfrak{s}_k g \circ \mathfrak{s}_k f$.

c) If A is a H-presheaf, whose restriction is compatible with its equality (26.6), *then, the rule*

$$\langle\, x, p\,\rangle_{|q} = \langle\, x_{|q}, \, p \wedge q\,\rangle$$

yields a restriction in $\mathfrak{s}_k A$, compatible with its equality and with which it is a H-presheaf.

d) If H is a frame, k is a $[\wedge, \bigvee]$-morphism and A is a H-sheaf, then $\mathfrak{s}_k A$ is a sheaf over H.

PROOF. a) Assume that $E\langle\, x, p\,\rangle \;=\; E\langle\, y, q\,\rangle \;=\; p \wedge q \wedge [x \equiv y]$. Then, $p \;=\; q \;=\; p \wedge [x \equiv y]$, that is

$$Ex \vee Ey \;\leq\; p \;\leq\; [x \equiv y]$$

and so, 28.2.(c) yields

$$(Ex \vee Ey) \;=\; (Ex \vee Ey) \wedge [x \equiv y] \;=\; [\![x = y]\!],$$

and the extensionality of A forces $x = y$, as desired. For $a, b \in |A|$, 28.2.(c) entails

$$\begin{aligned} [\![\mathfrak{s}_k a = \mathfrak{s}_k b]\!] &= [\![\langle\, a, Ea\,\rangle = \langle\, b, Eb\,\rangle]\!] \;=\; Ea \wedge Eb \wedge [a \equiv b] \\ &= [\![a = b]\!], \end{aligned}$$

and $\mathfrak{s}_k$ is a monic by 25.21. If k is injective, then $|\mathfrak{s}_k A| \;=\; \{\langle\, x, Ex\,\rangle : x \in |A|\}$, and it is clear that $\mathfrak{s}_k$ is an isomorphism.

b) It is obvious that $\mathfrak{s}_k f$ preserves extent. For $\langle\, x, p\,\rangle$, $\langle\, y, q\,\rangle \in |\mathfrak{s}_k A|$, item (2) in 28.2.(e) entails

$$\begin{aligned} [\![\mathfrak{s}_k f(\langle\, x, p\,\rangle) = \mathfrak{s}_k f(\langle\, y, q\,\rangle)]\!] &= [\![\langle\, fx, p\,\rangle = \langle\, fy, q\,\rangle]\!] \\ &= q \wedge p \wedge [fx \equiv fy] \\ &\geq p \wedge q \wedge [x \equiv y] \\ &= [\![\langle\, x, p\,\rangle = \langle\, y, q\,\rangle]\!], \end{aligned}$$

and $\mathfrak{s}_k f$ is a morphism of H-sets. The commutativity of the displayed diagram is clear. The remaining assertion in (b) is left to the reader.

c) We check only that restriction and equality are compatible, omitting other details. For $\langle\, x, p\,\rangle$, $\langle\, y, q\,\rangle \in |\mathfrak{s}_k A|$, r, $r' \in H$, first observe that

$$r \wedge r' \wedge ((Ex \wedge r) \vee (Ey \wedge r') \;=\; r \wedge r' \wedge (Ex \vee Ey). \tag{1}$$

Hence, (1) and 6.4.(i) furnish

$$\begin{aligned} &[\![\langle\, x_{|r}, p \wedge r\,\rangle = \langle\, y_{|r'}, q \wedge r'\,\rangle]\!] \;=\; p \wedge q \wedge r \wedge r' \wedge [x_{|r} \equiv y_{|r'}] \\ &\quad= p \wedge q \wedge r \wedge r' \wedge (((Ex \wedge r) \vee (Ey \wedge r')) \to [\![x_{|r} = y_{|r'}]\!]) \\ &\quad= p \wedge q \wedge r \wedge r' \wedge (((r \wedge r') \wedge (Ex \vee Ey)) \to (r \wedge r' \wedge [\![x = y]\!])) \\ &\quad= p \wedge q \wedge r \wedge r' \wedge ((Ex \vee Ey) \to [\![x = y]\!]) \\ &\quad= r \wedge r' \wedge [\![\langle\, x, p\,\rangle = \langle\, y, q\,\rangle]\!], \end{aligned}$$

establishing the compatibility of restriction and equality in $\mathfrak{s}_k A$.

d) Let $\langle\, x_i, p_i\,\rangle$, $i \in I$, be a compatible family of sections in $\mathfrak{s}_k A$. This means that for all i, $j \in I$,

$$p_i \wedge p_j = [\![\langle\, x_i, p_i\,\rangle = \langle\, x_j, p_j\,\rangle]\!] = p_i \wedge p_j \wedge [x_i \equiv x_j].$$

Hence, as in item (a), from

(C) $$Ex_i \wedge Ex_j \;\le\; p_i \wedge p_j \;\le\; [x_i \equiv x_j],$$

we conclude that $Ex_i \wedge Ex_j \;=\; [\![x_i = x_j]\!]$, i.e., the x_i are compatible; let x be their gluing in A. We have, for $i \in I$,

(G) $$Ex \;=\; \bigvee i \in I \; Ex_i \quad \text{and} \quad Ex_i \;=\; [\![x = x_i]\!].$$

Let $q \;=\; \bigvee_{i \in I} p_i$; we shall verify that $\langle x, q \rangle$ is the gluing of the originally given family in $\mathfrak{s}_k A$. Since h is a $[\wedge, \bigvee]$-morphism, the first equation in (G) entails

$$k(q) \;=\; \bigvee_{i \in I} k(p_i) = \bigvee_{i \in I} k(Ex_i) = k(Ex),$$

and so $\langle x, q \rangle \in |\mathfrak{s}_k A|$, with the required extent. To end the proof we establish

Fact. *For* $i \in I$, $\;p_i \wedge Ex \;\le\; Ex_i$.

Proof. We shall prove a bit more. Fix $i, j \in I$. Formula (C) above guarantees that

$$p_i \wedge Ex_j \;=\; p_i \wedge Ex_j \wedge [x_i \equiv x_j].$$

Since $p_i \wedge Ex_j \;\le\; (Ex_j \vee Ex_i)$, 6.4.(i) and *Modus Ponens* (6.4.(b)) yield

$$\begin{aligned} p_i \wedge Ex_j \wedge [x_i \equiv x_j] \;&=\; p_i \wedge Ex_j \wedge ((Ex_i \vee Ex_j) \to [\![x_i = x_j]\!]) \\ &=\; p_i \wedge Ex_j \wedge ((p_i \wedge Ex_j) \to [\![x_i = x_j]\!]) \\ &\le\; [\![x_i = x_j]\!] \;\le\; Ex_i, \end{aligned}$$

establishing the Fact.

If $i \in I$, the Fact above, the second equation in (G), together with items (a) and (i) in 6.4, imply

$$\begin{aligned} [\![\langle x, q \rangle = \langle x_i, p_i \rangle]\!] \;&=\; q \wedge p_i \wedge [x \equiv x_i] \;=\; p_i \wedge (Ex \to [\![x = x_i]\!]) \\ &=\; p_i \wedge ((p_i \wedge Ex) \to Ex_i) \;=\; p_i \wedge \top \\ &=\; p_i \;=\; E\langle x_i, p_i \rangle, \end{aligned}$$

verifying that $\langle x, q \rangle$ is the gluing of the $\langle x_i, p_i \rangle$ in $\mathfrak{s}_k A$. $\square$

The preceding construction has furnished endo-functors, all indicated by the same symbol,

$$\mathfrak{s}_k : \boldsymbol{Hset} \longrightarrow \boldsymbol{Hset}, \;\; \mathfrak{s}_k : \boldsymbol{pSh(H)} \longrightarrow \boldsymbol{pSh(H)},$$

and when H is a frame and k is a $[\wedge, \bigvee]$-morphism,

$$\mathfrak{s}_k : \boldsymbol{Sh(H)} \longrightarrow \boldsymbol{Sh(H)},$$

called the **$\mathfrak{s}$-functors** associated to k.

Exercises

28.6. Let $k : H \longrightarrow L$ be a semilattice morphism and A, B be extensional H-sets.

a) The map $\beta : |\mathfrak{s}_k(A \times B)| \longrightarrow |\mathfrak{s}_k A \times \mathfrak{s}_k B|$, defined by

$$\langle \langle a, b \rangle, p \rangle \longmapsto \langle \langle a, p \rangle, \langle b, p \rangle \rangle$$

is a H-set isomorphism from $\mathfrak{s}_k(A \times B)$ onto $\mathfrak{s}_k A \times \mathfrak{s}_k B$.

b) The functor $\mathfrak{s}_k$ preserves non-empty products, equalizers and monics. $\square$

Part 6

Change of Base

In applications of sheaf theory to Algebraic Geometry, Algebraic Topology and the Theory of Complex Analytic Spaces, being able to move, along a continuous function, from the category of sheaves over one space to the category of sheaves over another space is an essential part of the methods that have been developed in the last 50 years. Even a cursory glance at the classical way to define image and inverse image shows that the "stalk at a point" is used frequently. Since our base algebras might not have points at all, we shall here introduce another way to arrive at the concepts of image and inverse image along a map.

For sheaves (or presheaves) over a topological space with values in an Abelian category, there are other functors which are also central, e.g., images with proper support. The functors image, inverse image, image with proper supports and their derived functors constitute the basis of important methods, with applications in all of the above mentioned theories. For more information on these topics, the reader is referred to [**37**] and [**32**].

The first chapter discusses the definition of morphisms between objects with different bases. We then present the construction of image and inverse image, along a semilattice morphism, of $*$-sets and presheaves. The following table summarizes the pertinent constructions and associated functors, where $h : L \longrightarrow R$ is a semilattice morphism.

Construction	Functor	Reference
Image of a L-set	ε_h	Chapter 30
Base Extension for Presheaves	$\mathfrak{e}_h$	Chapter 31
Essential Image of a Presheaf along h	η_h	32.1, 32.2
Image of a L-presheaf along h	$\mathfrak{d}_h$	Chapter 32
Inverse image of R-presheaf along h	$\mathfrak{i}_h$	Chapter 33

The sixth chapter of this part is devoted to the notions of **localization, fibers and stalks**, generalizing well-known concepts in the topological setting.

The final chapter, **regularization**, gives an useful description of the inverse image of the image of an object along a semilattice morphism.

In all that follows, semilattices have $\perp$ and $\top$.

CHAPTER 29

Introduction

The following definition sets down the concept of morphism between objects with (possibly) different base algebras. In 29.3, we register the classical concept of morphism of presheaves over distinct topological spaces, while 29.4 indicates why it is a special case of the abstract version below.

DEFINITION 29.1. *Let L, R be semilattices.*
a) If A is a L-set and B is a R-set, a **morphism** *from A to B consists of a pair of maps $\langle f, h \rangle$, $L \xrightarrow{h} R$ and $|A| \xrightarrow{f} |B|$, such that*

$*$ *h is a semilattice morphism;*

$$* \text{ For all } x,\ y \in |A|, \quad \begin{cases} [\text{mor } 1]: & E_B f x = h(E_A x); \\ [\text{mor } 2]: & [\![fx = fy]\!]_B \geq h([\![x = y]\!]_A). \end{cases}$$

Let $[A, B]_h$ be the set of morphisms of the form $\langle f, h \rangle$ from A to B. When $h = Id_L$, we omit the subscript corresponding to h.

b) If P is a L-presheaf and Q is a R-presheaf, a **morphism** *from P to Q is a pair of maps, $\langle f, h \rangle$, $h : L \longrightarrow R$ and $f : |P| \longrightarrow |Q|$, such that*

$*$ *h is a semilattice morphism;*

$$* \text{ For all } \langle x, p \rangle \in |P| \times L, \quad \begin{cases} [\text{pmor } 1]: & Efx = h(Ex); \\ [\text{pmor } 2]: & f(x_{|p}) = (fx)_{|h(p)}. \end{cases}$$

As for $*$-sets, write $[A, B]_h$ for the set of morphisms of the form $\langle f, h \rangle$ from A to B, omitting the subscript h when it is Id_L.

When $h = Id_L$, we get back the notions of morphism of L-sets or of L-presheaves as in 25.10 and 26.1.(f), respectively. As usual, write

$$\langle f, h \rangle : A \longrightarrow B$$

to indicate that $\langle f, h \rangle$ is a morphism from A to B (be they $*$-sets or $*$-presheaves).

If $\langle g, k \rangle : B \longrightarrow C$ is a morphism from the R-set B to the T-set C, then **composition** of $\langle f, h \rangle$ and $\langle g, k \rangle$ is defined as

$$\langle g, k \rangle \circ \langle f, h \rangle = \langle g \circ f,\ k \circ h \rangle.$$

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, dash] & B \arrow[r, "g"] \arrow[d, dash] & C \arrow[d, dash] \\
L \arrow[r, "h"'] & R \arrow[r, "k"'] & T
\end{tikzcd}
$$

Note that the identity of A is $\langle Id_A, Id_L \rangle$. With this concept of morphism, we obtain a category, ***SLset***, whose objects are extensional L-sets, L a semilattice.

Similarly, if $\langle f, h \rangle : A \longrightarrow B$ and $\langle g, k \rangle : B \longrightarrow C$ are morphisms from the L-presheaf A to the R-presheaf B, and from B to the T-presheaf C, then **composition** of $\langle f, h \rangle$ and $\langle g, k \rangle$ is defined as

$$\langle g, k \rangle \circ \langle f, h \rangle = \langle g \circ f,\, k \circ h \rangle,$$

as for L-sets. As before, the identity of A is $\langle Id_A, Id_L \rangle$. ***pSh*** is the category whose objects are presheaves over some semilattice, with morphisms as above.

Example 29.2. This example shows that a homomorphism of commutative rings with identity, $f : A \longrightarrow B$, induces a morphism between the presheaves associated to A and B, as in Example 27.21, whose notation will be used here. Recall from 27.21 that

$*$ $R^* = R - \eta$, the complement of the ideal of nilpotent elements in R;

$*$ $L(A) = \{Z_a : a \in A^*\} \cup \{\emptyset\}$ and $L(B) = \{Z_b : b \in B^*\} \cup \{\emptyset\}$,

are the semilattices of basic compact opens in $Spec(A)$ and $Spec(B)$, respectively;

$*$ $\mathcal{A}$ and $\mathcal{B}$ are the presheaves over $L(A)$ and $L(B)$, respectively, constructed therein, whose domains are given by

$$\begin{cases} |\mathcal{A}| &=& \bigcup_{a \in A^*} A_a \times \{Z_a\} \cup \{\langle *, \emptyset \rangle\} \\ |\mathcal{B}| &=& \bigcup_{b \in B^*} B_b \times \{Z_b\} \cup \{\langle *, \emptyset \rangle\}, \end{cases}$$

with extent and restriction given by

$$\begin{cases} & E\langle \xi, Z_a \rangle \;=\; Z_a; \\ \langle \xi, Z_a \rangle_{|Z_c} &=\; \langle \xi, Z_{ac} \rangle \;=\; \langle \xi, Z_a \cap Z_c \rangle, \end{cases}$$

with a similar formulas holding for $\mathcal{B}$. For $\xi = x/a^n \in A_a = AS_a^{-1}$, write

$$f\xi \;=\; \frac{fx}{(fa)^n} \;\in\; BS_{fa}^{-1}. \tag{1}$$

By Proposition 19.5.(f), f induces a continuous map

$$f_Z : Spec(B) \longrightarrow Spec(A), \quad Q \mapsto f^{-1}(Q),$$

such that for all $a \in A$,

$$f_Z^{-1}(Z_a) = Z_{fa}.$$

Thus, $f_Z^* = f_Z^{-1}$ restricts to a semilattice morphism $L(A) \xrightarrow{h} L(B)$, $Z_a \longmapsto Z_{fa}$.

Define $\gamma : |\mathcal{A}| \longrightarrow |\mathcal{B}|$ by

$$\gamma(\langle \xi, Z_a \rangle) = \langle f\xi, Z_{fa} \rangle.$$

By (1) above the value of γ is indeed in $|\mathcal{B}|$. Since $h(Z_a) = Z_{fa}$, γ satisfies [*pmor* 1] in 29.1.(b). If $\langle \xi, Z_a \rangle \in |\mathcal{A}|$ and $Z_c \in L(A)$, we have

$$\begin{aligned} \gamma(\langle \xi, Z_a \rangle_{|Z_c}) &= \gamma(\langle \xi, Z_{ac} \rangle) = \langle f\xi, Z_{f(ac)} \rangle = \langle f\xi, Z_{fafc} \rangle \\ &= \gamma(\langle \xi, Z_a \rangle)_{|Z_{fc}} = \gamma(\langle \xi, Z_a \rangle)_{|h(Z_c)}, \end{aligned}$$

verifying [pmor 2]. Hence, $\langle \gamma, h \rangle$ is a morphism from $\mathcal{A}$ to $\mathcal{B}$, as desired. The reader will have noticed that γ is simply the "gluing" of the natural ring homomorphisms from AS_a^{-1} to BS_{fa}^{-1} of Lemma 9.42.(b), $a \in A$. $\square$

The classical concept of morphism of presheaves over distinct topological spaces follows. It has already appeared − without a formal definition −, in Propositions 20.9 and 24.64.

Definition 29.3. *Let A, B be presheaves over the topological spaces X and Y, respectively. A morphism from A to B consists of a pair $\langle \eta, f \rangle$ such that*

$*$ *$f : Y \longrightarrow X$ is a continuous map;*

$*$ *$\eta = \{\eta_U : U \in \Omega(X)\}$, where $\eta_U : A(U) \longrightarrow B(f^{-1}(U))$ is a map such that if $V \leq U$ in $\Omega(X)$, the following diagram is commutative :*

$$\begin{tikzcd} A(U) \arrow[r, "\eta_U"] \arrow[d, "\cdot_{|V}"'] & B(f^{-1}(U)) \arrow[d, "\cdot_{|f^{-1}(V)}"] \\ A(V) \arrow[r, "\eta_V"'] & B(f^{-1}(V)) \end{tikzcd}$$

Write $\langle \eta, f \rangle : A \longrightarrow B$ for a morphism from A to B. If we have morphisms,

$$A \overset{\langle \eta, f \rangle}{\longrightarrow} B \overset{\langle \mu, g \rangle}{\longrightarrow} C,$$

their composition is $\langle \mu \circ \eta,\ f \circ g \rangle$. The identity of A is $\langle Id_A, Id_X \rangle$[1]*. Write $\boldsymbol{pSh_t}$ for the category of presheaves over some topological space.*

The connection between the classical notion of morphism and that in 29.1 is described in

Lemma 29.4. *Let A, B be extensional presheaves over the spaces X, Y, respectively. The following conditions are equivalent :*

(1) *$\langle \eta, f \rangle$ is a morphism from A to B;*

(2) *$\langle \eta, f^* \rangle$ is a morphism from the $\Omega(X)$-presheaf A to the $\Omega(Y)$-presheaf B, according to* 29.1.(*b*)*;*

(3) *$\langle \eta, f^* \rangle$ is a morphism from the $\Omega(X)$-set A to the $\Omega(Y)$-set B, according to* 29.1.(*a*).

[1]Pointing in the appropriate directions.

PROOF. It is clear that (1) and (2) are equivalent; for (2) $\Leftrightarrow$ (3), the argument is the same as that used to show the equivalence in item (f) of Proposition 26.8. $\square$

DEFINITION 29.5. *Let A be a L-set and B a R-set. A morphism in **SLset**, $\langle f, h \rangle : A \longrightarrow B$, is a* **retract** *if it has a right inverse, that is, there is a morphism $\langle g, k \rangle : B \longrightarrow A$ in **SLset** such that*

$$\langle f, h \rangle \circ \langle g, k \rangle = \langle Id_B, Id_R \rangle.$$

*Analogously, one defines a retract in **pSh**.*

LEMMA 29.6. *Suppose L, R are frames, $h : L \longrightarrow R$ is a frame-morphism. If $\langle f, h \rangle : A \longrightarrow B$ is a retract in **SLset** and A is a sheaf over L, then B is a sheaf over R.*

PROOF. Let $\langle g, k \rangle : B \longrightarrow A$ be a morphism such that $\langle f \circ g,\, h \circ k \rangle = \langle Id_B, Id_R \rangle$. For $p \in R$, suppose $S \subseteq |B|$ is compatible over p. By 29.8, $gS = \{gs : s \in S\}$ is compatible over $g(p)$ (in $|A|$), and for all s, $s' \in S$,

$$Egs = k(Es) \quad \text{and} \quad k(p \wedge Es \wedge Es') = k(p) \wedge [\![gs = gs']\!].$$

Let t be a gluing of gS over $k(p)$ in $|A|$. Then,

$$\begin{cases} k(p) \wedge Et = \bigvee_{s \in S} k(p) \wedge k(Es); \\ \forall\, s \in S, \quad k(p) \wedge [\![t = gs]\!] = k(p) \wedge Egs = k(p \wedge Es). \end{cases}$$

Consequently, since h preserves joins, we obtain

$$\begin{aligned} p \wedge Eft &= h(k(p) \wedge Et) = h\big(k(p) \wedge \textstyle\bigvee_{s \in S} k(Es)\big) \\ &= h(k(p)) \wedge \textstyle\bigvee_{s \in S} h(k(Es)) = p \wedge \textstyle\bigvee_{s \in S} Es, \end{aligned}$$

as well as, for each $s \in S$,

$$\begin{aligned} p \wedge [\![ft = s]\!] &= h(k(p)) \wedge [\![ft = fgs]\!] \geq h(k(p)) \wedge h([\![t = gs]\!]) \\ &= h(k(p) \wedge [\![t = gs]\!]) = h(k(p \wedge Es)) = p \wedge Es, \end{aligned}$$

and so ft is the gluing of S over p in $|B|$. $\square$

COROLLARY 29.7. *If $f : A \longrightarrow B$ is a retract of presheaves over a frame, then $Im\, f = B$ is a sheaf, whenever the same is true of A.*

To understand the categorical content of direct and inverse image, fix a semilattice morphism $h : L \longrightarrow R$. If A is a L-set, consider the category of A-algebras over h [2], whose objects are diagrams

$$\mathcal{I} = (A \overset{\langle f,h \rangle}{\longrightarrow} B),$$

with B a R-set. If $\mathcal{J} = (A \overset{\langle g,h \rangle}{\longrightarrow} C)$ is an object of this type, a morphism, $\alpha : \mathcal{I} \longrightarrow \mathcal{J}$, consists of a morphism of R-sets, $\alpha : B \longrightarrow C$, such that the diagram below-left is commutative :

[2] Analogous to 16.6; sometimes called a *comma category*.

$$\begin{tikzcd} A \arrow[rr, "\langle f, h \rangle"] \arrow[dr, "\langle g, h \rangle"'] & & B \arrow[dl, "\langle \alpha,\, Id_R \rangle"] \\ & C & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[rr, "\langle \beta, Id_L \rangle"] \arrow[dr, "\langle f, h \rangle"'] & & A' \arrow[dl, "\langle g, h \rangle"] \\ & B & \end{tikzcd}$$

Dually, if B is a R-set, consider the category of B-bundles over h, whose objects are diagrams

$$\mathcal{D} = (A \xrightarrow{\langle f,h \rangle} B),$$

with A a L-set. If $\mathcal{E} = (A' \xrightarrow{\langle g,h \rangle} B)$ is an object of this type, a morphism, $\beta : \mathcal{D} \longrightarrow \mathcal{E}$, is a morphism of L-sets, $\beta : A \longrightarrow A'$, such that the diagram above-right is commutative. It is clear that these constructions can be rephrased for presheaves and sheaves.

An *initial object* in the category of A-algebras over h deserves to be considered as the *direct image* of A over h. Analogously, a *final object* in the category of B-bundles over h ought to be the *inverse image* of B over h.

It is a general fact that "arrows pointing in the right direction" induce natural constructions. Since equality in L-sets is a map $|A| \longrightarrow L$, a semilattice morphism $h : L \longrightarrow R$ makes $|A|$ the domain of a R-set, a good start to obtain the image of A along h. On the other hand, R-presheaves may be considered as contravariant functors (26.24)

$$P : L \longrightarrow \boldsymbol{Set}.$$

Hence, a semilattice morphism $h : L \longrightarrow R$ induces, by composition, a contravariant functor

$$L \xrightarrow{h} R \xrightarrow{P} \boldsymbol{Set},$$

describing the basic idea to get at the inverse image of P along h. It will not come as surprise that direct image of L-sets and inverse image of presheaves are easier to handle than direct image of presheaves and inverse image of L-sets.

The spatial context, that is, the classical setting of sheaves over topological spaces is important, and not just because it is the origin of the theory presented here. A continuous map, $f : X \longrightarrow Y$, induces (see 4.6) a dual pair

$$f^* : \Omega(Y) \longrightarrow \Omega(X) \quad \text{and} \quad f_* : \Omega(X) \longrightarrow \Omega(Y),$$

such that f^* is a frame morphism, g is a $\bigwedge$-morphism and the pair $\langle f^*, f_* \rangle$ satisfies the adjunction $[adj]$ in Theorem 7.8 (or (*) in 4.6). We shall, therefore, also discuss the special properties that arise in the presence of such adjoint pairs between our base algebras.

Exercises

29.8. a) If $\langle\, f, h\,\rangle : A \longrightarrow B$ is a morphisms in $\boldsymbol{LSset}$ or $\boldsymbol{pSh}$, and $S \subseteq |A|$ is compatible over p, then
$$fS = \{fs : s \in S\} \subseteq |B|$$
is compatible over $h(p)$.

b) If $L \xrightarrow{h} R$ is a $[\wedge, \bigvee]$-morphism, a morphism $\langle\, f, h\,\rangle : A \longrightarrow B$ in $\boldsymbol{SLset}$ or $\boldsymbol{pSh}$ preserves gluings of compatible families. $\square$

CHAPTER 30

Image of a L-set

Let A be L-set and $L \xrightarrow{h} R$ be a semilattice morphism. We can transport the L-equality in $|A|$ along h, to get an R-equality on $|A|$, given by

$$[\![x = y]\!]_{A_R} \quad = \quad h([\![x = y]\!]_A).$$

Write A_R for the R-set so defined. In general, A_R *is not* extensional, even if A is extensional. We now describe a different procedure, which furnishes an *extensional change of base.*

30.1. Let $h : L \longrightarrow R$ be a semilattice morphism and let A be a L-set. Define a binary relation θ_A on $|A|$, by

$$x \; \theta_A \; y \quad \text{iff} \quad h(E_A x) \quad = \quad h([\![x = y]\!]_A) \quad = \quad h(E_A y). \tag{I}$$

It is readily verified that θ_A is an equivalence relation on $|A|$; let

$*$ $|\boldsymbol{\varepsilon}_h A|$ be the set of θ_A-equivalence classes on $|A|$;

$*$ $\boldsymbol{\varepsilon}_h^A : |A| \longrightarrow |\boldsymbol{\varepsilon}_h A|$ be the canonical quotient map. Whenever A is clear from context, we omit the superscript A from the notation. $\square$

Remark 30.2. If h is injective and A is extensional, θ_A is the identity. Hence, in this case, $\boldsymbol{\varepsilon}_h A = A_R$. Furthermore, for all $p \in L$, ${\boldsymbol{\varepsilon}_h}_{|A(p)}$ is a bijection between $A(p)$ and $\boldsymbol{\varepsilon}_h A(h(p))$. $\square$

Proposition 30.3. *Let $h : L \longrightarrow R$ be a semilattice morphism and let A be a L-set.*
a) For x, $y \in |A|$, the map

$$\langle \, \boldsymbol{\varepsilon}_h x, \boldsymbol{\varepsilon}_h y \, \rangle \quad \longmapsto \quad h([\![x = y]\!]_A)$$

*defines an extensional equality on $|\boldsymbol{\varepsilon}_h A|$, with which it is a R-set. Furthermore, $\langle \, \boldsymbol{\varepsilon}_h, h \, \rangle : A \longrightarrow \boldsymbol{\varepsilon}_h A$ is a morphism in **SLset**, such that for all x, $y \in |A|$*

$$E(\boldsymbol{\varepsilon}_h x) \quad = \quad h(Ex) \qquad \textit{and} \qquad [\![\boldsymbol{\varepsilon}_h x \, = \, \boldsymbol{\varepsilon}_h y]\!] \quad = \quad h([\![x \, = \, y]\!]).$$

b) If B an extensional *R-set and $\langle \, f, h \, \rangle$ a morphism from A to B, then there is a unique morphism $\widehat{f} : A_h \longrightarrow B$ in **Rset** such that the following diagram commutes :*

$$\begin{tikzcd} A \arrow[rr, "\langle \varepsilon_h, h \rangle"] \arrow[dr, "\langle f, h \rangle"'] & & \varepsilon_h A \arrow[dl, "\langle \widehat{f}, Id_P \rangle"] \\ & B & \end{tikzcd}$$

Hence, for all L-sets A and extensional R-sets B, the map

$$\alpha \in [\varepsilon_h A, B] \longmapsto \langle \alpha, Id_R \rangle \circ \langle \varepsilon_h, h \rangle \in [A, B]_h$$

is a natural bijective correspondence.

PROOF. a) For x, y, t, $z \in |A|$, suppose $\varepsilon_h x = \varepsilon_h t$ and $\varepsilon_h y = \varepsilon_h z$. Then, the transitive law of equality in A yields

$$[\![x = y]\!]_A \wedge [\![x = t]\!]_A \wedge [\![y = z]\!]_A \;\; \leq \;\; [\![t = z]\!]_A,$$

and so the definition of θ_A in 30.1.(I) entails

$$\begin{aligned} h([\![t = z]\!]_A) \;\; &\geq \;\; h([\![x = y]\!]_A) \wedge h([\![x = t]\!]_A) \wedge h([\![y = z]\!]_A) \\ &= \;\; h([\![x = y]\!]_A) \wedge h(E_A x) \wedge h(E_A y) \\ &= \;\; h(E_A x \wedge E_A y \wedge [\![x = y]\!]_A) \;\; = \;\; h([\![x = y]\!]_A). \end{aligned}$$

Since the argument is symmetric in the pairs x, t and y, z, we conclude that $h([\![x = y]\!]_A) = h([\![t = z]\!]_A)$, that is, this value is independent of representatives. It is now straightforward that

$$\langle \varepsilon_h x, \varepsilon_h y \rangle \;\; \longmapsto \;\; h([\![x = y]\!])$$

defines an equality on $\varepsilon_h A$, satisfying the equations displayed in the statement of (a). These equations and the definition of the equivalence relation θ_A immediately imply that $\varepsilon_h A$ is an *extensional* R-set.

b) We first check that if $\varepsilon_h x = \varepsilon_h y$, then $fx = fy$. Recalling that

$$\varepsilon_h x = \varepsilon_h y \quad \text{iff} \quad h(E_A x) = h(E_A y) = h([\![x = y]\!]_A),$$

we obtain

$$\begin{aligned} [\![fx = fy]\!] \;\; &\geq \;\; h([\![x = y]\!]_A) \;\; = \;\; h(E_A x) \;\; = \;\; E_B fx \;\; = \;\; h(E_A y) \\ &= \;\; E_B fy, \end{aligned}$$

and the extensionality of B entails $fx = fy$. Define, for $x \in |A|$

$$\widehat{f}(\varepsilon_h x) = fx.$$

It is straightforward that $\widehat{f}$ is the unique morphism making the displayed diagram commutative. $\square$

DEFINITION 30.4. *The R-set $\varepsilon_h A$ of* 30.3 *is the* **extensional image of A along h**.

30.5. **ε_h is a covariant functor.** Let $f : A \longrightarrow B$ be a morphism of L-sets. If x, $y \in |A|$ satisfy $x \; \theta_A \; y$ (30.1.(I)), then

$$h(E_B fx) \;\; = \;\; h(E_A x) \;\; = \;\; h([\![x = y]\!]_A) \;\; = \;\; h(E_A y) \;\; = \;\; h(E_B fy),$$

which entails, since h is increasing and $[\![x = y]\!]_A \leq [\![fx = fy]\!]_B$, that $fx\ \theta_B\ fy$. Consequently, the map

$$\varepsilon_h f : |\varepsilon_h A| \longrightarrow |\varepsilon_h B|, \text{ given by } \varepsilon_h f(\varepsilon_h^A x) \;=\; \varepsilon_h^B(fx),$$

is well defined. Note that

$$[\![\varepsilon_h fx \,=\, \varepsilon_h fy]\!] \;=\; [\![\varepsilon_h^B(fx) \,=\, \varepsilon_h^B(fy)]\!] \;=\; h([\![fx \,=\, fy]\!]),$$

for all x, $y \in |A|$; hence, 30.3.(a) yields :

(i) $E(\varepsilon_h f(\varepsilon_h^A x)) \;=\; E(\varepsilon_h^B fx) \;=\; h(E_B fx) \;=\; h(E_A x) \;=\; E(\varepsilon_h^A x)$;

(ii) $[\![\varepsilon_h f(\varepsilon_h^A x) \,=\, \varepsilon_h f(\varepsilon_h^A y)]\!] \;=\; [\![\varepsilon_h^B fx \,=\, \varepsilon_h^B fy]\!] \;=\; h([\![fx = fy]\!]_B)$

$$\geq\; h([\![x = y]\!]_A) \;=\; [\![\varepsilon_h^A x \,=\, \varepsilon_h^A y]\!].$$

Thus, $\varepsilon_h f$ is a morphism of R-sets and the following diagram is commutative :

$$\begin{tikzcd} A \arrow[r, "\varepsilon_h"] \arrow[d, "f"'] & \varepsilon_h A \arrow[d, "\varepsilon_h f"] \\ B \arrow[r, "\varepsilon_h"'] & \varepsilon_h B \end{tikzcd}$$

It is straightforward that

$$\varepsilon_h(f \circ g) = \varepsilon_h f \circ \varepsilon_h g \quad \text{and} \quad \varepsilon_h(Id_A) = Id_{\varepsilon_h A}.$$

Hence, h induces a covariant functor, $\boldsymbol{\varepsilon_h} : \boldsymbol{Lset} \longrightarrow \boldsymbol{Rset}$, the **(extensional) change of base functor along $\boldsymbol{h}$**. □

LEMMA 30.6. *If $L \xrightarrow{h} R$ is a semilattice morphism, the functor ε_h preserves non-empty finite products. If h is surjective, then ε_h preserves final objects.*

PROOF. We give a sketch, leaving details to the reader. If A, B are L-sets, the map

$$\varepsilon_h\langle\, x, y\,\rangle \in |\varepsilon_h(A \times B)| \;\longmapsto\; \langle\, \varepsilon_h x,\, \varepsilon_h y\,\rangle \in |\varepsilon_h A \times \varepsilon_h B|$$

is an isomorphism. Recalling 25.5, it is easy to see that $|\varepsilon_h \mathbf{1}| \;=\; Im\, h$, with equality induced by R. Hence, if h is surjective, ε_h preserves final objects. □

REMARK 30.7. In case $h = Id_L$, ε_h is the **extensionalization functor**, written ε, from L-sets to the category of extensional L-sets. Generalizing 23.21, its properties can be read off the statement of 30.3; for future reference, we include the Corollary that follows. □

COROLLARY 30.8. *Let L be a semilattice.*

a) If A is L-set, the binary relation on $|A|$

$$x\ R\ y \quad \textit{iff} \quad Ex = Ey = [\![x \,=\, y]\!]$$

is an equivalence relation on $|A|$. Write εx for the equivalence class of x with respect to R and $|\varepsilon A| \;=\; \{\varepsilon x : x \in |A|\}$.

b) For $x,\, y \in |A|$, set

$$[\![\varepsilon x\ =\ \varepsilon y]\!]\ \ =\ \ [\![x\ =\ y]\!].$$

This defines an extensional *equality on $|\varepsilon A|$, such that $E\varepsilon x = Ex$, for all $x \in |A|$.*

c) The map $\varepsilon : A \longrightarrow \varepsilon A$, $x \mapsto \varepsilon x$ is a L-set morphism, with the following universal property :

If B is an extensional L-set and $f : A \longrightarrow B$ is a morphism of L-sets, there is a unique *morphism of L-sets, $\widehat{f} : \varepsilon A \longrightarrow B$, making the diagram below-left commutative :*

$$\begin{tikzcd} A \arrow[rr, "\varepsilon"] \arrow[dr, "f"'] & & \varepsilon A \arrow[dl, "\widehat{f}"] \\ & B & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\varepsilon"] \arrow[d, "f"'] & \varepsilon A \arrow[d, "\varepsilon f"] \\ B \arrow[r, "\varepsilon"'] & \varepsilon B \end{tikzcd}$$

d) If $A \xrightarrow{f} B$ is a morphism of L-sets, the map

$$\varepsilon f\ : |\varepsilon A| \longrightarrow |\varepsilon B|,\ \textit{defined by}\ \varepsilon x \mapsto \varepsilon f x,$$

is the unique morphism of L-sets making the diagram above-right commutative. □

The composition of ε_h with the completion functor in 27.9 yields

COROLLARY 30.9. *Let $L \xrightarrow{h} \Omega$ be a semilattice morphism, with Ω a frame. If A is a L-set, then there is a sheaf c_hA over Ω and a morphism*

$$\langle\, c_h, h\,\rangle\ : A \longrightarrow c_hA,$$

verifying the following conditions :

(1) *For all $a,\, b \in |A|$, $[\![c_ha\ =\ c_hb]\!] = h([\![a\ =\ b]\!])$;*

(2) *For all $s \in |cA|$, $Es = \bigvee_{a\in|A|} [\![c_ha\ =\ s]\!]$;*

(3) *If C is a sheaf over Ω and $\langle\, g, h\,\rangle : A \longrightarrow C$ is a morphism, there is a unique morphism $\widehat{g} : c_hA \longrightarrow C$ in $\boldsymbol{Sh(\Omega)}$ such that the following diagram is commutative :*

$$\begin{tikzcd} A \arrow[rr, "\langle\, c_h, h\,\rangle"] \arrow[dr, "\langle\, g, h\,\rangle"'] & & c_hA \arrow[dl, "\langle\, \widehat{g}, Id_\Omega\,\rangle"] \\ & C & \end{tikzcd}$$

□

The sheaf c_hA is the **completion of $\boldsymbol{A}$ along $\boldsymbol{h}$**. When $h = Id_\Omega$, we get back the completion cA of Theorem 27.9; furthermore,

$$\langle c_h, h \rangle = \langle c, Id_\Omega \rangle \circ \langle \boldsymbol{\varepsilon}_h, h \rangle,$$

with c the completion map of 27.9. Since every $[\wedge, \bigvee]$-semilattice can be regularly embedded in a frame (25.29), we get

Corollary 30.10. *If L is a $[\wedge, \bigvee]$-semilattice and A is a L-set, there is a regular embedding, $h : L \longrightarrow \Omega$, a sheaf $c_h A$ over Ω together with a morphism, $c_h : A \longrightarrow c_h A$, such that, if L is identified with $h(L)$, conditions* (1), (2) *and* (3) *in* 27.9 *are verified* [1].

In particular, if H is a HA, then any H-set gives rise to a sheaf over H_*, the frame completion of H (14.4). Exercise 30.14 shows that this completion deserves to be considered **the completion** of the H-set under consideration.

The functor c_h may also be used to define *image in the category of sheaves* :

Definition 30.11. *If L is a frame, $h : L \longrightarrow \Omega$ is a semilattice morphism and A is a sheaf over L, the sheaf $c_h A$ is the* **image of $\boldsymbol{A}$ along $\boldsymbol{h}$** *in the category of* $\boldsymbol{Sh(\Omega)}$.

We now show that image along a semilattice morphism preserves composition. This result has many applications, considerably simplifying the computation of the transfer of objects along maps between the base algebras.

Proposition 30.12. *Let $L \xrightarrow{f} R \xrightarrow{g} T$ be semilattice morphisms. Then,*

a) $\boldsymbol{\varepsilon}_{(g \circ f)} = \boldsymbol{\varepsilon}_g \circ \boldsymbol{\varepsilon}_f$ [2].

b) If L, R, T are frames, then $c_{(g \circ f)} = c_g \circ c_f$.

c) If $T = L$ and $g \circ f = Id_L$, $\boldsymbol{\varepsilon}_g \circ \boldsymbol{\varepsilon}_g$ and $c_g \circ c_f$ are the identity functors in the extensional subcategory of $\boldsymbol{Lset}$ *and in* $\boldsymbol{Sh(L)}$, *respectively.*

Proof. a) We prove (a), leaving (b) as an exercise. Let A be a L-set and $\langle \alpha, gf \rangle : A \longrightarrow B$ be a morphism in $\boldsymbol{pSh}$ into the extensional T-set B.

Fact. *The map* $\boldsymbol{\varepsilon}_f a \in |\boldsymbol{\varepsilon}_f A| \overset{\beta}{\longmapsto} \alpha a \in |B|$ *is the carrier of the* unique *morphism,* $\langle \beta, g \rangle : \boldsymbol{\varepsilon}_f A \longrightarrow B$, *making the following triangle commutative :*

$$\begin{tikzcd} A \arrow[rr, "\langle \boldsymbol{\varepsilon}_f, f \rangle"] \arrow[dr, "\langle \alpha, gf \rangle"'] & & \boldsymbol{\varepsilon}_f A \arrow[dl, "\langle \beta, g \rangle"] \\ & B & \end{tikzcd}$$

Proof. If a, $b \in |A|$ are such that $\boldsymbol{\varepsilon}_f a = \boldsymbol{\varepsilon}_f b$, then

$$f(Ea) = f(Eb) = f([\![a = b]\!])$$

[1] With c_h in place of c, of course.

[2] I.e., these functors are naturally isomorphic; the same comment applies to (b).

and so $gf(Ea) = gf(Eb) = gf([\![a = b]\!])$. Since $\langle\, \alpha, gf \,\rangle$ is a morphism, we obtain

$$(*) \qquad \begin{array}{rcl} [\![\alpha a = \alpha b]\!] & \geq & gf([\![a = b]\!]) \;=\; gf(Ea) \;=\; E_B(\alpha a) \\ & = & gf(Eb) \;=\; E_B(\alpha b). \end{array}$$

The extensionality of B entails $\alpha a = \alpha b$, as needed to establish that β is well-defined. Since $[\![\boldsymbol{\varepsilon}_f a = \boldsymbol{\varepsilon}_f b]\!] \;=\; f([\![a = b]\!])$, the inequality and equalities in (*) imply that $\langle\, \beta, g \,\rangle$ is a morphism in ***SLset***. Clearly, the displayed triangle is commutative, ending the proof of the Fact.

By the universal property of image along g in 30.3.(b), there is a *unique* morphism of T-sets, $\widehat{\beta} : \boldsymbol{\varepsilon}_g\boldsymbol{\varepsilon}_f A \longrightarrow B$, making the diagram below-left commutative :

$$\begin{tikzcd} \boldsymbol{\varepsilon}_f A \arrow[rr, "\langle\, \boldsymbol{\varepsilon}_g, g \,\rangle"] \arrow[dr, "\langle\, \beta, g \,\rangle"'] & & \boldsymbol{\varepsilon}_g\boldsymbol{\varepsilon}_f A \arrow[dl, "\langle\, \widehat{\beta}, Id_T \,\rangle"] \\ & B & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[rr, "\langle\, \boldsymbol{\varepsilon}_f \circ \boldsymbol{\varepsilon}_g, gf \,\rangle"] \arrow[dr, "\langle\, \alpha, gf \,\rangle"'] & & \boldsymbol{\varepsilon}_f\boldsymbol{\varepsilon}_g A \arrow[dl, "\langle\, \gamma, Id_T \,\rangle"] \\ & B & \end{tikzcd}$$

Consider the morphism $\langle\, \boldsymbol{\varepsilon}_g \circ \boldsymbol{\varepsilon}_f,\ gf \,\rangle : A \longrightarrow \boldsymbol{\varepsilon}_g\boldsymbol{\varepsilon}_f A$; by what has been proven, if B is an extensional T-set and $\langle\, \alpha, gf \,\rangle : A \longrightarrow B$ is a morphism, there is a *unique* morphism of T-sets, γ, such that the triangle above-right commutative. Since the morphism

$$\langle\, \boldsymbol{\varepsilon}_{g \,\circ\, f},\ gf \,\rangle : A \longrightarrow \boldsymbol{\varepsilon}_{g \,\circ\, f} A$$

has precisely that same universal property, there is a *unique* isomorphism of T-sets,

$$\sigma_A : \boldsymbol{\varepsilon}_g\boldsymbol{\varepsilon}_f A \longrightarrow \boldsymbol{\varepsilon}_{(g \,\circ\, f)} A,$$

such that the following diagram is commutative :

$$\begin{tikzcd} A \arrow[rr, "\langle\, \boldsymbol{\varepsilon}_f \circ \boldsymbol{\varepsilon}_g, gf \,\rangle"] \arrow[dr, "\langle\, \boldsymbol{\varepsilon}_{g \,\circ\, f},\ gf \,\rangle"'] & & \boldsymbol{\varepsilon}_g\boldsymbol{\varepsilon}_f A \arrow[dl, "\sigma_A"] \\ & \boldsymbol{\varepsilon}_{(g \,\circ\, f)} A & \end{tikzcd}$$

The family $\sigma = \{\sigma_A : A \text{ is a } L\text{-set}\}$ provides a natural isomorphism between the functors $\boldsymbol{\varepsilon}_g \circ \boldsymbol{\varepsilon}_f$ and $\boldsymbol{\varepsilon}_{g \,\circ\, f}$, as needed.

For (c), if $g \circ f = Id_L$, then $\boldsymbol{\varepsilon}_{g \,\circ\, f} = \boldsymbol{\varepsilon}_{Id_L} = \boldsymbol{\varepsilon}$, the extensionalization functor of 30.7. Hence, when restricted to extensional L-sets, $\boldsymbol{\varepsilon}$ is the identity functor, as claimed. $\square$

REMARK 30.13. Let $X \xrightarrow{f} Y$ be a continuous map of topological spaces. If A is a sheaf over Y, the *completion* of $\boldsymbol{\varepsilon}_{f^*}A$, that is, $c_{f^*}A$ (30.9), is called the **inverse image of $\boldsymbol{A}$ along $\boldsymbol{f}$**, written f^*A.

This distinction in terminology might seem confusing, but both have strong arguments to their side. In the spatial case, the main parameter is f and we are bringing a sheaf over Y "back" along f to a sheaf over X. In our setting, the main parameter is f^* and A is brought "forward" to a sheaf over X. Hence, *image along* f^*, which is our viewpoint, corresponds to *inverse image along* f, which is the spatial viewpoint.

It should be remarked that our rendering of the subject is (considerably) more general than the classical construction. Even for topological spaces, there are semilattice morphisms $h : \Omega(Y) \longrightarrow \Omega(X)$ that are not represented by continuous maps, i.e., h is distinct from f^*, for all continuous $f : X \longrightarrow Y$. □

Exercises

30.14. Let H be a HA and let A be a H-set. Consider the category $\mathcal{A}$ whose objects are diagrams

$$\langle\, \alpha, k \,\rangle : A \longrightarrow B,$$

where B is a sheaf over a frame D and k is a $[\wedge, \bigvee]$-morphism from L to D. The morphisms in $\mathcal{A}$ are the usual ones for algebras, i.e., $\langle\, \beta, g \,\rangle : B \longrightarrow B'$, where $g : D \longrightarrow D'$ is a frame-morphism, such that the following diagram is commutative :

$$\begin{tikzcd} A \arrow[rr, "\langle\, \alpha, k \,\rangle"] \arrow[dr, "\langle\, \alpha', k' \,\rangle"'] & & B \arrow[dl, "\langle\, \beta, g \,\rangle"] \\ & B' & \end{tikzcd}$$

If A_* is the completion of A over the regular embedding $H \xrightarrow{*} H_*$, then $\langle\, A_*, H_* \,\rangle$ is an **initial object** in $\mathcal{A}$. □

30.15. With notation as in 28.5, let $k : H \longrightarrow L$ be a semilattice morphism and let A be a H-set.

a) The map $\gamma_k : |\mathfrak{s}_k A| \longrightarrow |\boldsymbol{\varepsilon}_k A|$, defined by

$$\gamma_k(\langle\, x, p \,\rangle) = \boldsymbol{\varepsilon}_k x$$

determines a morphism $\langle\, \gamma_k, k \,\rangle : \mathfrak{s}_k A \longrightarrow \boldsymbol{\varepsilon}_k A$, making the following diagram commutative :

$$
\begin{tikzcd}
A \arrow[rr, "\mathfrak{s}_k"] \arrow[dr, "\langle\, \boldsymbol{\varepsilon}_k, k \,\rangle"'] & & \mathfrak{s}_k A \arrow[dl, "\langle\, \gamma_k, k \,\rangle"] \\
& \boldsymbol{\varepsilon}_k A &
\end{tikzcd}
$$

b) For all $\langle\, x, p \,\rangle$, $\langle\, y, q \,\rangle \in |\mathfrak{s}_k A|$,

$$k([\![\langle\, x, p \,\rangle \;=\; \langle\, y, q \,\rangle]\!]) \;\;=\;\; k([\![x \;=\; y]\!]) \;\;=\;\; [\![\boldsymbol{\varepsilon}_k x \;=\; \boldsymbol{\varepsilon}_k y]\!].$$

c) The map

$$\beta_A : \boldsymbol{\varepsilon}_k \mathfrak{s}_k A \longrightarrow \boldsymbol{\varepsilon}_k A, \text{ given by } \boldsymbol{\varepsilon}_k(\langle\, x, p \,\rangle) \longmapsto \boldsymbol{\varepsilon}_k x$$

is an isomorphism. $\square$

CHAPTER 31

Base Extension

By 26.5 and 26.25, whenever $L \subseteq R$, a R-presheaf may be restricted to L, originating a functor, $\mathfrak{r} : \boldsymbol{pSh(R)} \longrightarrow \boldsymbol{pSh(L)}$. This Chapter is devoted to the inverse operation. Section 16.4 and Chapter 17 are references for the colimits employed below.

31.1. We return to the context of Chapter 23. Let L be a semilattice, considered a category as in Example 16.4. Let $\mathcal{C}$ be a category that has colimits for all diagrams over up-directed posets and **fix** a covariant functor [1]

$$A : L^{op} \longrightarrow \mathcal{C}.$$

We write the values of A at $p \in L$ as $A(p)$ and if $p \leq q \in L$,

$$x \in A(q) \quad \longmapsto \quad x_{|p} \in A(p)$$

for the morphism corresponding to the pair $p \leq q$. For $p \leq q \leq r$ and $x \in A(r)$, the functoriality of A entails

$$x_{|r} = x \quad \text{and} \quad x_{|p} = (x_{|q})_{|p}.$$

These equalities imply that a down-directed subset F of L gives rise to an *inductive system in* $\mathcal{C}$, $\mathcal{A}(F)$, given by :

$$\mathcal{A}(F) \;\; = \;\; \langle\, A(q);\; \{ *_{|p} \;:\; p \leq q \text{ in } F \} \,\rangle.$$

Note that $\mathcal{A}(F)$ is an inductive system over the up-directed subset F^{op} of L. Write

$$\mathcal{A}_F \;\; = \;\; \varinjlim_{p \in F} A(p) \;\; = \;\; \varinjlim \; \mathcal{A}(F)$$

for the colimit of $\mathcal{A}(F)$. Recall that $\mathcal{A}_F$ comes equipped with $\mathcal{C}$-morphisms

$$f_p^F : A(p) \longrightarrow \mathcal{A}_F,\; p \in F,$$

making the following diagram commutative, for $p \leq q$ in F :

$$\begin{tikzcd}
A(q) \arrow[rr, "*_{|p}"] \arrow[dr, "f_q^F"'] & & A(p) \arrow[dl, "f_p^F"] \\
& \mathcal{A}_F &
\end{tikzcd}$$

---

[1] That is, a L-presheaf with values in $\mathcal{C}$.

It is clear that if F has a least element p, then $\mathcal{A}_F = A(p)$ and that the morphism f_q^F is the restriction from $A(q)$ to $A(p)$.

If G is another down-directed subset of L, with $F \subseteq G$, then for all $p \in F$, there is a map $f_p^G : A(p) \longrightarrow \mathcal{A}_G$, making the diagram below-left commutative, where $p \leq q$ in F :

$$\begin{tikzcd} A(q) \arrow[rr, "*|_p"] \arrow[dr, "f_q^G"'] & & A(p) \arrow[dl, "f_p^G"] \\ & \mathcal{A}_G & \end{tikzcd} \qquad \begin{tikzcd} A(p) \arrow[rr, "f_p^F"] \arrow[dr, "f_p^G"'] & & \mathcal{A}_F \arrow[dl, "\alpha_{FG}"] \\ & \mathcal{A}_G & \end{tikzcd}$$

Since $\mathcal{A}_F = \varinjlim \mathcal{A}(F)$, there is a *unique* $\mathcal{C}$-morphism,

$$\alpha_{FG} : \mathcal{A}_F \longrightarrow \mathcal{A}_G,$$

making the diagram above right commutative, for all $p \in F$. We call α_{FG} the **restriction** from $\mathcal{A}_F$ to $\mathcal{A}_G$.

A final comment : if F is empty, $\mathcal{A}_F$ is also empty. Thus, if G is a down-directed subset of L, the restriction map α_{FG} is the empty map from $\mathcal{A}_F$ to $\mathcal{A}_G$. $\square$

By Exercise 26.24, the category of L-presheaves is isomorphic to that of covariant functors from L^{op} to $\boldsymbol{Set}$, with natural transformations as morphisms. We may therefore apply the general scheme described in 31.1 to the extension problem.

31.2. Let A be a L-presheaf and suppose that R is a semilattice, containing L as a subsemilattice. For each $u \in R$, set

$$G(u) = \{p \in L : u \leq p\}.$$

Clearly, $G(u)$ is down-directed and, in fact, closed under meets. Let $\mathfrak{e}A(u)$ be the colimit of the system associated to $G(u)$ and A, which comes equipped with maps $f_{pu} : A(p) \longrightarrow \mathfrak{e}A(u)$, making the diagram below-left commutative :

$$\begin{tikzcd} A(q) \arrow[rr, "*|_p"] \arrow[dr, "f_{qu}"'] & & A(p) \arrow[dl, "f_{pu}"] \\ & \mathfrak{e}A(u) & \end{tikzcd} \qquad \begin{tikzcd} A(p) \arrow[rr, "f_{pu}"] \arrow[dr, "f_{pv}"'] & & \mathfrak{e}A(u) \arrow[dl, "\alpha_{uv}"] \\ & \mathfrak{e}A(v) & \end{tikzcd}$$

Note that if $v \leq u$ in R, then $G(u) \subseteq G(v)$, and 31.1 guarantees that there is a *unique* map, $\alpha_{uv} : \mathfrak{e}A(u) \longrightarrow \mathfrak{e}A(v)$, such that the diagram above-right is commutative, for all $p \in G(u)$. It is straightforward to verify that if $r \leq v \leq u$ in R, then

$$(*) \qquad \alpha_{uu} = Id_{\mathfrak{e}A(u)} \quad \text{and} \quad \alpha_{ur} = \alpha_{uv} \circ \alpha_{vr}.$$

Moreover, if $p \in L$, then $p = \min G(p)$ and we have :

$$(**) \qquad \begin{cases} (i)\ \mathfrak{e}A(p) = A(p); & (ii)\ \forall\ q \in G(p),\quad \alpha_{qp} = *_{|p}; \\ (iii)\ \text{If } p \in G(r), \text{ then } \quad \alpha_{pr} = f_{pr}. \end{cases}$$

We now define a R-presheaf, $\mathfrak{e}A$, as follows :

$$(\mathfrak{e}A) \qquad \begin{cases} \text{a) } |\mathfrak{e}A| = \coprod_{u \in R} \mathfrak{e}A(u); \\ \text{b) } E_{\mathfrak{e}A}(s) = u \quad \text{iff} \quad s \in \mathfrak{e}A(u); \\ \text{c) For } s \in |\mathfrak{e}A| \text{ and } v \in R, \quad s_{|v} = \alpha_{Es, Es \wedge v}(s). \end{cases}$$

□

PROPOSITION 31.3. *Notation as in* 31.2*, let A be a L-presheaf and R a semilattice containing L as a subsemilattice. Then,*

a) $\mathfrak{e}A$ is a R-presheaf.

b) The map $\mathfrak{e} : |A| \longrightarrow |\mathfrak{e}A|$ defined by

$$a \in |A| \longmapsto \mathfrak{e}a \in \mathfrak{e}A(Ea) = A(Ea)$$

is an injection, such that for all $\langle x, p \rangle \in |A| \times L$

$$E\mathfrak{e}x \ = \ Ex \quad \text{and} \quad \mathfrak{e}(x_{|p}) \ = \ (\mathfrak{e}x)_{|p}.$$

Moreover, for all $s \in |\mathfrak{e}A|$, there is $p \in G(Es)$ and $x \in A(p)$ such that $s = x_{|Es}$ [2].

c) If L is complete and A is extensional, then $\mathfrak{e}A$ is extensional.

d) If B is a R-presheaf and $|A| \xrightarrow{g} |B|$ is a map such that for all $\langle x, p \rangle \in |A| \times L$

$$Egx \ = \ Ex \quad \text{and} \quad g(x_{|p}) \ = \ (gx)_{|p},$$

then there is a unique *morphism of R-presheaves, $f : \mathfrak{e}A \longrightarrow B$, such that the diagram below-left is commutative :*

$$\begin{tikzcd} A \arrow[rr, "\mathfrak{e}"] \arrow[dr, "g"'] & & \mathfrak{e}A \arrow[dl, "f"] \\ & B & \end{tikzcd} \qquad\qquad \begin{tikzcd} A \arrow[rr, "\rho_A"] \arrow[dr, "\mathfrak{e}"'] & & (\mathfrak{e}A)_{|L} \arrow[dl, "\mathfrak{r}_{\mathfrak{e}A}"] \\ & \mathfrak{e}A & \end{tikzcd}$$

e) There is a natural **isomorphism**, *$\rho_A : A \longrightarrow (\mathfrak{e}A)_{|L}$, making the diagram above-right is commutative* [3].

PROOF. a) That $\mathfrak{e}A$ is a R-presheaf follows easily from the equalities in (*) of 31.2. For instance, if $x \in |\mathfrak{e}A|$ and $u, v \in R$, then

$$(x_{|u})_{|v} = x_{|u \wedge v}$$

is an immediate consequence of the second equation in 31.2.(*).

[2]Thus, the image of $\mathfrak{e}$ is *restriction dense* in $\mathfrak{e}A$; see 26.28.(d).

[3]Notation as in 26.5 and 26.25.

b) Clearly, $\mathfrak{e}$ has the stated properties. Fix $s \in \mathfrak{e}A(u)$; since $\mathfrak{e}A(u) = \lim\limits_{\rightarrow p\in G(u)} A(p)$, there is $p \in G(u)$ and $x \in A(p)$, such that $f_{pu}(x) = s$. But ($\mathfrak{e}A$) in 31.2 then entails $x_{|u} = s$ in $\mathfrak{e}A$ [4].

c) Assume that A is extensional and L is complete. Let $s,\ t \in \mathfrak{e}A(u)$ be sections such that there is $\alpha \subseteq R$ satisfying

$$\bigvee \alpha = u \quad \text{and} \quad s_{|v} = t_{|v}, \text{ for all } v \in \alpha.$$

By item (b), there are $p,\ q \in G(u)$, $x \in A(p)$ and $y \in A(q)$, such that $x_{|u} = s$ and $y_{|u} = t$. Since $G(u)$ is down-directed, we may as well assume that $p = q$ (otherwise, substitute both for $p \wedge q$). For $v \in \alpha$, we have

$$x_{|v} \;=\; x_{|u\wedge v} \;=\; (x_{|u})_{|v} \;=\; s_{|v} \;=\; t_{|v} \;=\; y_{|v}.$$

Since $\mathfrak{e}A(v) = \lim\limits_{\rightarrow r\in G(v)} A(r)$, there is $q_v \in G(v)$ such that $x_{|q_v} = y_{|q_v}$. Because $v \leq u \leq p$, we may assume that $q_v \leq p$ (otherwise, replace q_v by its meet with p). It is clear that

$$u \;\leq\; q \;=_{def}\; \textstyle\bigvee_{v\in\alpha} q_v \in L.$$

Consider $x' = x_{|q}$, $y' = y_{|q} \in A(q)$; since $s = x_{|u}$ and $t = y_{|u}$, we also have $s = x'_{|u}$ and $t = y'_{|u}$. Now observe that for all $v \in \alpha$,

$$x'_{|q_v} \;=\; (x_{|q})_{|q_v} \;=\; x_{|q\wedge q_v} \;=\; x_{|q_v} \;=\; y_{|q_v} \;=\; y'_{|q_v},$$

and the extensionality of A entails $x' = y'$. Thus, $s = x'_{|u} = y'_{|u} = t$, establishing the extensionality of $\mathfrak{e}A$.

d) For $u \in R$ and $p \in G(u)$, since $Egx = Ex$, define

$$g_{pu} : A(p) \longrightarrow B(u), \text{ by } \; x \in A(p) \longmapsto (gx)_{|u} \in B(u).$$

Since g commutes with the restrictions by elements in L, the diagram below left is commutative, where $q \leq p$, both in $G(u)$:

$$\begin{tikzcd}
A(p) \arrow[rr, "*_{|p}"] \arrow[dr, "g_{pu}"'] & & A(q) \arrow[dl, "g_{qu}"] \\
& B(u) &
\end{tikzcd}
\qquad
\begin{tikzcd}
A(p) \arrow[rr, "f_{pu}"] \arrow[dr, "g_{pu}"'] & & \mathfrak{e}A(u) \arrow[dl, "f_u"] \\
& B(u) &
\end{tikzcd}$$

Since $\mathfrak{e}A(u) = \lim\limits_{\rightarrow p\in G(u)} A(p)$, there is a *unique* $f_u : \mathfrak{e}A(u) \longrightarrow B(u)$, such that the diagram above-right is commutative, for all p in $G(u)$. The family $f = \{f_u : u \in R\}$ determines the required morphism of R-presheaves. Details are left to the reader.

[4] We are, of course, identifying A with its image by the injection $\mathfrak{e}$.

d) By 23.26.(a) (and (b) in the present result), there is a *unique* morphism of L-presheaves, $\rho_A : A \longrightarrow (\mathfrak{e}A)_{|L}$, such that the diagram in the statement is commutative. Since for all $p \in L$ we have

$$(\mathfrak{e}A)_{|L}(p) = \mathfrak{e}A(p) = A(p),$$

it is clear that ρ_A is a bijection when restricted to $A(p)$. Hence, ρ_A is an isomorphism between A and $(\mathfrak{e}A)_{|L}$. □

DEFINITION 31.4. *The R-presheaf $\mathfrak{e}A$ of* 31.3 *is the* **extension of A to R**.

EXAMPLE 31.5. Let $g : C \longrightarrow D$ be a *non-injective* map of sets. Fix a, b in C, such that $g(a) = g(b)$. Let

$$L \;=\; [0,\,1/2) \cup (1/2,\,1] \;\subseteq\; [0,\,1] \;=\; R,$$

the unit real interval. Both L and R have their natural orders. Note that R is a frame and L is a $[\wedge, \bigvee]$-sublattice of R. Define a L-presheaf A as follows :

(*i*) $|A| = \bigcup_{p \in [0,1/2)} D \times \{p\} \;\cup\; \bigcup_{p \in (1/2,1]} C \times \{p\}$;

(*ii*) $E\langle x, p\rangle = p$;

(*iii*) $\langle x, p\rangle_{|q} = \begin{cases} \langle g(x), p\rangle & \text{if } p \in [0,\,1/2) \text{ and } q \in (1/2,\,1] \\ \langle x, p \wedge q\rangle & \text{otherwise.} \end{cases}$

A is extensional for the simple reason that no element of $(1/2,\,1]$ is the join of elements in $[0,\,1/2)$. If $p \leq q$ in L, the restriction maps of A are given by

$$\rho_{qp} = \begin{cases} Id_C & \text{if } p,\, q \in (1/2,\,1] \\ Id_D & \text{if } p,\, q \in [0,\,1/2) \\ g & \text{if } p \in [0,\,1/2) \text{ and } q \in (1/2,\,1]. \end{cases}$$

Consequently, $\mathfrak{e}A(1/2) \;=\; \lim\limits_{\rightarrow p>1/2} A(p) \;=\; C$; further, for $p < 1/2$, the restriction map from $\mathfrak{e}A(1/2)$ to $A(p)$ is g. Hence, for all $p \in [0,\,1/2)$,

$$a_{|p} \;=\; b_{|p},$$

although $a \neq b$ in $\mathfrak{e}A(1/2)$. This shows that the hypothesis of completeness of L in 31.3.(c) is necessary. □

31.6. **Base extension is a covariant functor.** Let $L \subseteq R$ be semilattices and $f : A \longrightarrow B$ be a morphism of L-presheaves. It is straightforward that the composition

$$A \xrightarrow{f} B \xrightarrow{\mathfrak{e}} \mathfrak{e}B$$

verifies the conditions in item (d) of 31.3, and so there is a *unique* morphism of R-presheaves, $\mathfrak{e}f : \mathfrak{e}A \longrightarrow \mathfrak{e}B$, making the following diagram commutative :

$$
\begin{tikzcd}
A \arrow[r, "\mathfrak{e}"] \arrow[d, "f"'] & \mathfrak{e}A \arrow[d, "\mathfrak{e}f"] \\
B \arrow[r, "\mathfrak{e}"'] & \mathfrak{e}B
\end{tikzcd}
$$

The proof of 31.3.(d) will yield a concrete description of $\mathfrak{e}f$ on the elements of $|\mathfrak{e}A|$. It is straightforward that

$$\mathfrak{e}(Id_A) = Id_{\mathfrak{e}A} \quad \text{and} \quad \mathfrak{e}(g \circ f) = \mathfrak{e}g \circ \mathfrak{e}f,$$

and so base extension is a covariant functor, $\mathfrak{e} : \boldsymbol{pSh(L)} \longrightarrow \boldsymbol{pSh(R)}$. It has been noted (26.25) that restriction is also a covariant functor, $\mathfrak{r} : \boldsymbol{pSh(R)} \longrightarrow \boldsymbol{pSh(L)}$. The next result establishes the adjointness of these functors. $\square$

PROPOSITION 31.7. *Let $L \subseteq R$ be semilattices, A be a L-presheaf and B a R-presheaf.*
a) There is a natural morphism of R-presheaves, $\mathfrak{j}_B : \mathfrak{e}(B_{|L}) \longrightarrow B$, making the following diagram commutative :

$$
\begin{tikzcd}
B_{|L} \arrow[rr, "\mathfrak{e}"] \arrow[dr, "\mathfrak{r}_B"'] & & \mathfrak{e}(B_{|L}) \arrow[dl, "\mathfrak{j}_B"] \\
& B &
\end{tikzcd}
$$

b) Extension is left adjoint to restriction, that is, there a bijective correspondence, natural in A and B,

$$k_{AB} : [\mathfrak{e}A,\, B] \approx [A,\, B_{|L}],$$

defined by $g \longmapsto \mathfrak{r}g \circ \rho_A$, whose inverse is given by $f \longmapsto \mathfrak{j}_B \circ \mathfrak{e}f$ [5].

PROOF. The existence and naturality of $\mathfrak{j}_B$ follows from the universal property of the morphism $\mathfrak{e} : B_{|L} \longrightarrow \mathfrak{e}(B_{|L})$ in 31.3.(d). Item (b) is a straightforward consequence of the universal properties of extension and restriction (26.25). $\square$

EXAMPLE 31.8. Let $\mathcal{M}$ be the sheaf of 23.11, whose base algebra is the complete Boolean algebra 2^I. Recall that for all $u \subseteq I$,

$$\mathcal{M}(u) = \textstyle\prod_{i \in u} M_i,$$

and the restriction maps are the projections that forget the coordinates outside the larger domain.

[5] ρ_A as in 31.3.(e).

Let βI be the Stone-Čech compactification of the discrete set I (21.6). Since βI is the Stone space of 2^I, Stone duality yields a lattice injection

$$2^I \longrightarrow \Omega(\beta I),$$

which identifies each $u \subseteq I$ with the compact clopen S_u [6]. We may, therefore, consider 2^I as a sublattice of $\Omega(\beta I)$. To determine the extension of $\mathcal{M}$ to $\Omega(\beta I)$, recall that by 21.2.(b), βI is *extremally disconnected*, that is, the closure of every open set is clopen. Hence, if $U \in \Omega(\beta I)$, there is a *unique* $u \in 2^I$, with $\overline{U} = S_u$. Hence, if $v \subseteq I$, elementary topology and Stone duality yield

$$U \subseteq S_v \quad \text{iff} \quad S_u \subseteq S_v \quad \text{iff} \quad u \subseteq v.$$

Whence, in the notation of 31.2,

$$G(U) = \{v \subseteq I : u \subseteq v\}.$$

and so

$$\mathfrak{e}\mathcal{M}(U) = \lim_{\rightarrow v \in G(U)} \mathcal{M}(v) = \mathcal{M}(u) = \textstyle\prod_{i \in u} M_i,$$

with restriction maps induced by $\mathcal{M}$. In particular,

$$\mathfrak{e}\mathcal{M}(\beta I) = \textstyle\prod_{i \in I} M_i = \mathcal{M}(I).$$

Let $T = \{t_k : k \in K\}$ be a compatible set of sections in $|\mathfrak{e}\mathcal{M}|$. For each $k \in K$, there is $u_k \subseteq I$ and $s_k \in A(u_k)$ such that

$$\overline{Et_k} \;=\; S_{u_k} \quad \text{and} \quad s_k \;=\; t_k.$$

Since $\mathcal{M}$ is a sheaf, the s_k have a unique gluing in $\mathcal{M}$, s, whose extent is $v \;=\; \bigcup_{k \in K} u_k$. By 19.7.(d).(2), we have

$$U =_{def} \textstyle\bigcup_{k \in K} Et_k \;\subseteq\; \bigcup_{k \in K} S_{u_k} \;\subseteq\; \overline{\bigcup_{k \in K} S_{u_k}} \;=\; S_v.$$

Thus, if $u \subseteq I$ is such that $U = S_u$, then $S_u \subseteq S_v$, that is, $u \subseteq v$. Then, $s_{|u}$ is the unique gluing of the t_k in $\mathfrak{e}\mathcal{M}$. We have shown that $\mathfrak{e}\mathcal{M}$ is a *sheaf* over βI. Thus, a sheaf over the discrete space I has given rise, by base extension, to a sheaf over the non-discrete space βI.

Finally, if s, $t \in |\mathfrak{e}\mathcal{M}|$, then

$$[\![s = t]\!]_{\mathfrak{e}\mathcal{M}} \;=\; Es \wedge Et \wedge S_{[\![s=t]\!]},$$

where, in the second member of the above equation, the index $[\![s = t]\!]$ is the equality of these terms as elements of $|\mathcal{M}|$. $\Box$

We now describe how restriction (26.5, 26.25) and extension behave on categories of sheaves. By 31.11.(a), extension *does not, in general*, preserve completeness (but see Exercise 31.14). If $L \subseteq R$ are frames, to construct a functor, $c\mathfrak{e} : \boldsymbol{Sh(L)} \longrightarrow \boldsymbol{Sh(R)}$, we follow the extension functor $\mathfrak{e}$ of 31.3 by completion over R, obtaining the **complete extension functor**. Hence, we have functors

$$\mathfrak{r} : \boldsymbol{Sh(R)} \longrightarrow \boldsymbol{Sh(L)} \quad \text{and} \quad c\mathfrak{e} : \boldsymbol{Sh(L)} \longrightarrow \boldsymbol{Sh(R)},$$

which are still adjoint, as in 31.7.

[6] S_u is the set of ultrafilters F in 2^I such that $u \in F$.

Corollary 31.9. *If $L \subseteq R$ are frames, then complete extension is left adjoint to restriction, i.e., if A is a L-sheaf and B is a R-sheaf, there is a bijective correspondence, natural in A and B,*

$$[c\mathfrak{e}A,\, B] \;\approx\; [A,\, B_{|L}],$$

which is induced, as in 31.7*, by natural isomorphisms*

$$\rho_A : A \approx (c\mathfrak{e}A)_{|L} \quad \textit{and} \quad \jmath_B : c\mathfrak{e}(B_{|L}) \longrightarrow B.$$

Proof. Left to the reader. □

Exercises

31.10. Prove 31.3.(d) making use of 26.21 and of the fact that A is restriction dense in $\mathfrak{e}A$. □

31.11. Let L be a semilattice with $\perp$ and $\top$. Let $2 = \{\perp, \top\} \subseteq L$; clearly, 2 is a complete subsemilattice of L.

a) Show that if A is a set (i.e., a 2-presheaf, 26.3), then the extension of A to L is the constant A presheaf on L (26.4).

b) If B is a L-presheaf, then $B_{|L}$ can be naturally identified with the set of global sections of B over L.

c) "Global section" is a functor from $\boldsymbol{pSh(L)}$ to $\boldsymbol{Set} = \boldsymbol{pSh(2)}$, right adjoint to the constant presheaf functor. □

31.12. Let L be a distributive lattice and let A be a L-presheaf. With 31.8 as a model, study the extension of A to the Stone space, $S(L)$, of L. Show that if U is a compact open in $S(L)$, there is $p \in L$, such that $\mathfrak{e}A(U) = A(p)$. □

31.13. If L is a subsemilattice of R and A is a *flabby* presheaf over R (26.28.(e)), then $A_{|L}$ is a flabby L-presheaf. □

31.14. If $L \subseteq R$ are frames, then $B \in \boldsymbol{Sh(R)} \;\Rightarrow\; B_{|L} \in \boldsymbol{Sh(L)}$. □

CHAPTER 32

Image of a Presheaf

As a preliminary to the construction in the title, we discuss the *essential image* of a presheaf along h.

32.1. Let $L \xrightarrow{h} R$ be a semilattice morphism and let A be a L-presheaf. Set

$$\mathcal{A} = \{\langle a, q \rangle \in |A| \times (Im\, h) : h(Ea) = q\}.$$

Define a binary relation ϑ_A on $\mathcal{A}$, as follows :

$$\langle a, q \rangle \ \vartheta_A \ \langle b, q' \rangle \quad \text{iff} \quad q = q' \quad \text{and} \quad \exists\, u \in h^{-1}(q) \text{ such that } a_{|_u} = b_{|_u}.$$

ϑ_A is an equivalence relation on $\mathcal{A}$. For instance, if $\langle a, q \rangle \ \vartheta_A \ \langle b, q' \rangle$ and $\langle b, q' \rangle \ \vartheta_A \ \langle c, q'' \rangle$, then $q = q' = q''$; moreover, there are u, $v \in L$ such that

$$h(u) = h(v) = q, \quad a_{|_u} = b_{|_u} \quad \text{and} \quad b_{|_v} = c_{|_v}.$$

Hence, $(u \wedge v) \in h^{-1}(q)$, $a_{|_{u\wedge v}} = c_{|_{u\wedge v}}$, and so $\langle a, q \rangle \ \vartheta_A \ \langle c, q'' \rangle$. If $r \in Im\, h$ and v, $w \in h^{-1}(r)$, then for all $\langle a, q \rangle \in \mathcal{A}$,

$$(\vartheta) \qquad \begin{cases} (i)\ \langle a_{|_v}, q \wedge r \rangle \in \mathcal{A}; \\ (ii)\ \langle a_{|_v}, q \wedge r \rangle \ \vartheta_A \ \langle a_{|_w}, q \wedge r \rangle \ \vartheta_A \ \langle a_{|_{v\wedge w}}, q \wedge r \rangle. \end{cases}$$

For (i), we have $Ea_{|_v} = Ea \wedge v$, and therefore $h(Ea \wedge v) = q \wedge r$. For (ii), let $u = Ea \wedge v \wedge w$. Then, $h(u) = q \wedge r$ and we get

$$(a_{|_v})_{|_u} \ = \ a_{|_{Ea\wedge v\wedge w}} \ = \ a_{|_{v\wedge w}} = (a_{|_w})_{|_u},$$

as needed. Let $|\eta_h A|$ be the set of equivalence classes of $\mathcal{A}$ by ϑ_A :

$$|\eta_h A| = \{\langle a, q \rangle / \vartheta_A : \langle a, q \rangle \in \mathcal{A}\}.$$

To avoid overloading notation, and whenever no confusion is possible, write $\langle a, q \rangle$ (or $\langle a, h(Ea) \rangle$) both for the element of $\mathcal{A}$ and for its class in $|\eta_h A|$. Define a structure of $Im\, h$-presheaf on $|\eta_h A|$ by the following prescriptions :

$[\eta_h\ 1]$: $E\langle a, q \rangle = q$;

$[\eta_h\ 2]$: For $r \in Im\, h$, $\ \langle a, q \rangle_{|_r} = \langle a_{|_v}, q \wedge r \rangle$, where $v \in h^{-1}(r)$.

By (ii) in (ϑ) above, $[\eta_h\ 2]$ is independent of the element v in $h^{-1}(r)$. It is straightforward that the maps defined by $[\eta_h\ 1]$ and $[\eta_h\ 2]$ endow $|\eta A_h|$ with the structure of $Im\, h$-presheaf, written $\eta_h A$. For instance,

$*$ $\langle a, q \rangle_{|_q} = \langle a_{|_{Ea}}, q \rangle = \langle a, q \rangle$, verifying [rest 1] in 26.1;

$*$ If r, $r' \in Im\, h$ and v, $v' \in L$ satisfy $h(v) = r$, $h(v') = r'$, then

$$(\langle a, q \rangle_{|r})_{|r'} = (\langle a_{|v}, q \wedge r \rangle)_{|r'} = \langle (a_{|v})_{|v'}, q \wedge r \wedge r' \rangle$$
$$= \langle a_{|v \wedge v'}, q \wedge r \wedge r' \rangle = \langle a, q \rangle_{|r \wedge r'},$$

verifying [rest 3] in 26.1. □

Definition 32.2. *The $(Im\, h)$-presheaf $\eta_h A$ constructed in* 32.1 *is the* **essential image of A along h**.

The basic properties of essential image are described in

Proposition 32.3. *Let $L \xrightarrow{h} R$ be a semilattice morphism and let A be a L-presheaf. With notation as above,*

a) $\eta_h A$ is a $(Im\, h)$-presheaf and the map [1]

$$\eta_h^A : |A| \longrightarrow |\eta_h A|, \text{ given by } \eta_h^A(a) = \langle a, h(Ea) \rangle,$$

*makes the pair $\langle \eta_h^A, h \rangle$ a morphism from A to $\eta_h A$ in **pSh**.*

*b) If B is a R-presheaf and $\langle f, h \rangle : A \longrightarrow B$ is a morphism in **pSh**, there is a* unique *map, $\widehat{f}: |\eta_h A| \longrightarrow |B|$, satisfying the following properties :*

(1) *For all $\langle a, q \rangle \in |\eta_h A|$ and $r \in Im\, h$,*

$$E\widehat{f}(\langle a, q \rangle) = q \quad \text{and} \quad \widehat{f}(\langle a, q \rangle_{|r}) = \widehat{f}(\langle a, q \rangle_{|r}),$$

that is, $\widehat{f}$ is a morphism of $(Im\, h)$-presheaves.

(2) *The following diagram commutes :*

$$\begin{tikzcd} A \arrow[rr, "\langle \eta_h, h \rangle"] \arrow[dr, "\langle f, h \rangle"'] & & \eta_h A \arrow[dl, "\langle \widehat{f},\ Id_{Im\, h} \rangle"] \\ & B & \end{tikzcd}$$

Proof. a) For $v \in L$ and $a \in |A|$,

$$\eta_h(a_{|v}) \;=\; \langle a_{|v}, h(Ea \wedge v) \rangle \;=\; \langle a, h(Ea) \rangle_{|h(v)} \;=\; \eta_h(a)_{|h(v)},$$

as needed to show that $\langle \eta_h, h \rangle$ is a morphism in ***pSh***.

b) Note that if $a, b \in |A|$ and $\eta_h(a) = \eta_h(b)$, then $f(a) = f(b)$. Indeed, the hypothesis means that $\langle a, h(Ea) \rangle\, \vartheta_A\, \langle b, h(Eb) \rangle$, that is,

$$h(Ea) = h(Eb) \quad \text{and} \quad \exists\, u \in L, \text{ with } h(u) = Ea \text{ and } a_{|u} = b_{|u}.$$

Hence, since $Efa = h(Ea)$, we get

$$f(a_{|u}) = (fa)_{|h(u)} = fa_{|h(Ea)} = f(a),$$

with similar relations for b, as desired. Set $\widehat{f}(\eta_h a) = f(a)$, to get the unique map with the properties in the statement. □

[1] In fact, $\langle a,\ h(Ea) \rangle / \vartheta_A$; but the reader should keep in mind our standing notational convention. Moreover, as was the case of image for L-sets, we omit the mention of A in the map η_h when it is clear from context.

If h is *surjective*, then $\eta_h A \in \boldsymbol{pSh(R)}$. In the general case, to obtain the *image of A along h*, we follow η_h by base extension from $Im\, h$ to R; 31.3 and 32.3 yield

COROLLARY 32.4. *Let $L \xrightarrow{h} R$ be a semilattice morphism and let A be a L-presheaf.*
a) There is a R-presheaf, $\mathfrak{d}_h A$, and a morphism $\langle\, \mathfrak{d}_h,\, h\,\rangle : A \longrightarrow \mathfrak{d}_h A$, such that if B is a R-presheaf and $\langle\, f, h\,\rangle : A \longrightarrow B$ is a morphism, there is a unique *morphism of R-presheaves, $g : \mathfrak{d}_h A \longrightarrow B$, making the triangle below-left is commutative.*

$$\begin{tikzcd} A \arrow[rr, "\langle\, \mathfrak{d}_h A,\, h\,\rangle"] \arrow[dr, "\langle\, f, h\,\rangle"'] & & \mathfrak{d}_h A \arrow[dl, "\langle\, g, Id_R\,\rangle"] \\ & B & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[rr, "\langle\, c_h, h\,\rangle"] \arrow[dr, "\langle\, f, h\,\rangle"'] & & c_h A \arrow[dl, "g"] \\ & B & \end{tikzcd}$$

b) If R is a frame, there is a sheaf $c_h A$ over R and a morphism
$$\langle\, c_h, h\,\rangle : A \longrightarrow c_h A,$$
such that if B is a R-sheaf $\langle\, f, h\,\rangle : A \longrightarrow B$ is a morphism, there is a unique *$g : c_h A \longrightarrow B$, making the triangle above-right commutative.*

PROOF. a) Define $\mathfrak{d}_h A = \mathfrak{e}(\eta_h A)$ and $\mathfrak{d}_h = \mathfrak{e} \circ \eta_h$; the existence and uniqueness of g comes from 32.3.(b) and 31.3.(d). For (b), just follow the preceding construction by completion over R. Details are left to the reader. $\square$

DEFINITION 32.5. *a) The diagram $\langle\, \mathfrak{d}_h,\, h\,\rangle : A \longrightarrow \mathfrak{d}_h A$ of Corollary* 32.4.(a) *is the* **image of $\boldsymbol{A}$ along $\boldsymbol{h}$**.
b) $\langle\, c_h, h\,\rangle : A \longrightarrow c_h A$ in 32.4.(b) *is the* **completion of $\boldsymbol{A}$ along $\boldsymbol{h}$**.

REMARK 32.6. The constructions η_h, $\mathfrak{d}_h$ and c_h are functors. We discuss the case of $\mathfrak{d}_h$, the others being analogous. If $f : A \longrightarrow A'$ is a morphism of L-presheaves, composition with $\langle\, \mathfrak{d}_h,\, h\,\rangle : A' \longrightarrow \mathfrak{d}_h A'$ yields a morphism from A to $\mathfrak{d}_h A'$. The universal property in 32.4 yields a *unique* morphism of R-presheaves, $\mathfrak{d}_h f : \mathfrak{d}_h A \longrightarrow \mathfrak{d}_h A'$, such that the following square is commutative :

$$\begin{tikzcd} A \arrow[r, "\langle\, \mathfrak{d}_h,\, h\,\rangle"] \arrow[d, "f"'] & \mathfrak{d}_h A \arrow[d, "\mathfrak{d}_h f"] \\ A' \arrow[r, "\langle\, \mathfrak{d}_h,\, h\,\rangle"'] & \mathfrak{d}_h A' \end{tikzcd}$$

$\mathfrak{d}_h f$ is obtained by base extension of the morphism $\eta_h f : \eta_h A \longrightarrow \eta_h B$, given explicitly as

(*) $$\eta_h f(\langle\, a,\, h(Ea)\,\rangle) = \langle\, fa,\, h(Ea)\,\rangle.$$

For ease of reading, let k stand for $\eta_h f$. Let $\langle a, q \rangle$, $\langle b, q' \rangle$ be sections in $\eta_h A$, such that for some $r \in R$ $\langle a, q \rangle_{|r} = \langle b, q' \rangle_{|r}$. We may as well suppose that $r \leq q \wedge q'$. Since

$$\mathfrak{d}_h A(r) = \lim_{\rightarrow w \in G(r)} \eta_h A(w)$$

there is $p \in L$ such that $r \leq h(p) \leq q \wedge q'$ and $\langle a, q \rangle_{|h(p)} = \langle b, q' \rangle_{|h(p)}$. But then,

$$\begin{aligned} k(\langle a, q \rangle)_{|r} &= (k(\langle a, q \rangle)_{|h(p)})_{|r} = k(\langle a_{|p}, h(p) \rangle)_{|r} \\ &= k(\langle b_{|p}, h(p) \rangle)_{|r} = (k(\langle b, q' \rangle)_{|h(p)})_{|r} = k(\langle b, q' \rangle)_{|r}, \end{aligned}$$

verifying the hypothesis of 26.21. Hence, $k = \eta_h f$ has a *unique* extension to morphism from $\mathfrak{d}_h A$ to $\mathfrak{d}_h B$. □

There is an important situation in which restriction and equality coexist in harmony : presheaves over a frame (26.8). It will be shown that in this case image as a Ω-set coincides with image as a Ω-presheaf. The results in 26.8 will be used without explicit reference.

Proposition 32.7. *Let $h : \Omega \longrightarrow R$ be a semilattice morphism. If A is a Ω-presheaf, then for all $x, y \in |A|$ and $p, q \in L$,*

a) $\boldsymbol{\varepsilon}_h(x) = \boldsymbol{\varepsilon}_h(y)$ and $h(p) = h(q) \Rightarrow \boldsymbol{\varepsilon}_h(x_{|p}) = \boldsymbol{\varepsilon}_h(y_{|q})$.

b) Let coker $h = \{p \in \Omega : h(p) = \top\}$ [2]; then, for $p \in$ coker h, $\boldsymbol{\varepsilon}_h(x) = \boldsymbol{\varepsilon}_h(x_{|p})$

c) The maps

$$\boldsymbol{\varepsilon}_h x \in |\boldsymbol{\varepsilon}_h A| \mapsto h(Ex) \quad \text{and} \quad \langle \boldsymbol{\varepsilon}_h x, r \rangle \in |\boldsymbol{\varepsilon}_h A| \times Im\, h \mapsto \boldsymbol{\varepsilon}_h(x_{|p})$$

where $p \in h^{-1}(r)$, define the structure of a $(Im\, h)$-presheaf on $|\boldsymbol{\varepsilon}_h A|$, compatible with its equality. Moreover, $\langle \boldsymbol{\varepsilon}_h, h \rangle$ is a morphism from A to $\boldsymbol{\varepsilon}_h A$.

d) For $p \leq q$ in Ω, let $\rho_{qp} : A(q) \longrightarrow A(p)$ be the restriction, $x \mapsto x_{|p}$. For $r \in Im\, h$, the system

$$\langle A(q); \{\rho_{qp} : p \leq q \text{ and } p, q \in h^{-1}(r)\} \rangle$$

is inductive over the left-directed subset $h^{-1}(r)$ of Ω. If A is extensional *then*

$$\boldsymbol{\varepsilon}_h A(r) = \lim_{\rightarrow p \in h^{-1}(r)} A(p).$$

Proof. a) If $\boldsymbol{\varepsilon}_h(x) = \boldsymbol{\varepsilon}_h(y)$ then

$$h(Ex) = h(Ey) = h([\![x = y]\!]). \tag{I}$$

Consequently, since $Ex_{|p} = p \wedge Ex$ and $Ey_{|q} = q \wedge Ey$, we get

$$h(Ex \wedge p) = h(Ex) \wedge h(p) = h(Ey) \wedge h(q) = h(Ey \wedge q). \tag{II}$$

Furthermore, by 26.8.(b) we have

$$h([\![x_{|p} = y_{|q}]\!]) = h(p \wedge q \wedge [\![x = y]\!]) = h(p) \wedge h([\![x = y]\!]),$$

[2]Similar to 4.9.(d); clearly, if it is non-empty, *coker h* is a *filter* in Ω.

and it is clear from (I) and (II) that $x_{|p}\ \theta_A\ y_{|q}$, i.e., $\boldsymbol{\varepsilon}_h(x) = \boldsymbol{\varepsilon}_h(y)$. Item (b) is an immediate consequence of (a), because $h(Ex \wedge p) = h(Ex)$.
c) By item (a), the definition of the restriction map in the statement is independent of representatives. Moreover, by 30.3.(a), $E(\boldsymbol{\varepsilon}_h x) = h(Ex)$, $x \in |A|$. It is now straightforward that $\boldsymbol{\varepsilon}_h A$ is a presheaf over the semilattice $Im\, h$ and that $\langle\, \boldsymbol{\varepsilon}_h, h\,\rangle$ is a morphism in $\boldsymbol{pSh}$.
d) Since h is a semilattice morphism, it is clear that

$$p,\, q \in h^{-1}(r) \;\;\Rightarrow\;\; p \wedge q \in h^{-1}(r),$$

and $h^{-1}(r)$ is down-directed (or left-directed) in Ω. For $p \in h^{-1}(r)$, let

$$f_p : A(p) \longrightarrow \boldsymbol{\varepsilon}_h A(r), \text{ be given by } f_p(x) = \boldsymbol{\varepsilon}_h x.$$

Hence, $f_p = (\boldsymbol{\varepsilon}_h)_{|A(p)}$. We shall verify that

$$\mathcal{I} = \langle\, \boldsymbol{\varepsilon}_h A(r);\ \{f_p\ :\ p \in h^{-1}(r)\}\,\rangle$$

is the inductive limit of the system in the statement. First, observe that $\mathcal{I}$ is a dual cone over the base diagram, that is, for $p \le q$ in $h^{-1}(r)$, the following diagram is commutative :

$$\begin{tikzcd}
A(q) \arrow[rr, "\rho_{qp}"] \arrow[dr, "f_q"'] & & A(p) \arrow[dl, "f_p"] \\
& \boldsymbol{\varepsilon}_h A(r) &
\end{tikzcd}$$

Indeed, the equation $f_p \circ f_{qp} = f_q$ follows easily from the fact, proven in (c), that $\langle\, \boldsymbol{\varepsilon}_h,\, h\,\rangle$ is a morphism from the Ω-presheaf A to the $Im\, h$-presheaf $\boldsymbol{\varepsilon}_h A$. To finish the proof, it suffices to check that for $p,\, q \in h^{-1}(r)$, $x \in A(p)$ and $y \in A(q)$,

$$f_p(x) = f_q y \quad \text{iff} \quad \exists\; u \in h^{-1}(r), \text{ such that } u \le p,\, q \text{ and } x_{|u} = y_{|u}.$$

By the definition of $\boldsymbol{\varepsilon}_h A$,

$$\boldsymbol{\varepsilon}_h x = \boldsymbol{\varepsilon}_h y \quad \text{iff} \quad h(Ex) = h(Ey) = h([\![x\, =\, y]\!]),$$

and so $[\![x\, =\, y]\!] \in h^{-1}(r)$. Since A is extensional, $x_{|[\![x=y]\!]} = y_{|[\![x=y]\!]}$, as needed. Conversely, if there is $u \le p \wedge q$ with $h(u) = r$ and $x_{|u} = y_{|u}$, the definition of the equality in A yields $u \le [\![x\, =\, y]\!]$. Thus,

$$r \;=\; h(u) \;\le\; h([\![x\, =\, y]\!]) \;\le\; h(Ex) \;=\; h(Ey) \le r,$$

establishing that $\boldsymbol{\varepsilon}_h x \;=\; \boldsymbol{\varepsilon}_h y$ and ending the proof. $\square$

We now have

Lemma 32.8. *Let $h : L \longrightarrow R$ be a semilattice morphism and let A be an extensional Ω-presheaf.*

a) For $a,\, b \in |A|$, $\quad a\ \theta_A\ b \quad$ iff $\quad \langle\, a,\, h(Ea)\,\rangle\ \vartheta_A\ \langle\, b,\, h(Eb)\,\rangle$.

b) The map $\beta_A : |\varepsilon_h A| \longrightarrow |\eta_h A|$, defined by $\varepsilon_h a \mapsto \langle a, h(Ea) \rangle/\vartheta_A$, is an isomorphism of $(Im\, h)$-presheaves. Moreover, if $f : A \longrightarrow A'$ is a morphism of L-presheaves, the following diagram is commutative :

$$\begin{tikzcd} \varepsilon_h A \arrow[r, "\varepsilon_h f"] \arrow[d, "\beta_A"'] & \varepsilon_h A' \arrow[d, "\beta_{A'}"] \\ \eta_h A \arrow[r, "\eta_h f"'] & \eta_h A' \end{tikzcd}$$

PROOF. a) If $a\ \theta_A\ b$ then $h(Ea) = h(Eb) = h([\![a = b]\!])$. Since A is extensional,

$$a_{|[\![a=b]\!]} \ = \ b_{|[\![a=b]\!]},$$

and so $\langle a, h(Ea) \rangle\ \vartheta_A\ \langle b, h(Eb) \rangle$. Conversely, if this relation holds, $h(Ea) = h(Eb)$ and there is $p \le (Ea \wedge Eb)$ such that $h(p) = h(Ea)$ and $a_{|p} = b_{|p}$. It follows that $p \le [\![a = b]\!]$, wherefrom we conclude that

$$h(Ea) \ \ge \ h([\![a = b]\!]) \ \ge \ h(p) \ = \ h(Ea),$$

showing that $a\ \theta_A\ b$.

b) It follows immediately from (a) that β_A is well-defined and bijective. It remains to check that for all $p \in \Omega$,

$$\beta_A((\varepsilon_h a)_{|h(p)}) \ = \ [\beta_A(\varepsilon_h a)]_{|h(p)}.$$

By 30.3.(c), we get

$$\begin{aligned} \beta_A((\varepsilon_h a)_{|h(p)}) \ &= \ \beta_A(\varepsilon_h(a_{|p})) \ = \ \langle a_{|p}, h(Ea \wedge p) \rangle \\ &= \ \langle a_{|p}, h(Ea) \wedge h(p) \rangle \ = \ \langle a, h(Ea) \rangle_{|h(p)} \ = \ [\beta_A(\varepsilon_h a)]_{|h(p)}, \end{aligned}$$

as desired. For $a \in |A|$, the explicit formula (*) for η_h in 32.6 and the definition of $\varepsilon_h f$ in 30.5 yield

$$\begin{aligned} \beta_{A'}(\varepsilon_h f(\varepsilon_h a)) \ &= \ \beta_{A'}(\varepsilon_h(fa)) \ = \ \langle fa, h(Efa) \rangle \ = \ \eta_h f(\langle a, h(Ea) \rangle) \\ &= \ \eta_h f(\beta_A(\varepsilon_h a)), \end{aligned}$$

and the displayed diagram is commutative, ending the proof. □

REMARK 32.9. Let $h : L \longrightarrow R$ be a semilattice morphism and let A be a L-presheaf with a compatible structure of L-set. By 31.3.(b), $\varepsilon_h A$ is restriction dense in $\mathfrak{d}_h A$. If R is a frame, it follows from 26.28.(a) and (e) that ε_h is *dense* in $\mathfrak{d}_h A$. Thus, both have the same completion over R, namely the R-sheaf $c_h A$ of 30.9 and 30.11. Thus, $\varepsilon_h A$ contains *essentially* the same information as $\mathfrak{d}_h A$ and $c_h A$. □

EXAMPLE 32.10. If $L \subseteq R$ are semilattices, write ι for the canonical injection of L into R. Then, image along ι is precisely base extension from L to R, that is, in the notation of 31.3, $\mathfrak{d}_\iota = \mathfrak{e}$. □

EXAMPLE 32.11. Let Ω be a frame and $p \in \Omega$. Consider the map

$$h : \Omega \longrightarrow p^{\leftarrow} \text{ given by } h(u) = p \wedge u.$$

The ideal $p^{\leftarrow}$ is a frame and h is a surjective $[\bigwedge, \bigvee]$-morphism. If X is a topological space and U is an open set in X, then h corresponds to the dual of the canonical continuous injection of U into X [3].

I. If A is an extensional presheaf over Ω, then $\boldsymbol{\varepsilon}_h A$ is an extensional presheaf over $p^{\leftarrow}$. To describe the sections of $\boldsymbol{\varepsilon}_h A$, let $q \leq p$. Then,

$$h^{-1}(q) \;=\; \{u \in \Omega : u \wedge p = q\}$$

and so $q = \min\, h^{-1}(q)$. Consequently, 32.7.(d) yields

$$\boldsymbol{\varepsilon}_h A(q) \;=\; \lim_{\rightarrow u \in h^{-1}(q)} A(u) \;=\; A(q).$$

Hence, image along h "forgets" the part of A that is outside p. Note that for all $s,\, t \in |A|$,

$$[\![\boldsymbol{\varepsilon}_h s \,=\, \boldsymbol{\varepsilon}_h t]\!] \;=\; p \wedge [\![s \,=\, t]\!].$$

Thus, if $Es,\, Et \in p^{\leftarrow}$, $\boldsymbol{\varepsilon}_h s \;=\; \boldsymbol{\varepsilon}_h t$ iff $s = t$, whence $\boldsymbol{\varepsilon}_h$ is injective when restricted to the subset $\bigcup_{q \leq p} A(q)$ of $|A|$.

II. By 7.8, h has a right adjoint $\rho : p^{\leftarrow} \longrightarrow \Omega$, determined by the adjunction

$$\text{For all } \langle u, q \rangle \in \Omega \times (p^{\leftarrow}), \quad u \wedge p \leq q \quad \text{iff} \quad u \leq \rho(q).$$

Thus, $[\rightarrow]$ in 6.1 implies that for all $q \leq p$, $\rho(q) = (p \rightarrow q)$.

Since h is surjective, by 7.9.(a), ρ is injective. Recalling that for all $q \in p^{\leftarrow}$

$$(p \rightarrow \bot) \;=\; \neg\, p \;\leq\; (p \rightarrow q) \;\leq\; (p \rightarrow p) \;=\; \top,$$

we get $\min(Im\,\rho) = \neg\, p$, $\max(Im\,\rho) = \top$ and $Im\,\rho \subseteq (\neg\, p)^{\rightarrow}$. In general, the latter containment is strict : for instance, in the algebra of opens of $(0,\, 1) \subseteq \mathbb{R}$, if $p = (0,\, 1/2) \cup (1/2,\, 1)$, then $\neg\, p = \emptyset$, but p *is not* of the form $(p \rightarrow q)$, $q \leq p$.

If B is an extensional $p^{\leftarrow}$-set, then $\boldsymbol{\varepsilon}_\rho B$ is the Ω-set defined by

(i) $|\boldsymbol{\varepsilon}_\rho B| = |B|$;

(ii) For $s,\, t \in |B|$, $[\![s = t]\!]_{\varepsilon_\rho B} \;=\; p \rightarrow ([\![s = t]\!]_B) \;\geq\; \neg\, p$.

Thus, a section of extent $q \leq p$ in B, is considered as a section of extent $(p \rightarrow q)$ in the Ω-set $\boldsymbol{\varepsilon}_\rho B$. In particular,

$*$ A global section in B (of extent p) is a global section (of extent $\top$) in $\boldsymbol{\varepsilon}_\rho B$;

$*$ The unique section over $\bot$ in B, yields the unique section of extent $\neg\, p$ in $\boldsymbol{\varepsilon}_\rho B$.

Thus, image along g "spreads out" B over Ω, without producing sections of extent strictly less than $\neg\, p$.

Now suppose that B is an extensional presheaf over $p^{\leftarrow}$. Then, by 32.7, $\boldsymbol{\varepsilon}_\rho B$ is a presheaf over $Im\,\rho$. If $q \leq r \leq p$, the restriction map of B is transferred directly to $\boldsymbol{\varepsilon}_\rho B$, that is, if $s \in \boldsymbol{\varepsilon}_\rho B(\rho(r))$, then

$$s_{|\rho(q)} = s_{|q},$$

regarded as an element of $\boldsymbol{\varepsilon}_\rho B(\rho(q))$. We take the opportunity to comment on the extension of $\boldsymbol{\varepsilon}_\rho B$ to the Ω-presheaf $\mathfrak{d}_\rho B$, discussed in 31.3. For $u \in \Omega$,

[3] That is, $h = \iota^*$, where $\iota : U \hookrightarrow X$.

$$\begin{aligned} G(u) &= \{q \in p^{\leftarrow} : u \leq (p \to q)\} = \{q \in p^{\leftarrow} : u \wedge p \leq q\} \\ &= \{q \leq p : h(u) \leq q\}. \end{aligned}$$

and $\mathfrak{d}_\rho B(u) = \lim_{\to q \in G(u)} B(q)$. Since for all $u \in \Omega$, $G(u) = G(p \wedge u)$ and $(p \wedge u) \in p^{\leftarrow}$, it follows that [4]

$$\mathfrak{d}_\rho B(u) = \mathfrak{d}_\rho B(p \wedge u) = \lim_{\to p \wedge u \leq q \leq p} B(q) = B(p \wedge u).$$

This fits well with the preceding results because if $u = (p \to q)$, then

$$p \wedge u = p \wedge (p \to q) = p \wedge q = q,$$

whence $\mathfrak{d}_\rho B(p \to q) = B(q)$. If $v \leq u$ in Ω, the restriction map from $\mathfrak{d}_\rho B(u)$ to $\mathfrak{d}_\rho B(v)$ is induced by B, i.e., if $s \in \mathfrak{d}_\rho B(u) = B(p \wedge u)$, then $s_{|v}$ in $|\mathfrak{d}_\rho B|$ is $s_{|p \wedge v}$ in $|B|$. Therefore,

* $(\mathfrak{d}_\rho B)_{|p}$ is a copy of B, that is, for all r, $q \leq p$,

$$\mathfrak{d}_\rho B(q) = B(q) \quad \text{and} \quad \cdot_{|q} \text{ is the restriction map of } B;$$

* If $u \leq \neg p$, then $\mathfrak{d}_\rho B(u) = B(\perp)$ and the restriction maps into $\mathfrak{d}_\rho B(u)$ are the only possible ones, taking all elements to $*$;
* If $s \in \mathfrak{d}_\rho B(u)$ and $t \in \mathfrak{d}_\rho B(v)$, then

$$[\![s = t]\!]_{\mathfrak{d}_\rho B} = u \wedge v \wedge \rho([\![s = t]\!]_B),$$

the verification of which is left to the reader.

* $\boldsymbol{\varepsilon}_h \boldsymbol{\varepsilon}_\rho B = B$.

III. By 7.8, h has a *left adjoint*, $\lambda : p^{\leftarrow} \longrightarrow \Omega$, determined by the adjunction

$$\text{For all } \langle x, q \rangle \in p^{\leftarrow} \times \Omega, \quad \lambda(x) \leq q \text{ iff } x \leq h(q) = p \wedge q.$$

Hence, λ is the natural inclusion $x \in p^{\leftarrow} \longmapsto x \in \Omega$.

If D is an extensional $p^{\leftarrow}$-set then $\boldsymbol{\varepsilon}_\lambda D$ is the extensional Ω-set, whose domain is that of D and whose equality is $[\![* = *]\!]_D$, considered as an element of Ω. In this case, $\boldsymbol{\varepsilon}_\lambda A$ may be identified with A, equality included.

If D is a $p^{\leftarrow}$-presheaf, as observed in 32.10, $\mathfrak{d}_\lambda$ is base extension from $p^{\leftarrow}$ to Ω. To describe the sections of $\mathfrak{d}_\lambda D = \mathfrak{e} D$, let $u \in \Omega$. Then,

$$G(u) = \{q \in p^{\leftarrow} : u \leq q\} = \begin{cases} u^{\leftarrow} & \text{if } u \in p^{\leftarrow} \\ \emptyset & \text{otherwise.} \end{cases}$$

Therefore,

$$\mathfrak{e} D(u) = \begin{cases} D(u) & \text{if } u \leq p \\ \emptyset & \text{otherwise,} \end{cases}$$

with restriction induced by D. Therefore, $\mathfrak{e} D$ is a copy of D over the elements in $p^{\leftarrow}$, with empty set of sections over all elements of Ω outside $p^{\leftarrow}$. If A is a presheaf over Ω, the reader can check that the preceding description and the results in **(I)** above entail

[4] Or put differently, since $u \leq (p \to (p \wedge u))$, $p \wedge u = \min(G(u)) \in p^{\leftarrow}$.

$$\mathfrak{d}_\lambda \varepsilon_h A \;=\; \mathfrak{e}\mathfrak{d}_h A \;=\; A_{|p}.$$

The examples above are a special − although important −, instances of 30.12 and 33.9. □

The description of the base extension along the natural embedding $p^{\leftarrow} \hookrightarrow \Omega$ in 32.11.**(III)** yields

COROLLARY 32.12. *Let Ω be frame and $p \in \Omega$. If A is a sheaf over $p^{\leftarrow}$, then $\mathfrak{e}A$ is a sheaf over Ω, such that for $u \in \Omega$*

$$\mathfrak{e}A(u) \;=\; \begin{cases} A(u) & \text{if } u \le p \\ \emptyset & \text{otherwise.} \end{cases}$$ □

Exercises

32.13. Let $L \xrightarrow{h} R$ be a semilattice morphism, A be a L-presheaf and $r \in Im\, h$.
a) $h^{-1}(r)$ is a down-directed subset of L.
b) For $u \le v$ in $h^{-1}(r)$, let $\rho_{uv} : A(u) \longrightarrow A(v)$ be the restriction map of A. The system $\langle\, A(u); \{\rho_{uv} \;:\; v \le u \text{ in } h^{-1}(r)\}\,\rangle$ is inductive and

$$\eta_h A(r) \;=\; \varinjlim_{u \in h^{-1}(r)} A(u).$$ □

32.14. Show that Proposition 30.12 holds for the image functor, that is, if $L \xrightarrow{f} R \xrightarrow{g} T$ are semilattice morphisms, then $\mathfrak{d}_{g \circ f} = \mathfrak{d}_g \circ \mathfrak{d}_f$. □

32.15.[5] Let $h : \Omega \longrightarrow R$ be a surjective semilattice morphism and let $\mathfrak{C}(R)$ be the category whose objects are extensional R-presheaves with compatible structure of R-set and arrows R-set morphisms. By 32.7.(c), ε_h is a functor from $\boldsymbol{pSh(\Omega)}$ to $\mathfrak{C}(R)$. Prove that this functor preserves all finite limits. □

[5]This exercise complements 30.6.

CHAPTER 33

Inverse Image of a Presheaf

Let $h : L \longrightarrow R$ be a semilattice morphism. If B is a R-presheaf, set

(i) $|\mathfrak{i}_h B| \;=\; \coprod_{p \in L} B(h(p)) \;=\; \bigcup_{p \in L} B(h(p)) \times \{p\}$;

(ii) $E_{\eta_h B}\langle s, p \rangle \;=\; p$; (iii) $\langle s, p \rangle_{|q} \;=\; \langle s_{|h(q)}, p \wedge q \rangle$.

The properties [rest 1] $-$ [rest 3] in 26.1 are readily verified. For instance

$$\langle s, p \rangle_{|p} \;=\; \langle s_{|h(p)}, p \rangle \;=\; \langle s, p \rangle,$$

because $s \in B(h(p))$ and B is a R-presheaf. Similarly, for p, q, $r \in L$,

$$\begin{aligned} (\langle s, p \rangle_{|q})_{|r} \;&=\; \langle (s_{|h(q)})_{|h(r)}, p \wedge q \wedge r \rangle \;=\; \langle s_{|h(q) \wedge h(r)}, p \wedge q \wedge r \rangle \\ &=\; \langle s_{|h(q \wedge r)}, p \wedge q \wedge r \rangle \;=\; \langle s, p \rangle_{|q \wedge r}. \end{aligned}$$

Definition 33.1. *The presheaf* $\mathfrak{i}_h B$ *is the* **inverse image of B along h**.

Remark 33.2. As was the case in 30.13, for sheaves over topological spaces, classical nomenclature is dual to ours. If $f : X \longrightarrow Y$ is a continuous map and A is a sheaf over X, *inverse image along* f^* is called **image along f**, written $f_* A$. The arguments for this and for our usage are the same as in 30.13. Additional arguments will emerge in 33.9. A clue to what is happening is already in 33.11. $\Box$

Proposition 33.3. *Let $h : L \longrightarrow R$ be a semilattice morphism and B a R-presheaf.*

a) $\mathfrak{i}_h B$ is extensional whenever h is a $[\wedge, \bigvee]$-morphism and B is extensional.

b) The map

$$\mathfrak{i}_h : |\mathfrak{i}_h B| \longrightarrow |B|, \text{ given by } \langle s, p \rangle \mapsto s,$$

makes $\langle \mathfrak{i}_h, h \rangle$ a morphism from $\mathfrak{i}_h B$ to B in ***pSh*** (29.1.(b))*, with the following universal property :*

If A is a L-presheaf and $\langle f, h \rangle : A \longrightarrow B$ is a morphism in ***pSh****, there is a* unique *morphism of L-presheaves, $g : A \longrightarrow \mathfrak{i}_h B$, making the following diagram commutative :*

$$\begin{tikzcd} A \arrow[rr, "\langle\, g, Id_L \,\rangle"] \arrow[dr, "\langle\, f, h \,\rangle"'] & & \mathfrak{i}_h B \arrow[dl, "\langle\, \mathfrak{i}_h, h \,\rangle"] \\ & B & \end{tikzcd}$$

c) If L and R are frames, h is a frame morphism and B is a sheaf, then $\mathfrak{i}_h B$ is a sheaf over L.

Proof. a) For $\langle\, s, p \,\rangle$, $\langle\, t, p \,\rangle \in |\mathfrak{i}_h B|$, suppose that there is $\alpha \subseteq L$ is such that

$$\bigvee \alpha = p \quad \text{and} \quad \langle\, s_{|t(q)}, q \,\rangle = \langle\, t_{|h(q)}, q \,\rangle, \text{ for all } q \in \alpha.$$

Since h preserves joins, we have $\bigvee_{q \in \alpha} h(q) \;=\; h(p)$ and $s_{|h(q)} \;=\; t_{|h(q)}$, for all $q \in \alpha$. Since $E_B s = E_B t = h(p)$, the extensionality of B implies that $s = t$ and so $\langle\, s, p \,\rangle = \langle\, t, p \,\rangle$, as needed.

b) Clearly, $\langle\, \mathfrak{i}_h, h \,\rangle$ is a morphism in $\boldsymbol{pSh}$. If $\langle\, f, h \,\rangle : A \longrightarrow B$ is a morphism in $\boldsymbol{pSh}$, then for all $x \in |A|$, we have $E_B f x = h(E_A x)$. Hence, $\langle\, fx, E_A x \,\rangle \in |\mathfrak{i}_h B|$, and the map

$$g : |A| \longrightarrow |\mathfrak{i}_h B|, \text{ given by by } x \longmapsto \langle\, fx, E_A x \,\rangle$$

is well-defined. If $p \in L$ and $x \in |A|$, we have

$$\begin{aligned} g(x_{|p}) \;&=\; \langle\, f(x_{|p}),\, p \wedge E_A x \,\rangle \;=\; \langle\, (fx)_{|h(p)},\, p \wedge E_A x \,\rangle \;=\; \langle\, fx, E_A x \,\rangle_{|p} \\ &=\; (gx)_{|p}, \end{aligned}$$

and g is a morphism of L-presheaves. Uniqueness is clear.

c) Let $\{\langle\, x_i, p_i \,\rangle : i \in I\}$ be a compatible set of sections in $\mathfrak{i}_h B$, i.e., for all i, $j \in I$

$$\langle\, x_i, p_i \,\rangle_{|p_j} \;=\; \langle\, x_{i|h(p_j)},\, p_i \wedge p_j \,\rangle \;=\; \langle\, x_{j|h(p_j)},\, p_i \wedge p_j \,\rangle \;=\; \langle\, x_j, p_j \,\rangle_{|p_i}.$$

Consequently, $x_{i|h(p_j)} \;=\; x_{j|h(p_i)}$; since $E_B x_i = h(p_i)$, the collection x_i, $i \in I$, is compatible in B. Let x be the *unique* gluing of the x_i in B. Since h preserves joins, it follows that

$$E_B x \;=\; \bigvee_{i \in I} h(p_i) \;=\; h(\bigvee_{i \in I} p_i),$$

and so $\langle\, x, q \,\rangle$, with $q = \bigvee_{i \in I} p_i$, is an element in the domain of $\mathfrak{i}_h B$. It is easily established that $\langle\, x, q \,\rangle$ is a gluing of the $\langle\, x_i, p_i \,\rangle$ in $\mathfrak{i}_h B$. Uniqueness of $\langle\, x, q \,\rangle$ is a consequence of item (a), ending the proof. $\square$

Illustrating the interplay between direct and inverse image we have

PROPOSITION 33.4. *Consider the commutative square below left,*

$$\begin{tikzcd} L \arrow[r, "h"] \arrow[d, "\alpha"'] & R \arrow[d, "\beta"] \\ P \arrow[r, "\widehat{h}"'] & Q \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\langle f, h \rangle"] \arrow[d, "\langle c_\alpha, \alpha \rangle"'] & B \arrow[d, "\langle c_\beta, \beta \rangle"] \\ c_\alpha A \arrow[r, "\langle \widehat{f}, \widehat{h} \rangle"'] & c_\beta B \end{tikzcd}$$

where L, R are semilattices, P, Q are frames, h, α, β are semilattice morphisms and $\widehat{h}$ is a frame-morphism. Suppose $\langle f, h \rangle : A \longrightarrow B$ is a morphism in $\boldsymbol{pSh}$. Then, there is a unique *morphism, $\langle \widehat{f}, \widehat{h} \rangle : c_\alpha A \longrightarrow c_\beta B$, making the diagram above-right commutative.*

PROOF. Recall that $c_\alpha A$ is the completion of A along α, as in 30.9. Write $\gamma = \beta \circ h : L \longrightarrow Q$ and $f_0 = c_\beta \circ f$. Then, $\langle f_0, \gamma \rangle$ is a morphism in $\boldsymbol{pSh}$ from A to $c_\beta B$. Hence, 32.4 yields a *unique* morphism, $f_1 : \mathfrak{d}_\gamma A \longrightarrow c_\beta$B, such that the following triangle is commutative :

$$\begin{tikzcd} A \arrow[rr, "\langle \mathfrak{d}_\gamma, \gamma \rangle"] \arrow[dr, "\langle f_0, \gamma \rangle"'] & & \mathfrak{d}_\gamma A \arrow[dl, "f_1"] \\ & c_\beta B & \end{tikzcd}$$

Since $\gamma = \widehat{h} \circ \alpha$, it follows from Exercise 32.14 that $\mathfrak{d}_\gamma A = \mathfrak{d}_{\hat{h}}(\mathfrak{d}_\alpha A)$ and so we have a morphism $\langle \mathfrak{d}_{\hat{h}}, \widehat{h} \rangle : \mathfrak{d}_\alpha A \longrightarrow \mathfrak{d}_\gamma A$, which upon composition with f_1, yields a morphism $\langle f_2, \widehat{h} \rangle : \mathfrak{d}_\alpha A \longrightarrow c_\beta B$. By 33.3.(b), there is a *unique* morphism $f_3 : \mathfrak{d}_\alpha A \longrightarrow \mathfrak{i}_{\hat{h}} c_\beta B$, such that the triangle below-left is commutative :

$$\begin{tikzcd} \mathfrak{d}_\alpha A \arrow[rr, "f_3"] \arrow[dr, "\langle f_2, \widehat{h} \rangle"'] & & \mathfrak{i}_{\hat{h}} c_\beta B \arrow[dl, "\langle \mathfrak{i}_{\hat{h}}, \widehat{h} \rangle"] \\ & c_\beta B & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[rr, "\langle c_\alpha, \alpha \rangle"] \arrow[dr, "\langle f_4, \alpha \rangle"'] & & c_\alpha A \arrow[dl, "f_5"] \\ & \mathfrak{i}_{\hat{h}} c_\beta B & \end{tikzcd}$$

Hence, with $f_4 = f_3 \circ \mathfrak{d}_\alpha$, we obtain a morphism in $\boldsymbol{pSh}$, $\langle f_4, \alpha \rangle : A \longrightarrow \mathfrak{i}_{\hat{h}} c_\beta B$. Since, $\mathfrak{i}_{\hat{h}} c_\beta B$ is a sheaf over P (33.3.(c)), Corollary 32.4.(b) yields a *unique* morphism $f_5 : c_\alpha A \longrightarrow \mathfrak{i}_{\hat{h}} c_\beta B$, such that the triangle above-right is commutative. Composing f_5 with $\langle \mathfrak{i}_{\hat{h}}, \widehat{h} \rangle$ one obtains the morphism $\langle \widehat{f}, \widehat{h} \rangle : c_\alpha A \longrightarrow c_\beta B$ with the desired properties. $\square$

Proposition 33.4 and Examples 27.21 and 29.2 yield

Corollary 33.5. *A homomorphism of commutative rings with identity, $f : A \longrightarrow B$, induces a morphism of their structure sheaves (or affine schemes)*

$$\langle f, f^*_Z \rangle : c\mathcal{A} \longrightarrow c\mathcal{B}$$

such that for all $a \in R$ the restriction of f to the ring RS_a^{-1} of sections of $c\mathcal{R}$ over Z_a is given by $f(\langle x/a^n, Z_a \rangle) \ = \ \langle fx/(fa)^n, Z_{fa} \rangle$.

Proof. With notation as in 29.2, we have a commutative square

$$\begin{tikzcd} L(A) \arrow[r, "h"] \arrow[d, "\text{can.}"'] & L(B) \arrow[d, "\text{can.}"] \\ \Omega_A \arrow[r, "f^*_Z"'] & \Omega_B \end{tikzcd}$$

where Ω_A, Ω_B are the frames of opens in $Spec(A)$ and $Spec(B)$, and *can.* are the inclusions [1]. Now apply 33.4 to $\langle \gamma, h \rangle : \mathcal{A} \longrightarrow \mathcal{B}$ of 29.2 to complete the proof. $\square$

Corollary 33.5 and Theorem 27.22 show that there is a natural covariant functor from the category of commutative rings with identity into $\boldsymbol{Sh_t}$, whose image are the affine schemes.

Example 33.6. If $L \subseteq R$ are semilattices and ι is the injection of L into R, then inverse image along ι is precisely restriction to L, that is, $\eta_\iota B = B_{|L}$. This, Example 32.10 and Proposition 31.7 suggest the development that follows. $\square$

Theorem 33.7. *Let $h : L \longrightarrow R$ be a semilattice morphism, A be a L-presheaf and B be a R-presheaf.*

a) There are unique *morphism of L and R presheaves, respectively,*

$$\mathfrak{f}_A : A \longrightarrow \mathfrak{i}_h\mathfrak{d}_h A \quad \textit{and} \quad \mathfrak{b}_B : \mathfrak{d}_h\mathfrak{i}_h B \longrightarrow B,$$

natural in A and B, making the following diagrams commutative :

$$\begin{tikzcd} A \arrow[rr, "\langle \mathfrak{f}_A, Id_L \rangle"] \arrow[dr, "\langle \mathfrak{d}_h, h \rangle"'] & & \mathfrak{i}_h\mathfrak{d}_h A \arrow[dl, "\langle \mathfrak{i}_h, h \rangle"] \\ & \mathfrak{d}_h A & \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{i}_h B \arrow[rr, "\langle \mathfrak{d}_h, h \rangle"] \arrow[dr, "\langle \mathfrak{i}_h, h \rangle"'] & & \mathfrak{d}_h\mathfrak{i}_h B \arrow[dl, "\langle \mathfrak{b}_B, Id_R \rangle"] \\ & B & \end{tikzcd}$$

If L and R are frames and A, B are sheaves, then

(i) $\mathfrak{f}_A$ is a morphism from A to $\mathfrak{i}_h c_h A$;

[1] Recall that h is the restriction of f^*_Z to $L(A)$.

(ii) $\mathfrak{b}_B$ *has a* unique *extension to a morphism from* $c_h\mathfrak{i}_hB$ *to* B*, still denoted by the same symbol.*

b) Image along h *is left adjoint to inverse image along* h*, that is, there is a bijective correspondence, natural in* A *and* B*,*

$$[\mathfrak{d}_hA,\, B] \quad\approx\quad [A,\, \mathfrak{i}_hB],$$

given by $\alpha \mapsto \mathfrak{i}_h\alpha \circ \mathfrak{f}_A$*, whose inverse is* $\beta \mapsto \mathfrak{b}_B \circ \mathfrak{d}_h\beta$*.*

c) If L *and* R *are frames and* h *is a frame morphism, the functor* $c_h : \boldsymbol{Sh(L)} \longrightarrow \boldsymbol{Sh(R)}$ *is left adjoint to* $\mathfrak{i}_h : \boldsymbol{Sh(R)} \longrightarrow \boldsymbol{Sh(L)}$ *and the natural bijective correspondence in* (*b*) *is valid for sheaves, with* c_h *in place of* $\mathfrak{d}_h$ *and* $\mathfrak{f}_A$*,* $\mathfrak{b}_B$ *interpreted as in* (*a*).(*i*) *and* (*a*).(*ii*)*, respectively.*

PROOF. The hard work has all been done. For (a), existence and uniqueness of $\mathfrak{f}_A$ and $\mathfrak{b}_B$ follows from 33.3.(b) and 32.4. For instance, since $\langle\, \mathfrak{d}_h, h\,\rangle$ is a morphism from A to $\mathfrak{d}_hA$, 33.3.(b) yields a unique morphism of L-presheaves, $\mathfrak{f}_A$, making commutative the diagram displayed on the left. Similarly, applying 32.4 to the morphism $\langle\, \mathfrak{i}_h, h\,\rangle$ yields $\mathfrak{b}_B$.

Since $\mathfrak{d}_hA$ may be considered a subpresheaf of c_hA, we get $\mathfrak{i}_h\mathfrak{d}_hA \subseteq \mathfrak{i}_hc_hA$, as needed for (*a*).(*i*). The result in (*a*).(*ii*) is a consequence of unique extension of a morphism to the completion of a presheaf (27.9.(3) or 30.9.(3)).

Items (b) and (c) are straightforward consequences of (a) and the uniqueness of morphisms making certain diagrams commute. Details are left to the reader. □

REMARK 33.8. In the proof of 33.7, there is no explicit formula for the morphisms $\mathfrak{f}_A$ and $\mathfrak{b}_B$, because the proof does not depend on this information. However, in certain situations it may be useful to have such formulas. In the setting of 33.7, let A is a L-presheaf. Then, the map

$$a \in |A| \quad \longmapsto \quad \langle\, \eta_h a, Ea\,\rangle \;\in\; |\mathfrak{i}_h\eta_hA| \;\subseteq\; |\mathfrak{i}_h\mathfrak{d}_hA|,$$

is a morphism of L-presheaves, making the left-displayed diagram in the statement of 33.7 commutative. By uniqueness, it must be $\mathfrak{f}_A$.

Now let B be a R-presheaf, $p \in L$ and $s \in |B|$, with $Es = h(p)$. Consider the map α defined by

$$\langle\, \langle\, s, p\,\rangle,\, h(p)\,\rangle \in |\eta_h\mathfrak{i}_hB| \longmapsto s \in |B|.$$

To show that α is well-defined, assume that $\langle\, \langle\, s, p\,\rangle,\, h(p)\,\rangle \;\vartheta_{\eta_hB}\; \langle\, \langle\, t, q\,\rangle,\, h(q)\,\rangle$. Then, $h(p) = h(q)$ and there is $u \leq p$ such that $h(u) = h(p)$ and

$$\langle\, s, p\,\rangle_{|u} = \langle\, t, q\,\rangle_{|u};$$

whence, $s_{|h(u)} = t_{|h(u)}$. Since $Es = h(u) = h(p) = Et$, we obtain $s = t$, as needed. Clearly, α preserves extent. If $q \in Im\,h$, since η_h commutes with restriction to q, we have, for $r \in h^{-1}(q)$,

$$\alpha(\eta_h(\langle\, s, p\,\rangle)_{|h(r)}) \;=\; \alpha(\eta_h(\langle\, s, p\,\rangle_{|r})) \;=\; \alpha(\eta_h(\langle\, s_{|h(r)}, p \wedge r\,\rangle) \;=\; s_{|q},$$

and α preserves restriction by all $q \in Im\,h$. It is clear that α makes the right-displayed diagram in the statement of 33.7 commutative, with $\mathfrak{d}_h$ replaced by η_h.

By 31.3.(d), α has a *unique* extension to a morphism from $\mathfrak{d}_h \mathfrak{i}_h B$ to B. This extension makes the right-displayed diagram in the statement of 33.7 commute, and so is the required morphism $\mathfrak{b}_B$. $\square$

In the spatial setting, if $f : X \longrightarrow Y$ is a continuous map, then f^* is a frame morphism, with right adjoint f_*. We now consider an abstract counterpart of this situation, already previewed in 32.11.

As shown by (a) and (d) in 33.3, inverse image has pleasant preservation properties when h is a $[\wedge, \bigvee]$-morphism. Additionally, 7.8 guarantees that h has a right adjoint, $g : R \longrightarrow L$, that is, g is a $\bigwedge$-morphism such that

[*adj*] For all $\langle u, v \rangle \in L \times R$, $hu \leq v \;\Leftrightarrow\; u \leq gv$.

When L and R are frames, there is a connection between the inverse image along h and the image along g, for *extensional presheaves* :

Theorem 33.9. *Let $h : L \longrightarrow R$ be a frame morphism, with right adjoint g. Let A, B be an extensional presheaves over L and R, respectively. Then,*

a) For all $\langle s, p \rangle, \langle t, q \rangle \in |\mathfrak{i}_h B|$

$$[\![\langle s, p \rangle = \langle t, q \rangle]\!] \;=\; g([\![s = t]\!]_B) \wedge p \wedge q.$$

b) There is an injective L-set morphism, natural in B,

$$\iota_B : \boldsymbol{\varepsilon}_g B \longrightarrow \mathfrak{i}_h B, \text{ given by } \boldsymbol{\varepsilon}_g s \longmapsto \langle s_{|hg(Es)}, g(Es) \rangle,$$

whose image is restriction dense [2] *in $\mathfrak{i}_h B$. Furthermore, if $p \in Im\, g$ and $x \in |\boldsymbol{\varepsilon}_g B|$, then* $\iota_B(x_{|p}) = \iota_B(x)_{|p}$.

c) If B is a R-sheaf, ι_B has a unique extension to a natural isomorphism from $c_g B$ to $\mathfrak{i}_h B$, that in turn induces a natural isomorphism between the functors c_g and $\mathfrak{i}_h$.

d) The functor c_h is left adjoint to c_g.

e) If h is surjective, then

(1) $c_h \mathfrak{i}_h B \approx c_h c_g B \approx B$;

(2) *The map $\langle \mathfrak{i}_h, h \rangle : \mathfrak{i}_h \boldsymbol{\varepsilon}_h A \longrightarrow \boldsymbol{\varepsilon}_h A$ is a retract* (29.5).

f) If h is injective, then $\mathfrak{i}_h c_h A \approx c_g c_h A \approx A$.

Proof. a) Since $E\langle s, p \rangle = p$ in $\mathfrak{i}_h B$, while $E_B s = h(p)$, we have

$$\begin{aligned}
[\![\langle s, p \rangle = \langle t, q \rangle]\!] &= \bigvee \{u \leq p \wedge q : \langle s, p \rangle_{|u} = \langle t, q \rangle_{|u}\} \\
&= \bigvee \{u \leq p \wedge q : \langle s_{|h(u)}, u \rangle = \langle t_{|h(u)}, u \rangle\} \\
&= \bigvee \{u \leq p \wedge q : s_{|h(u)} = t_{|h(u)}\} \\
&= \bigvee \{u \leq p \wedge q : h(u) \leq [\![s = t]\!]_B\},
\end{aligned}$$

where the last step in the above equalities follows from extensionality and 26.8.(d).(2). Since h preserves arbitrary joins, it follows that

[2]Defined in 26.28.(d).

$$h([\![\langle s,p\rangle = \langle t,q\rangle]\!]) \;\leq\; [\![s=t]\!]_B,$$

and the adjointness relation $[adj]$, $E\langle s,p\rangle = p$ and $E\langle t,q\rangle = q$, entail

$$[\![\langle s,p\rangle = \langle t,q\rangle]\!] \;\leq\; g([\![s=t]\!]_B) \wedge p \wedge q.$$

Let $w = g([\![s=t]\!]_B) \wedge p \wedge q$; clearly, $h(w) \leq h(p \wedge q) = h(p) \wedge h(q)$. By 7.8.(a), we have

$$\text{For all } v \in R \text{ and } p \in L,\quad hg(v) \leq v \quad\text{ and }\quad p \leq gh(p), \tag{*}$$

Thus,

$$\begin{aligned} h(w) &= h(g([\![s=t]\!]_B) \wedge p \wedge q) = h(g([\![s=t]\!]_B)) \wedge h(p) \wedge h(q) \\ &\leq [\![s=t]\!]_B \wedge h(p) \wedge h(q) = [\![s=t]\!]_B, \end{aligned}$$

and so $s_{|h(w)} = t_{|h(w)}$ in B. Hence,

$$\langle s,p\rangle_{|w} = \langle s_{|h(w)}, w\rangle = \langle t_{|h(w)}, w\rangle = \langle t,q\rangle_{|w},$$

showing that $w \;\leq\; [\![\langle s,p\rangle = \langle t,q\rangle]\!]$, as needed.

b) For $s \in |B|$, (*) entails $h(g(Es)) \leq Es$ and so $\langle s_{|hg(Es)}, g(Es)\rangle$ is in $|\mathfrak{i}_h A|$. To show that ι_B is well-defined, let $t \in |B|$ verify $\boldsymbol{\varepsilon}_g s = \boldsymbol{\varepsilon}_g t$; then

$$g(Es) = g(Et) = g([\![s = t]\!])$$

and so $h(g([\![s = t]\!])) \leq [\![s = t]\!]$ ((*), again), which entails $s_{|hg(Es)} = t_{|hg(Et)}$, as needed. Clearly,

$$E\iota_B\boldsymbol{\varepsilon}_g s = g(Es) = E\boldsymbol{\varepsilon}_g s.$$

Furthermore, item (a) yields

$$\begin{aligned} [\![\iota_B\boldsymbol{\varepsilon}_g s = \iota_B\boldsymbol{\varepsilon}_g t]\!] &= [\![\langle s_{|hg(Es)}, g(Es)\rangle = \langle t_{|hg(Et)}, g(Et)\rangle]\!] \\ &= g([\![s_{|hg(Es)} = \langle t_{|hg(Et)}\rangle]\!]) \wedge g(Es) \wedge g(Et) \\ &= g([\![s = t]\!]) \wedge g(Es) \wedge g(Et) \\ &= g([\![s = t]\!]) = [\![\boldsymbol{\varepsilon}_g s = \boldsymbol{\varepsilon}_g t]\!], \end{aligned}$$

and ι_B is, by 25.21, a monic morphism from $\boldsymbol{\varepsilon}_G B$ into $\mathfrak{i}_h B$. For $\langle s,p\rangle \in |\mathfrak{i}_h B|$, we have, because of $Es = h(p)$ and $hgh = h$ (7.8.(b)),

$$\begin{aligned} \xi &=_{def} \iota_B\boldsymbol{\varepsilon}_g s = \langle s_{|hg(Es)}, g(Es)\rangle = \langle s_{|hgh(p)}, gh(p)\rangle \\ &= \langle s_{|h(p)}, gh(p)\rangle = \langle s, gh(p)\rangle, \end{aligned}$$

and so, since $p \leq gh(p)$, it follows that

$$\xi_{|p} = \langle s_{|h(p)},\ p \wedge gh(p)\rangle = \langle s,p\rangle.$$

Hence, the image of ι_B is *restriction dense* (26.28.(d)) in $\mathfrak{i}_h B$. Similar techniques will establish the remaining statements in (b).

c) By 27.17, ι_B has a unique monic extension, γ_B, to $c_g B$, the completion of $\boldsymbol{\varepsilon}_g B$ over L. Since the image of ι_B is restriction dense in $\mathfrak{i}_h B$, it is clear that γ_B is onto $|\mathfrak{i}_h B|$, and hence an isomorphism. Just as in the proof of 30.12, the family

$$\gamma = \{\gamma_B : B \text{ is a } R\text{-sheaf}\}$$

is a natural isomorphism between the functors c_g and $\mathfrak{i}_h$. Hence, sheaf-theoretical inverse image by h comes to the same thing as sheaf-theoretical image by its adjoint

g. This also establishes (d). For (e), observe that if h is surjective, 7.9 guarantees that $h \circ g = Id_R$ and (1) is an immediate consequence of (b) and 30.12.(c). For (2), consider the morphism

$$\langle\, \iota_{(\varepsilon_h A)} \circ \varepsilon_g,\ g\,\rangle : \varepsilon_h A \longrightarrow \mathfrak{i}_h \varepsilon_h A,$$

where ι_* is defined in item (b). Since $h \circ g = Id_R$, it remains to check that $(\mathfrak{i}_h \circ \iota_{(\varepsilon_h A)} \circ \varepsilon_g)$ is the identity in $\varepsilon_h A$. The definitions of ι_* in item (b) and of $\mathfrak{i}_h$ in 33.3.(b) yield, for $x \in |A|$,

$$\begin{array}{rcl} \mathfrak{i}_h(\iota_{(\varepsilon_h A)}(\varepsilon_g(\varepsilon_h x))) & = & \mathfrak{i}_h\Big(\langle\, (\varepsilon_h x)_{|hg(Ex)},\ g(E\varepsilon_h x)\,\rangle\Big) \\ & = & \mathfrak{i}_h(\langle\, \varepsilon_h x,\ g(E\varepsilon_h x)\,\rangle) \ = \ \varepsilon_h x, \end{array}$$

as needed. Item (f) is similar to (e).(1), ending the proof. $\square$

The reader will have noticed that Propositions 30.12 and Theorem 33.9 apply to the spatial situation, in fact generalizing it considerably. All the classical relations involving image and inverse image along a continuous map are straightforward consequences of the results presented above.

EXAMPLE 33.10. Let $h : \Omega \longrightarrow 2 = \{\bot, \top\}$ be a *pure state* in Ω (12.14), i.e., a surjective frame morphism; h has a right adjoint g (7.8), given by

$$g(\top) = \top \quad \text{and} \quad g(\bot) \ = \ \bigvee \{p \in \Omega : h(p) = \bot\}.$$

Since $F = coker\, h = \{q \in \Omega : h(q) = \top\}$ is a completely prime filter in Ω (12.1), the join of all elements outside F is outside F, constituting the value of g at $\bot$. Write

$$q_F \ = \ g(\bot) \ = \ \bigvee \{p \in \Omega : p \notin F\}.$$

In the spatial setting, if X is a topological space and x is point in X, the (completely prime) filter of open neighborhoods of x in X induces a pure state p_x in $\Omega(X)$, given by

$$p_x(U) = \top \quad \text{iff} \quad u \in \nu_x.$$

The right adjoint of p_x takes $\bot$ to $(X - \overline{\{x\}})$, the largest open in X to which x does not belong. If B is a set (i.e., a 2-sheaf), inverse image of B along h is, by 33.3.(c), a sheaf over Ω, which by 33.9.(c), is the same as the completion of $\varepsilon_g B$. We discuss both constructions, to illustrate the relations between them. One should keep in mind that, as a 2-sheaf, the domain of B is $B \cup \{*\}$, where $* \notin B$, with equality given by

$$[\![x \ = \ y]\!] \ = \ \begin{cases} \top & \text{if } x = y \text{ in } B, \\ \bot & \text{otherwise.} \end{cases}$$

In particular, $[\![* \ = \ *]\!] \ = \ [\![x \ = \ *]\!] \ = \ \bot$.

The Ω-set $\varepsilon_g B$ has sections only over $g(\bot) = q_F$ and $g(\top) = \top$, as follows :

$$\varepsilon_G B(\top) = B \quad \text{and} \quad \varepsilon_G B(q_F) = \{*\}.$$

Moreover, since ε_g is injective and B is extensional, we may identify $\varepsilon_G x$ with $x \in |B|$ (30.2). Thus, for $x, y \in |\varepsilon_g B|$,

(I) $$[\![x = y]\!]_{\varepsilon_g B} \;=\; g([\![x \,=\, y]\!]) \;=\; \begin{cases} \top & \text{if } x = y \text{ in } B; \\ q_F & \text{otherwise.} \end{cases}$$

The definition in the first paragraph of this Chapter yields, since $B(\perp) = \{*\}$,

$$\begin{aligned} |\mathfrak{i}_h B| \;&=\; \textstyle\bigcup_{p \in \Omega} B(h(p)) \times \{p\} \\ &=\; \textstyle\bigcup_{p \in F} B \times \{p\} \;\cup\; \bigcup_{p \le q_F} \{*\} \times \{p\}. \end{aligned}$$

$\mathfrak{i}_h B$ is a sheaf over Ω, the **skyscraper sheaf over $\boldsymbol{\Omega}$, centered at $\boldsymbol{F}$ and generated by $\boldsymbol{B}$**.

Note that $\boldsymbol{\varepsilon}_g B$ is indeed restriction dense in $\mathfrak{i}_h B$ and so the latter is the former's completion. In view of 33.9.(a) and (I) above, equality in $\mathfrak{i}_h B$ is given by

$$\begin{aligned} [\![\langle x, p \rangle \,=\, \langle y, q \rangle]\!] \;&=\; p \wedge q \wedge g([\![x \,=\, y]\!]) \\ &=\; \begin{cases} p \wedge q & \text{if } x = y \text{ in } B, \\ p \wedge q \wedge q_F & \text{otherwise.} \end{cases} \end{aligned}$$

It is instructive to go back to the spatial setting and to verify, directly, that $\mathfrak{i}_h B$ is a sheaf. $\square$

Exercises

33.11. In the setting of 32.11, show that if B is a $p^{\leftarrow}$-presheaf, then $\mathfrak{d}_g B$ can be naturally identified with $\mathfrak{i}_h B$ [3]. $\square$

33.12. Let $h : L \longrightarrow R$ be a semilattice morphism and B a flabby R-presheaf (26.28.(e)). Then, $\mathfrak{i}_h B$ is a flabby L-presheaf. $\square$

33.13. If $L \xrightarrow{h} R \xrightarrow{k} T$ are semilattice morphisms, then $\mathfrak{i}_{k \circ h} \;=\; \mathfrak{i}_h \circ \mathfrak{i}_k$. $\square$

[3] So inverse image along h is image along g.

CHAPTER 34

Localization, Fibers and Stalks

As an application of the image functor, we present the notions of *localization*, *fiber* and *stalk*.

DEFINITION 34.1. *Let L be a distributive lattice and F a* proper *filter in L. Let $\pi_F : L \longrightarrow L/F$ be the natural quotient map. If A is a L-set, write*

$$A/F =_{def} \varepsilon_{\pi_F} A$$

for the extensional image of A along π_F (30.4). The L/F-set A/F is the **localization** *of A at F; its set of global sections is the* **fiber** *of A at F, written A_F. The morphism*

$$\langle \varepsilon_F, \pi_F \rangle =_{def} \langle \varepsilon_{\pi_F}, \pi_F \rangle : A \longrightarrow A/F$$

is the **localization morphism** *of A at F. Whenever context allows, write x/F for $\varepsilon_F x$, $x \in |A|$. If $u \in F$ and $s \in A(u)$, then $\varepsilon_F s$ is a global section of A/F* [1], *written s_F and called the* **germ of s at F** [2].

If $f : A \longrightarrow B$ is a morphism of L-sets, write

$$f/F : A/F \longrightarrow B/F,$$

for the morphism $\varepsilon_{\pi_F} f$ (the localization of f at F) and f_F for the map induced by f/F on global sections (the fiber map induced by f at F).

REMARK 34.2. Recall that for $a \in L$, a/F stands for the equivalence class of a by the congruence generated by F. By a standard abuse of notation, we have

$$a/F = \pi_F^{-1}(a/F) \quad \text{and} \quad a/F = \pi_F(a).$$

We collect here some of the formulas that describe explicitly the concepts introduced above. Let $f : A \longrightarrow B$ be a morphism of L-sets and F a (proper) filter in L. For all $x, y \in |A|$:

$$(1) \begin{cases} [\![x/F = y/F]\!] = \pi_F([\![x = y]\!]) = [\![x = y]\!]/F; \\ E(x/F) = \pi_F(Ex) = (Ex)/F; \end{cases}$$

(2) $f/F(x/F) = (fx)/F$;

(3) $[\![f/F(x/F) = f/F(y/F)]\!] = \pi_F([\![fx = fy]\!]) = [\![fx = fy]\!]/F$.

(4) If $Ex \in F$, then $f_F(x_F) = f/F(x/F) = (fx)/F = (fx)_F$, i.e., the germ of a section at F is taken to the germ of fx at F.

---

[1] Recall that $u \in F$ iff $\pi_F(u) = \top$.

[2] By analogy with the classical construction; see 34.10.

(5) If A is a L-presheaf, then A/F is a *presheaf* over L/F (32.7.(c)).
These observations will be used forthwith, without explicit reference. $\square$

PROPOSITION 34.3. *Let L be a distributive lattice. If A is a L-set and F is a proper filter in L, then for all $x, y \in |A|$*

a) If $(Ex \vee Ey) \in F$, then $x/F = y/F$ iff $[\![x = y]\!] \in F$.

b) $x/F \in A_F$ iff $Ex \in F$. If $Ex, Ey \in F$, $x_F = y_F$ iff $[\![x = y]\!] \in F$.

c) If L is a Heyting algebra, $x/F = y/F$ iff $[x \equiv y] \in F$ [3].

PROOF. Recall (4.9) that for $p, q \in L$

$$\pi_F p \;=\; \pi_F q \text{ iff } \exists\, u \in F \text{ such that } u \wedge p \;=\; u \wedge q. \qquad (*)$$

Since the equivalence relation θ_A defining $|A/F|$ is given by

$$\pi_F(Ex) \;=\; \pi_F(Ey) \;=\; \pi_F([\![x = y]\!]),$$

(*) and the fact that a filter is closed under meets yield

$$(**) \qquad x/F = y/F \quad \text{iff} \quad \begin{cases} \exists\, u \in F \text{ such that} \\ u \wedge Ex = u \wedge Ey = u \wedge [\![x = y]\!]. \end{cases}$$

a) If $(Ex \vee Ey) \in F$ and $x/F = y/F$, there is $u \in F$ verifying the conditions in the right-hand side of (**). Hence,

$$\begin{aligned}(Ex \vee Ey) \wedge u &= (Ex \wedge u) \vee (Ey \wedge u) = Ex \vee u = u \wedge [\![x = y]\!] \\ &\leq [\![x = y]\!],\end{aligned}$$

and so $[\![x = y]\!] \in F$. The converse is clear.

b) Since $E(x/F) = \pi_F(Ex)$ and $F = \pi_F^{-1}(\top)$, it follows that x/F is a global section of A/F iff $Ex \in F$. The remainder of (b) is an immediate consequence of (a).

c) If L is a HA and $x/F = y/F$, then (**) entails that for some $u \in F$

$$u \wedge (Ex \vee Ey) = u \wedge [\![x = y]\!] \leq [\![x = y]\!].$$

The adjunction $[\rightarrow]$ in 6.1 implies $u \leq [x \equiv y]$, and $[x \equiv y] \in F$. For the converse, 28.2.(c) entails that, with $u = [x \equiv y]$, the conditions in the right-hand side of (**) are satisfied, whence $x/F = y/F$. $\square$

For Ω-presheaves, the fiber construction is closer to its classical counterpart. From Propositions 34.3, 32.7 and 26.8 we obtain :

COROLLARY 34.4. *If A is an* extensional *Ω-presheaf, then for all $x, y \in |A|$*

a) The following conditions are equivalent :

(1) $x/F = y/F$; (2) $[x \equiv y] \in F$; (3) $\exists\, p \in F$ *with* $x_{|p} = y_{|p}$.

b) $E(x/F) = \top$ iff $Ex \in F$. If $Ex, Ey \in F$, then

$$x_F = y_F \quad \textit{iff} \quad \exists\, p \in F \cap (Ex \wedge Ey)^{\leftarrow} \textit{ such that } x_{|p} = y_{|p}.$$

In particular, if $p, Ex \in F$, then $(x_{|p})_F = x_F$.

[3]For the definition of *strict equality*, $[\cdot \equiv \cdot]$, and its basic properties see 28.1 and 28.2.

c) For $p \leq q$, let $\rho_{qp} : A(q) \longrightarrow A(p)$ be the restriction map $x \mapsto x_{|p}$. For $r \in \Omega$, let $D_r \;=\; \{q \in L : q \leq r$ and $q/F = r/F\}$. The system

$$\langle\, A(q); \{\rho_{qp} \;:\; p \leq q \text{ with } p,\, q \text{ in } D_r\}\,\rangle$$

is an inductive system of sets and maps such that $A/F(r/F) \;=\; \varinjlim_{q \in D_r} A(q)$. In particular, $A_F \;=\; \varinjlim_{p \in F} A(p)$ i.e., the fiber of A at F is the inductive limit of the sections with extent in F, by the restriction maps of A [4].

d) If $B \subseteq |A|$ is restriction dense in A (26.28.(d))*, then*

$$A_F \;=\; \{b_F : b \in B \text{ and } Eb \in F\}.$$

If A is flabby (26.28.(e))*, then $A_F = \{s_F : s \in A(\top)\}$.*

Proof. We comment only on the first assertions in (c) and (d). By 30.3.(d), we have $A/F(r/F) \;=\; \varinjlim_{q \in r/F} A(q)$. Since D_r is cofinal in r/F, the desired conclusion follows from 17.15.(a). For (d), if $p \in F$ and $s \in A(p)$, there is $x \in B$ such that $x_{|Es} = s$. By 34.3.(b), $s_F = x_F$. □

Regarding completeness, we state

Lemma 34.5. *Let F be a proper filter in the distributive lattice L. If A is a finitely complete L-set* (25.34)*, then A/F is finitely complete over L/F.*

Proof. By Exercise 26.29, A has a compatible structure of L-presheaf. Hence, A/F is a L/F-presheaf ((5) in 34.2) and to show that it is fc it is enough to check that a *finite* compatible subset has a gluing.

Let $T = \{s/F : s \in S\}$ be compatible in A/F, with $S \subseteq_f |A|$. This means that for $s,\, s' \in S$,

$$\pi_F(Es \wedge Es') \;=\; \pi_F([\![s \,=\, s']\!]).$$

Relation (**) in the proof of 34.3 implies that for $\langle\, s, s'\,\rangle \in S \times S$, there is $u(s,\, s') \in F$, such that

$$\text{(I)} \qquad Es \wedge Es' \wedge u(s,\, s') \;=\; u(s, s') \wedge [\![s \,=\, s']\!].$$

Since S is finite, $u \;=\; \bigwedge_{s,s' \in S} u(s, s') \in F$. Hence, (I) entails

$$\forall\, s,\, s' \in S, \quad Es \wedge Es' \wedge u \;=\; u \wedge [\![s \,=\, s']\!],$$

i.e., the finite set $\{s_{|u} : s \in S\}$ is compatible in A. Since A is fc, there is $t \in |A|$, such that

$$\begin{cases} Et = \bigvee_{s \in S} Es_{|u} = u \wedge \bigvee_{s \in S} Es; \\ Es_{|u} = [\![t \,=\, s_{|u}]\!] = u \wedge [\![t \,=\, s]\!] = u \wedge Es. \end{cases}$$

Applying π_F to these equations and recalling that $u \in F$, we obtain [5]

$$E(t/F) = \bigvee_{s \in S} E(s/F) \quad \text{and} \quad \forall\, s \in S, \quad E(s/F) = [\![t/F \,=\, s/F]\!],$$

whence t/F is the gluing of T in A/F, as needed. □

[4] Generalizing the classical case of “stalks at a point in a topological space”.

[5] The joins are *finite* and π_F is a lattice morphism.

EXAMPLE 34.6. Let A be a H-set (H a HA) and let D be the filter of dense elements in H (6.19). We know that H/D is a BA, isomorphic to the BA $Reg(H)$ of regular elements in H (6.21). The localization at D furnishes a H/D-set, A/D, whose set of global sections, A_D, is the fiber of A at D. If $x, y \in |A|$ then it follows from 34.3.(c) and 6.8.(j) that

$$x/D = y/D \quad \text{iff} \quad \neg\neg\,[x \equiv y] \in D \quad \text{iff} \quad \neg\neg\,(Ex \vee Ey) = \neg\neg\,[\![x = y]\!].$$

In particular, if Ex, $Ey \in D$, then $x_F = y_F$ iff $\neg\neg\,[\![x = y]\!] = \top$.

If A is a H-presheaf and $p \in H$, recall (6.19.(a)) that

$$D(p) = \{q \in H : p \leq q \leq \neg\neg p\}$$

is the set of elements *dense in* p. Hence, 34.4.(c) entails

$$AD(p/D) \;=\; \lim_{\to q \in D(p)} A(q).$$

In particular, $A_D = \lim_{\to d \in D} A(d)$.

If A is finitely complete over H, 34.5 guarantees that A/D is finitely complete over H/D. We shall prove latter that if H is a frame and A is a sheaf over H, then A/D is a sheaf over H/D (35.9).

Given the well-known connection between intuitionistic logic and Heyting algebras, on one hand, and classical logic and Boolean algebras, on the other, localization at D associates to the intuitionistic structure A, a "classical" structure, A/D. We shall explore this further, when discussing the regularization functor associated to π_D in 35.7. The reader might also consult [**11**] to see how this construction is connected to the notion of *ultrasheaf of first-order structures.* $\square$

We now describe the concept of *stalk*.

DEFINITION 34.7. *Let L be a semilattice with $\top$ and $2 = \{\bot, \top\}$ be the 2-element Boolean algebra. Let $h : L \longrightarrow 2$ is a* **surjective** *semilattice morphism. If A is a L-set, the image of A along h is the* **stalk of $\boldsymbol{A}$ at $\boldsymbol{h}$**, *written A_h. If $f : A \longrightarrow B$ is a morphism of L-set, write f_h for the morphism $\boldsymbol{\varepsilon}_h f : A_h \longrightarrow B_h$, called the* **stalk morphism induced by $\boldsymbol{f}$ at $\boldsymbol{g}$**. *If $s \in |A|$, with $h(Es) = \top$, write $s_h \in A_h$ for the* **germ of $\boldsymbol{s}$ at $\boldsymbol{h}$**.

Since the categories $\boldsymbol{2set}$, $\boldsymbol{Sh(2)}$, $\boldsymbol{pSh(2)}$ and $\boldsymbol{Set}$ are *isomorphic*, the stalk construction is a covariant functor from $\boldsymbol{Lset}$ to $\boldsymbol{Set}$.

The next result shows that when L is a distributive lattice, stalk at h is naturally isomorphic to fiber at the cokernel of h.

LEMMA 34.8. *Let L be a distributive lattice and $h : L \longrightarrow 2$ a surjective semilattice morphism. Let A be a L-set and write F for $coker\,h$* [6].

a) There is a unique *semilattice morphism, $g : L/F \longrightarrow 2$, making the following diagram commutative, where π_F is canonical quotient morphism from L to L/F :*

[6] $coker\,h = \{p \in L : h(p) = \top\}$ is the *cokernel of* h; it is a proper filter in L.

$$\begin{tikzcd} L \arrow[rr, "\pi_F"] \arrow[dr, "h"'] & & L/F \arrow[dl, "g"] \\ & 2 & \end{tikzcd}$$

b) For all $s \in |A|$ such that $Es \in F$, the map α_A given by

$$s_F \in A_F \longmapsto \boldsymbol{\varepsilon}_g s_F \in A_h$$

is an isomorphism, which induces a natural identification between the functors fiber at F and stalk at h.

c) If L is a frame, h is a frame morphism and $D \subseteq |A|$ is dense in A, then

$$A_F = \{s_F : s \in D \text{ and } Es \in F\}.$$

PROOF. a) For $a \in L$, define

$$g(a/F) = h(a).$$

To see that g is well-defined, assume that $a/F = b/F$. Then, there is $v \in F$ such that $a \wedge v = b \wedge v$. Hence, $h(a \wedge v) = h(a) \wedge h(v) = h(a) \wedge \top = h(a)$, and so $h(a) = h(b)$, as needed. We also have

$$\begin{aligned} g(a/F \wedge b/F) &= g((a \wedge b)/F) = h(a \wedge b) = h(a) \wedge h(b) \\ &= g(a/F) \wedge g(b/F), \end{aligned}$$

showing that g is a semilattice morphism; clearly, g is unique and surjective.

b) By 30.12.(a), we have $\boldsymbol{\varepsilon}_h = \boldsymbol{\varepsilon}_g \circ \boldsymbol{\varepsilon}_\pi$, and so α_A is indeed well-defined. Moreover, it is surjective, because $\boldsymbol{\varepsilon}_h$ is onto A_h. To check that it is injective, assume that $\boldsymbol{\varepsilon}_g s_F = \boldsymbol{\varepsilon}_g t_F$; then,

$$\top = E\boldsymbol{\varepsilon}_g s_F = g(Es_F) = g(\pi_F(Es)) = h(Es),$$

verifying that $Es \in F$. Similarly, we get $Et \in F$. Next, equality in A_h yields

$$\top = [\![\boldsymbol{\varepsilon}_g s_F = \boldsymbol{\varepsilon}_g t_F]\!] = g([\![s_F = t_F]\!]) = g(\pi_F([\![s = t]\!])) = h([\![s = t]\!]),$$

and $[\![s = t]\!] \in F$. By 34.3.(b), $s_F = t_F$, as desired. It is straightforward that if $f : A \longrightarrow B$ is a morphism of L-sets, the following square is commutative :

$$\begin{tikzcd} A_F \arrow[r, "f_F"] \arrow[d, "\alpha_A"'] & B_F \arrow[d, "\alpha_B"] \\ A_h \arrow[r, "f_h"'] & B_h \end{tikzcd}$$

Hence, $\alpha = \{\alpha_A : A \text{ is a } L\text{-set}\}$ is a natural isomorphism between the functors fiber at F and stalk at h.

c) If h preserves joins, then F is completely prime, that is,

$$\bigvee_{i \in I} p_i \in F \;\;\Rightarrow\;\; \exists\, i \in I \text{ with } p_i \in F.$$

To see this, suppose $p = \bigvee_{i \in I} p_i$ exists in L. Then,

$$\top \;=\; h(p) \;=\; h(\bigvee_{i \in I} p_i) \;=\; \bigvee_{i \in i} h(p_i).$$

It is immediate that the $h(p_i)$ cannot all be equal to $\bot$. For $x \in |A|$, with $Ex \in F$, we may write

$$Ex \;=\; \bigvee_{d \in D} [\![x = d]\!],$$

and so there is $d \in D$ such that $[\![x = d]\!] \in F$. But then, $Ed \in F$ and 34.3.(b) entails $s_F = d_F$, ending the proof. $\square$

Example 34.9. Let Ω be a frame and F a point (or completely prime filter) in Ω (12.1). The point F originates a *pure state* (12.14)

$$p_F : \Omega \longrightarrow 2, \text{ given by } p_F(p) = \top \;\text{ iff }\; a \in F.$$

However, 2 may be very far from the quotient Ω/F.

If A is a Ω-set, the stalk of A at p_F is the image of A by p_F. By Lemma 34.8, it is isomorphic to the fiber of A at F, but it will be important in applications to Logic to know that there is a frame morphism involved in the construction of A_F. The isomorphism in 34.8 justifies the usage of "stalk" both for the fiber of A at F and its image under p_F. In Example 33.10 it was shown that *inverse image* along p_F corresponds to the construction known as the "skyscraper" sheaf.

To exemplify the concept in the spatial setting, let X be a topological space and x a point in X. Let f be the *unique* continuous map from $\{\emptyset\}$ into X, that takes $\emptyset$ to x. Then, $f^* : \Omega(X) \longrightarrow 2$ is precisely the pure state associated to the completely prime filter ν_x, of open neighborhoods of x in X. Hence, in classical notation (30.13, 33.2)

$*$ If A is a sheaf over X, f^*A is the stalk of A at the point x;

$*$ If B is a set (i.e., a 2-sheaf), f_*B is the skyscraper sheaf on X, centered at x and generated by B. $\square$

Example 34.10. Let X be a topological space, A be $\Omega(X)$-set and $x \in X$. Localizing A at the completely prime filter ν_x, yields an extensional $(\Omega(X)/\nu_x)$-set, A/ν_x, whose set of global sections is written A_x (the stalk of A at x); the elements of A_x are written s_x. If s, t are sections in A such that Es, Et are open neighborhoods of x, then

$$s_x \;=\; t_x \quad \text{iff} \quad [\![s = t]\!] \in \nu_x.$$

If f, g are sections of the sheaf $\mathbb{C}(X, Y)$ of 23.5, with Ef, $Eg \in \nu_x$, then

$$f_x \;=\; g_x \quad \text{iff} \quad \exists\, u \leq Ef \cap Eg \text{ such that } f_{|u} \;=\; g_{|u},$$

corresponding to the classical notion of stalk at x : f and g have the same germ at x iff they coincide in an open neighborhood of x. Note that in the context of $\Omega(X)$-sets, there is more to this construction than just the stalk, even in the classical setting, for we can consider the whole localization at x as a $(\Omega(X)/\nu_x)$-set in its own right. $\square$

Since a L-set is dense in its completion, 34.8.(c), 34.9 and 34.10 yield

COROLLARY 34.11. *If F is a point in the frame Ω, then for all Ω-sets A*

$$A_F = (cA)_F,$$

where cA is the completion of A. In particular, if A is a presheaf over a topological space X, the stalk of A and that of its completion are the same at all $x \in X$. $\square$

EXAMPLE 34.12. Let L be a distributive lattice with $\perp$ and $\top$. Let $\mathcal{U}$ be a maximal filter in L. If $a \in L$, then

$$a/\mathcal{U} = \top/\mathcal{U} \quad \text{or} \quad a/\mathcal{U} = \perp/\mathcal{U}.$$

To see this, note that if $\mathcal{U} \cup \{a\}$ has the fip [7], then $a \in \mathcal{U}$, by maximality; otherwise, a is equivalent to $\perp$ modulo $\mathcal{U}$. Hence, the quotient $L/\mathcal{U}$ may be identified with $2 = \{\perp, \top\}$, and $\mathcal{U}$ originates a surjective lattice morphism

$$\pi_{\mathcal{U}} : L \longrightarrow 2, \text{ given by } \quad \pi_{\mathcal{U}}(a) = \top \quad \text{iff} \quad a \in \mathcal{U}.$$

There are many familiar constructions that are instances of "stalk at $\mathcal{U}$". As an example, consider the sheaf $\mathcal{M}$ of 23.11 and 31.8; $\mathcal{M}$ is a *flabby* sheaf over L, that is, every section is the restriction of a global section in $\mathcal{M}$ (26.28.(e)).

Let $\mathcal{U}$ be an ultrafilter in L (i.e., an ultrafilter on I). By 34.4.(d), to compute the stalk of $\mathcal{M}$ at $\mathcal{U}$ it is enough to determine the germs of global section at $\mathcal{U}$. For $s,\, t \in \mathcal{M}(I) = \prod_{i \in I} M_i$,

$$s_{\mathcal{U}} = t_{\mathcal{U}} \quad \text{iff} \quad [\![s = t]\!] \in \mathcal{U} \quad \text{iff} \quad \{i \in I : s(i) = t(i)\} \in \mathcal{U}.$$

Thus, $\mathcal{M}_{\mathcal{U}}$ is the **ultraproduct** of the family M_i by the ultrafilter $\mathcal{U}$. When $\mathcal{U}$ is a principal ultrafilter generated by $i \in I$,

$$\mathcal{U} = \{u \subseteq I : i \in u\},$$

then $\mathcal{M}_{\mathcal{U}}$ can be canonically identified with M_i.

In 31.8 we described the extension of $\mathcal{M}$ to a β I-sheaf, $\mathfrak{e}\mathcal{M}$. Now, an ultrafilter on I is a *point* in the topological space βI. Furthermore, for every $U \in \Omega(\beta I)$, $\mathfrak{e}\mathcal{M}(U) = \mathcal{M}(u)$, where u is the unique element of 2^I such that $\overline{U} = S_u$. It follows that the stalk of $\mathfrak{e}\mathcal{M}$ at a point $\mathcal{U}$ of βI is still the ultraproduct of the M_i at $\mathcal{U}$.

If instead of an ultrafilter $\mathcal{U}$ on I, we had considered a *filter* F on I, then the *fiber* of $\mathcal{M}$ at F is the **reduced product** of the M_i by F, another important construct in Model Theory. $\square$

Certain properties between Ω-presheaves are faithfully reflected by fiber and localization, as is the case for commutative rings [8]. An example is Proposition 34.13, proven below. Recall that $S(L)$ is the Stone space of the distributive lattice L (19.2); the definition of frame *with enough points* appears in 12.6.

PROPOSITION 34.13. *Let $f,\, g : A \longrightarrow B$ be morphisms of extensional Ω-presheaves.*

a) The following are equivalent :

(1) $f = g$; (2) $\forall\, F \in S(\Omega),\ f/F = g/F$; (3) $\forall\, F \in S(\Omega),\ f_F = g_F$.

[7] The finite intersection property; see 3.13.(c).

[8] That is the reason for adopting the term here.

If Ω is a frame with enough points, these conditions are equivalent to

(4) *For all points P in Ω, $f_P = g_P$.*

b) The following are equivalent :

(1) *f is monic in $\boldsymbol{pSh(\Omega)}$;*

(2) *For all $F \in S(L)$, f/F is monic in $\boldsymbol{pSh(\Omega/F)}$;*

(3) *For all $F \in S(L)$, f_F is injective.*

If Ω is a frame with enough points, these conditions are equivalent to

(4) *For all points P in Ω, f_P is injective*

c) If Ω is a frame with enough points, the following are equivalent :

(1) *f is epic in $\boldsymbol{pSh(\Omega)}$;* (2) *For all points P in Ω, f_P is surjective.*

PROOF. a) It is enough to show that (3) $\Rightarrow$ (1) and, in case Ω has enough points, that (4) $\Rightarrow$ (1).

(3) $\Rightarrow$ (1) : For $x \in |A|$, $fx \neq gx$ implies $[\![fx = gx]\!] < Ex$; by Corollary 4.26.(a), there is a prime filter F in L such that $Ex \in F$, but $[\![fx = gx]\!] \notin F$. Consequently, $f_F(x_F) \neq g_F(x_F)$, contradicting (3).

(4) $\Rightarrow$ (1) : The argument is the same as above, except that since Ω has enough points, there is a point P in Ω such that $Ex \in P$, with $[\![fx = gx]\!] \notin P$.

b) (1) $\Rightarrow$ (2) : Let F be a prime filter in Ω and $x, y \in |A|$. By 26.17.(b), (1) is equivalent to

$$[\![x = y]\!] = [\![fx = fy]\!].$$

Consequently, recalling the pertinent definitions in 30.5, we have

$$\begin{aligned}[\![x/F = y/F]\!] &= \pi_F([\![x = y]\!]) = \pi_F([\![fx = fy]\!]) = [\![(fx)/F = (fy)/F]\!] \\ &= [\![f/F(x/F) = f/F(y/F)]\!],\end{aligned}$$

and the desired conclusion follows from the extensionality of A/F and 25.21. The entailment (2) $\Rightarrow$ (3) is an immediate consequence of 26.17.(a).

(3) $\Rightarrow$ (1) : Assume, to get a contradiction, that $[\![x = y]\!] < [\![fx = fy]\!]$, where $x, y \in |A|$. By 4.26.(a), there is a prime filter F in L such that

$$[\![fx = fy]\!] \in F \quad \text{and} \quad [\![x = y]\!] \notin F.$$

Then, $Ex \wedge Ey = Efx \wedge Efy \in F$ and Proposition 34.3.(b) yields $x_F \neq y_F$ and $(fx)_F = (fy)_F$, proving that f_F is not injective, the desired contradiction. If Ω has enough points, the reasoning is analogous, replacing prime filters by points.

c) (1) $\Rightarrow$ (2) : If $b_F \in A_F$, then $Eb \in F$ (34.3.(b)). By 25.24, we get

$$Eb = \bigvee_{a \in |A|} [\![b = fa]\!],$$

and so, F being a point in Ω, we conclude that for some $a \in |A|$, $[\![b = fa]\!] \in F$. An application of 34.3.(b) yields

$$f_F(a/F) = (fa)_F = b_F,$$

establishing that f_F is surjective [9].

[9]Note that (1) $\Rightarrow$ (2) is true for *all* Ω.

<u>(2) ⇒ (1) :</u> Suppose, to get a contradiction, that

$$\bigvee_{a \in |A|} [\![b = fa]\!] \; < \; Eb.$$

Because Ω has enough points, Lemma 12.3 implies that there is a point F in Ω such that $Eb \in F$ and $[\![b = fa]\!] \notin F$, for all $a \in |A|$. It follows from 34.3.(a) that b_F is distinct from $(fa)_F$, for all $a \in |A|$, contradicting (2). □

In the remainder of this Chapter we describe how stalk, fiber and localization behave with respect to image and inverse image along a semilattice morphism.

Our first observation is a version, for semilattice morphisms, of the fundamental theorem for lattice morphisms (3.8, 4.13), analogous to 34.8.

Lemma 34.14. *Let $h : L \longrightarrow R$ be a semilattice morphism, with L, R distributive lattices with $\bot$, $\top$. Let F be a proper filter in R and assume that $G = h^{-1}(F)$ is not empty in L.*

a) G is a (not necessarily proper) filter in L.

b) There is a unique *semilattice morphism, $j_F : L/G \longrightarrow R/F$, such that :*

(1) *For all $p \in L$, $\;j_F(p/G) = \top \;$ iff $\; h(p) \in F$;*

(2) $\pi_F \circ h \; = \; j_F \circ \pi_G$.

$$\begin{tikzcd} L \arrow[r, "h"] \arrow[d, "\pi_G"'] & R \arrow[d, "\pi_F"] \\ L/G \arrow[r, "j_F"'] & R/F \end{tikzcd}$$

Proof. Item (a) is straightforward because h is increasing and preserves finite meets. For (b), define, for $p \in L$,

$$j_F(p/G) = h(p)/F.$$

If $p/G = q/G$, then there is $u \in G$ such that $p \wedge u = q \wedge u$; thus,

$$h(p \wedge u) = h(p) \wedge h(u) = h(q) \wedge h(u),$$

and $h(p)/F = h(q)/F$, showing that j_F is well-defined. It is clear that j_F is a semilattice morphism, satisfying the stated conditions. □

Theorem 34.15. *Let $h : L \longrightarrow R$ a semilattice morphism where L, R are distributive lattices with $\bot$, $\top$. Let F be a filter in R, such that $G = h^{-1}(F) \neq \emptyset$. If A is a L-set, then with notation as in* 34.14,

a) $(\boldsymbol{\varepsilon}_h A)/F \; = \; \boldsymbol{\varepsilon}_{j_F}(A/G)$. *b) $(\boldsymbol{\varepsilon}_h A)_F \; \approx \; A_G$.*

Proof. Item (a) is an immediate consequence of 30.12.(a) and item (2) in 34.14.(b). For (b), we verify that $\boldsymbol{\varepsilon}_{j_F}$ induces an isomorphism *on global sections.* To start off, $\boldsymbol{\varepsilon}_{j_F}$ is the following map on global sections, where $x \in |A|$:

$$\boldsymbol{\varepsilon}_{j_F} : A_G \longrightarrow (\boldsymbol{\varepsilon}_h A)_F, \text{ given by } x_G \longmapsto (\boldsymbol{\varepsilon}_h x)_F,$$

that is, it sends the germ of x at G, to the germ of $\boldsymbol{\varepsilon}_h x$ at F. Clearly, $\boldsymbol{\varepsilon}_{j_F}$ is surjective. If $x,\, y \in |A|$ are such that $(\boldsymbol{\varepsilon}_h x)_F = (\boldsymbol{\varepsilon}_h y)_F$ in the fiber of $\boldsymbol{\varepsilon}_h A$ at F, 34.3.(b) entails

$$\begin{cases} h(Ex) \;=\; E\boldsymbol{\varepsilon}_h x \in F; & h(Ey) \;=\; E\boldsymbol{\varepsilon}_h y \in F; \\ h([\![x = y]\!]) \;=\; [\![\boldsymbol{\varepsilon}_h x = \boldsymbol{\varepsilon}_h y]\!] \in F. & \end{cases}$$

These relations imply Ex, Ey, $[\![x = y]\!] \in G$, i.e., $x_G = y_G$, showing that $\boldsymbol{\varepsilon}_{j_F}$ is also injective and completing the proof. □

Remark 34.16. If G in 34.15 is empty, then the fiber of $\boldsymbol{\varepsilon}_h A$ at F is also empty, for there is no $x \in |A|$ for which $E\boldsymbol{\varepsilon}_h x \in F$. If the fiber at the empty set is defined to be empty, then item (b) will hold in a trivial way. □

All the classical relations between fibers and stalks along change of base are consequences of 34.15. We register a sample of these as illustration.

Corollary 34.17. *a) Let L, R be frames and $h : L \longrightarrow R$ be a frame-morphism. If F is a point in R, then*

(1) *$G = h^{-1}(F)$ is a point in L.*

(2) *For all L-sets A, $\;(cA)_G \;\approx\; A_G \;\approx\; (\boldsymbol{\varepsilon}_h A)_F \;\approx\; (c_h A)_F$.*

b) If $f : X \longrightarrow Y$ is a continuous map of topological spaces and B is a presheaf over Y, then, for all $x \in X$, $\;(f^*B)_x \;\approx\; B_{fx}$. □

Corollary 34.18. *Let $h : L \longrightarrow R$ be a semilattice morphism of distributive lattices with $\perp$, $\top$. Assume that h has a right adjoint g and let K be a proper filter in L.*

a) If B is a R-presheaf and K is a proper filter in L, then

$$(\mathfrak{i}_h B)_K \;\approx\; (\boldsymbol{\varepsilon}_g B)_K \;\approx\; B_{(g^{-1}K)}.$$

b) If h is surjective, F is a proper filter in R and A is a L-presheaf, then

$$(\mathfrak{i}_h \boldsymbol{\varepsilon}_h A)_{(h^{-1}F)} \;\approx\; (\boldsymbol{\varepsilon}_h A)_F.$$

c) If h is injective, F is a proper filter in L and B is a R-presheaf, then

$$(c_h \mathfrak{i}_h B)_{(g^{-1}F)} \;\approx\; (c_g B)_F \;\approx\; (\boldsymbol{\varepsilon}_g B)_F.$$

Proof. a) Since $gh \geq Id_L$, it follows that

$$u \in K \;\Rightarrow\; ghu \in K.$$

Hence, $g^{-1}(K) \neq \emptyset$ and 34.15 entails $(\boldsymbol{\varepsilon}_g B)_K \approx B_{g^{-1}K}$. The other isomorphism is a consequence of the fact that $\boldsymbol{\varepsilon}_g B$ is restriction dense in $\mathfrak{i}_h B$ (33.9.(b)) and 34.4.(d).

b) Since $h \circ g = Id_R$, we have $g^{-1}(h^{-1}(F)) = F$. Hence, item (a) applies to yield

$$(\mathfrak{i}_h \boldsymbol{\varepsilon}_h A)_{h^{-1}F} \;\approx\; (\boldsymbol{\varepsilon}_h A)_{(g^{-1}h^{-1}F)} \;=\; (\boldsymbol{\varepsilon}_h A)_F,$$

as needed. Item (c) is analogous and left to the reader. □

Exercises

34.19. Let A be a commutative ring with identity and $c\mathcal{A}$ be the structure sheaf of A over $Spec(A)$, as in 27.21.

a) For $P \in Spec(A)$, let $N_P = \{Z_a : a \notin P\}$ be the set of basic open neighborhoods of P in $Spec(A)$. Then, with $S_a = \{a^n : n \geq 0\}$,

$$\mathcal{P} = \langle\, AS_a^{-1};\ \{\rho_{ba} : Z_a \subseteq Z_b \text{ in } N_P\}\,\rangle$$

is an inductive system of rings over the up-directed poset N_P^{op}, with $\varinjlim \mathcal{P} = A_P$, the localization of A at P (9.41).

b) Determine the stalk of $c\mathcal{A}$ at the point $P \in Spec(A)$.

c) If A is a regular ring (19.19), then for all $P \in Spec(A)$, A_P is (isomorphic to) the residue field A/P. □

34.20. Let A be an extensional presheaf over the topological space X. Show that there is a *geometrical* sheaf over X, $\mathfrak{c}A$, together with a presheaf morphism, $\mathfrak{c} : A \longrightarrow \mathfrak{c}A$, which satisfies the conditions in Theorem 27.9 [10]. □

34.21. Let Ω be a frame with enough points (12.6).

a) If $A \subseteq B$ are *sheaves* over Ω, the following are equivalent :

(1) $A = B$; (2) For all points F in Ω, $A_F = B_F$.

b) Let $f : A \longrightarrow B$ be a morphism of sheaves over Ω. The following are equivalent :

(1) f is an isomorphism; (2) For all points F in Ω, f_F is a bijection. □

34.22. State and prove a result similar to 34.18, with semilattice morphism onto 2 (34.7) in place of filters. □

34.23. Let R be a commutative ring with identity and let $\mathcal{R}$ be the structure sheaf of R over $X = Spec(R)$ as in 27.21 and 27.22. Let $\iota : X_c \longrightarrow X$ be the identity map, where X_c is $Spec(R)$ endowed with the constructible topology (as in Theorem 20.10). Describe the inverse image of $\mathcal{R}$ along ι, establishing, in particular that for all $P \in X$, the stalk of $\iota^*\mathcal{R}$ at P is the localization R_P. □

[10] This is the classical version of completion.

CHAPTER 35

Regularization Functors

In this Chapter it will be shown that if $k : L \longrightarrow R$ is a semilattice morphism with a right adjoint, there is a functor

$$k^r : \boldsymbol{Lset} \longrightarrow \boldsymbol{Lset},$$

such that for all L-sets A, with a compatible structure of presheaf (26.6), $k^r A$ is naturally isomorphic to $\mathfrak{i}_k \boldsymbol{\varepsilon}_k A$. We shall assume the reader familiar with the image and inverse image constructions, discussed in the preceding Chapters.

35.1. **The $\boldsymbol{L}$-set $\boldsymbol{k^r A}$.** Assume that $k : L \longrightarrow R$ has a right adjoint, that is, there is a semilattice morphism, $g : R \longrightarrow L$, such that

[adj] $\qquad$ For all $\langle p, r \rangle \in L \times R$, $\;kp \le r\;$ iff $\;p \le gr$.

As usual in this situation (7.8), we have

$$(*) \qquad \begin{cases} (i) \;\; \forall \, \langle p, r \rangle \in L \times R, \;\; p \le gk(p) \;\text{ and }\; kg(r) \le r; \\ (ii) \; kgk = k \;\text{ and }\; gkg = g. \end{cases}$$

Let

$$|\mathcal{A}| = \{\langle x, p \rangle \in |A| \times H : Ex \le p \text{ and } k(Ex) = k(p)\}.$$

For $\langle x, p \rangle$, $\langle y, q \rangle \in |\mathcal{A}|$ set [1]

$$(**) \qquad [\![\langle x, p \rangle = \langle y, q \rangle]\!]_r \;=_{def}\; p \wedge q \wedge gk([\![x = y]\!])).$$

It is straightforward that (**) defines an equality in $|\mathcal{A}|$, with which it becomes a L-set $\mathcal{A}$ [2]. Furthermore, the first relation in $(*).(i)$ above entails

$$\begin{aligned} E_r\langle x, p \rangle \;&=_{def}\; [\![\langle x, p \rangle = \langle x, p \rangle]\!]_r \;=\; p \wedge gk(Ex) \;=\; p \wedge gk(p) \\ &=\; p. \end{aligned}$$

In general, $\mathcal{A}$ is not extensional. In fact, we have

Fact 35.2. *For $\langle x, p \rangle$, $\langle y, q \rangle \in |\mathcal{A}|$, the following are equivalent*

(1) $\;E_r\langle x, p \rangle \;=\; E_r\langle y, q \rangle \;=\; [\![\langle x, p \rangle = \langle y, q \rangle]\!]_r$;

(2) $\;p = q\;$ *and* $\;k(Ex) \;=\; k(Ey) \;=\; k([\![x = y]\!])$.

Proof. If (1) holds, then, $p = q$ and $p \le gk([\![x = y]\!])$. Since $k(Ex) = p = k(Ey)$, the equations in $(*).(ii)$ imply $k(Ex) \;=\; k(Ey) \;\le\; k([\![x = y]\!])$, and (2) follows immediately. Conversely, if (2) holds, then applying g to all terms yields

[1] Of course, $[\![x = y]\!]$ is the equality in A.

[2] Clearly, $|\mathcal{A}| = |\mathfrak{s}_k A|$ (28.4), and so (**) defines another equality in the same domain. However, we have preferred to maintain notations distinct.

$$gk(Ex) \;=\; gk(Ey) \;=\; gk([\![x \,=\, y]\!]),$$

and the first inequality in $(*).(i)$ entails $p \;=\; q \;=\; p \wedge q \wedge gk([\![x \,=\, y]\!])$, which is equivalent to (1). $\square$

It is clear that (2) defines an equivalence relation on $|\mathcal{A}|$, and that the domain of the extensionalization (30.8) of $\mathcal{A}$ is the set of these equivalence classes. Let

$$k^r A = \boldsymbol{\varepsilon}\mathcal{A}$$

be the extensionalization of $\mathcal{A}$. **We shall use the same name for an element of $|\mathcal{A}|$ and for its equivalence class in $|k^r A|$.** Hence, for $\langle\, x, p\,\rangle$, $\langle\, y, q\,\rangle \in |k^r A|$

$$\begin{cases} \langle\, x, p\,\rangle \;=\langle\, y, q\,\rangle & \text{iff} \quad p = q \text{ and } k(Ex) = k(Ey) = k([\![x \,=\, y]\!]); \\ [\![\langle\, x, p\,\rangle = \langle\, y, q\,\rangle]\!]_r & = \quad p \wedge q \wedge gk([\![x \,=\, y]\!]); \\ E_r\langle\, x, p\,\rangle & = \quad p. \end{cases}$$

Since $E_r\langle\, a, Ea\,\rangle = Ea$ and

$$[\![\langle\, a, Ea\,\rangle = \langle\, b, Eb\,\rangle]\!]_r \;=\; Ea \wedge Eb \wedge gk([\![a \,=\, b]\!]) \;\geq\; [\![a \,=\, b]\!],$$

$k^r A$ comes equipped with a L-set morphism, $k^r : A \longrightarrow k^r A$, $\;a \longmapsto \langle\, a, Ea\,\rangle$.

A morphism $f : A \longrightarrow B$ of L-sets, induces a morphism

$$k^r f : k^r A \longrightarrow k^r B, \;\; \langle\, x, p\,\rangle \longmapsto \langle\, fx, p\,\rangle.$$

Indeed, clearly $k^r f$ preserves extent. For $\langle\, x, p\,\rangle$, $\langle\, y, q\,\rangle \in |k^r A|$, since $g \circ k$ is increasing, it follows that

$$\begin{aligned} [\![\langle\, fx, p\,\rangle = \langle\, fy, q\,\rangle]\!]_r \;&=\; p \wedge q \wedge gk([\![fx \,=\, fy]\!]) \\ &\geq\; p \wedge q \wedge gk([\![x \,=\, y]\!]) \;=\; [\![\langle\, x, p\,\rangle = \langle\, y, q\,\rangle]\!]_r, \end{aligned}$$

as needed. It is straightforward that

$$k^r(Id_A) = Id_{k^r A} \quad \text{and} \quad k^r(f \circ g) = k^r f \circ k^r g. \qquad \square$$

Definition 35.3. *Let $k : L \longrightarrow R$ be a semilattice morphism, with a right adjoint g. The construction in* 35.1 *is the* **regularization functor** *associated to k, $\;k^r$: $\boldsymbol{Lset} \longrightarrow \boldsymbol{Lset}$.*

We now prove that

$*$ The image of $k^r A$ along k is isomorphic to $\boldsymbol{\varepsilon}_k A$ (35.4.(a));

$*$ In case A has a presheaf structure compatible with its equality, $k^r A$ is naturally isomorphic to $\mathfrak{i}_k \boldsymbol{\varepsilon}_k A$ (35.4.(c)). For this, it is enough to show that $k^r A$ has the corresponding universal property (33.3.(b)).

Theorem 35.4. *Let $k : L \longrightarrow R$ be a semilattice morphism with a right adjoint g. Let A be a L-set.*
a) The map $j_A : |k^r A| \longrightarrow |\boldsymbol{\varepsilon}_k A|$, given by $\langle\, x, p\,\rangle \mapsto \boldsymbol{\varepsilon}_k x$ is surjective and determines a morphism in $\boldsymbol{SLset}$, $\langle\, j_A, k\,\rangle : k^r A \longrightarrow \boldsymbol{\varepsilon}_k A$, making the triangle below-left commutative :

$$\begin{tikzcd} A \arrow[rr, "k^r"] \arrow[dr, "\boldsymbol{\varepsilon}_k"'] & & k^r A \arrow[dl, "\langle j_A, k \rangle"] \\ & \boldsymbol{\varepsilon}_k A & \end{tikzcd} \qquad \begin{tikzcd} k^r A \arrow[rr, "\langle \boldsymbol{\varepsilon}_k, k \rangle"] \arrow[dr, "\langle j_A, k \rangle"'] & & \boldsymbol{\varepsilon}_k(k^r A) \arrow[dl, "\langle \phi_A, Id_L \rangle"] \\ & \boldsymbol{\varepsilon}_k A & \end{tikzcd}$$

b) There is a unique *isomorphism of R-sets,* $\phi_A : \boldsymbol{\varepsilon}_k(k^r A) \longrightarrow \boldsymbol{\varepsilon}_k A$*, making the triangle above-right commutative.*

c) If A has a compatible structure of L-presheaf (26.6)*, then the rule*

$$\langle x, p \rangle_{|r} = \langle x_{|r}, p \wedge r \rangle$$

defines a restriction in $k^r A$*, compatible with its equality, with which it is a L-presheaf. Moreover, the maps* k^r *and* $\langle j_A, k \rangle$ *are presheaf morphisms.*

d) Assume that A has a compatible L-presheaf structure. If B is a L-set and $\langle f, k \rangle : B \longrightarrow \boldsymbol{\varepsilon}_k A$ *is a morphism in* ***SLset****, there is a* unique *morphism of H-sets,* $\widehat{f} : B \longrightarrow k^r A$*, making the following diagram commutative :*

$$\begin{tikzcd} B \arrow[rr, "\langle \widehat{f}, Id_H \rangle"] \arrow[dr, "\langle f, k \rangle"'] & & k^r A \arrow[dl, "\langle j_A, k \rangle"] \\ & \boldsymbol{\varepsilon}_k A & \end{tikzcd}$$

PROOF. Recall our convention (35.1, page 386) that $\langle x, p \rangle \in |k^r A|$ stands for the equivalence class of $\langle x, p \rangle$ under the equivalence relation in 35.2.(2).

a) If $\langle x, p \rangle = \langle y, q \rangle$ in $|k^r A|$, then 35.2.(2) implies that $\boldsymbol{\varepsilon}_k x = \boldsymbol{\varepsilon}_k y$, and so j_A is well-defined; $\langle j_A, k \rangle$ is a morphism because

$$E\boldsymbol{\varepsilon}_k x \;=\; k(Ex) \;=\; k(p) \;=\; k(E_r\langle x, p \rangle),$$

and the first equation in (ii) of (*) in 35.1 yields

$$\begin{aligned} [\![\boldsymbol{\varepsilon}_k x = \boldsymbol{\varepsilon}_k y]\!] &= k([\![x = y]\!]) \;=\; k(Ex \wedge Ey \wedge [\![x = y]\!]) \\ &= k(Ex) \wedge k(Ey) \wedge k([\![x = y]\!]) \\ &= k(p) \wedge k(q) \wedge k([\![x = y]\!]) \;=\; k(p \wedge q \wedge gk([\![a = b]\!])) \\ &= k([\![\langle x, p \rangle = \langle y, q \rangle]\!]_r), \end{aligned}$$

as needed (29.1.(a)). It is clear from the definitions that j_A is surjective and the displayed diagram is commutative.

b) By the universal property of image along k (30.3.(b)), there is a *unique* morphism of R-sets, $\phi_A : \boldsymbol{\varepsilon}_k(k^r A) \longrightarrow \boldsymbol{\varepsilon}_k A$, making the displayed triangle commutative. Hence, for all $\langle x, p \rangle \in k^r A$, ϕ_A is given by

$$\phi_A(\boldsymbol{\varepsilon}_k\langle x, p \rangle) = \boldsymbol{\varepsilon}_k x.$$

Since j_A is surjective, ϕ_A is also surjective. To show that it is injective, we use 25.21, recalling the equations in $(*).(ii)$ in 35.1, to obtain

$$\begin{aligned}[\![\phi_A(\varepsilon_k\langle x,p\rangle) = \phi_A(\varepsilon_k\langle y,q\rangle)]\!] \;&=\; [\![\varepsilon_k x = \varepsilon_k y]\!] \;=\; k([\![x = y]\!]) \\ &=\; k(Ex \wedge Ey \wedge [\![x = y]\!]) \;=\; k(p \wedge q \wedge [\![x = y]\!]) \\ &=\; k(p \wedge q \wedge gk([\![x = y]\!])) \;=\; k([\![\langle x,p\rangle = \langle y,q\rangle]\!]_r) \\ &=\; [\![\varepsilon_k\langle x,p\rangle = \varepsilon_k\langle y,q\rangle]\!],\end{aligned}$$

completing the proof of (b). Item (c) is straightforward.

c) Fix a morphism $\langle f,k\rangle : B \longrightarrow \varepsilon_k A$; for $b \in |B|$, let x^b in $|A|$ be such that $fb = \varepsilon_k x^b$. Since $\langle f,k\rangle$ is a morphism, for all c, $b \in |B|$, we have

$$\text{(F)}\qquad \begin{cases} (i)\ Efb = k(Eb) = k(Ex^b); \\ (ii)\ k([\![b = c]\!]) \;\le\; [\![fb = fc]\!] \;=\; [\![\varepsilon_k x^b = \varepsilon_k x^c]\!] \\ \qquad\qquad\qquad\;\;=\; k([\![x^b = x^c]\!]). \end{cases}$$

Applying g to inequality (ii) in (F) yields, by $(*).(i)$ in 35.1,

$$\text{(G)}\qquad [\![b = c]\!] \;\le\; gk([\![b = c]\!]) \;\le\; gk([\![x^b = x^c]\!]).$$

With these preliminaries, note that (i) in (F) entails

$$Ex^b \wedge Eb \;=\; E(x^b_{|Eb}) \;\le\; Eb \quad\text{and}\quad k(Ex^b \wedge Eb) \;=\; k(Eb).$$

Hence, $\langle x^b_{|Eb}, Eb\rangle \in |k^rA|$. Define $\widehat{f}b \;=\; \langle x^b_{|Eb}, Eb\rangle$.

We have, $E\widehat{f}b = E_r\langle x^b_{|Eb}, Eb\rangle = Eb$, showing that $\widehat{f}$ verifies [mor 1] in 25.10.

If b, $c \in |B|$, (G) yields

$$\begin{aligned}[\![\widehat{f}b = \widehat{f}c]\!] \;&=\; [\![\langle x^b_{|Eb}, Eb\rangle = \langle x^c_{|Ec}, Ec\rangle]\!] \\ &=\; Eb \wedge Ec \wedge gk([\![x^b_{|Eb} = x^c_{|Ec}]\!]) \\ &=\; Eb \wedge Ec \wedge gk(Eb \wedge Ec \wedge [\![x^b = x^c]\!]) \\ &=\; Eb \wedge Ec \wedge gk([\![x^b = x^c]\!]) \\ &\ge\; Eb \wedge Ec \wedge [\![b = c]\!] \;=\; [\![b = c]\!],\end{aligned}$$

verifying [mor 2] in 25.10. For $b \in |B|$, (i) in (F) above yields

$$j_A(\langle x^b_{|Eb}, Eb\rangle) \;=\; \varepsilon_k(x^b_{|Eb}) \;=\; (\varepsilon_k x^b)_{|k(Ep)} \;=\; fb_{|h(Ep)} \;=\; fb,$$

verifying the commutativity of the displayed triangle. It remains to check uniqueness. Let $\beta : B \longrightarrow k^rA$ be a morphism of L-sets, for which the triangle in the statement is commutative. For $b \in |B|$, write $\beta b = \langle y,q\rangle$. Clearly, $q = Eb \ge Ey$. Since, $j_A(\langle y,q\rangle) \;=\; \varepsilon_k y$, we conclude that $\varepsilon_k y \;=\; \varepsilon_k x^b$. Consequently,

$$k(Ey) = k(Ex^b) = k(Eb) = k([\![x^b = y]\!]).$$

These equations and $(*).(i)$ in 35.1 furnish

$$\begin{aligned}[\![\langle x^b_{|Eb}, Eb\rangle = \langle y, Eb\rangle]\!] \;&=\; Eb \wedge gk([\![x^b_{|Eb} = y]\!]) \;=\; Eb \wedge gk(Eb \wedge [\![x^b = y]\!]) \\ &=\; Eb \wedge gk([\![x^b = y]\!]) \;=\; Eb\end{aligned}$$

and the extensionality of k^rA guarantees that $\beta b = \widehat{f}b$, as needed. □

From item (d) in Theorem 35.4 we get

COROLLARY 35.5. *Let $k : L \longrightarrow R$ be a semilattice morphism with a right adjoint. If A is has compatible structures of L-set and L-presheaf, then $k^r A$ is naturally isomorphic to $\mathfrak{i}_k \varepsilon_k A$.* $\square$

LEMMA 35.6. *If $k : L \longrightarrow R$ is a semilattice morphism with a right adjoint, the functor k^r : $\boldsymbol{Lset} \longrightarrow \boldsymbol{Lset}$ preserves all finite products.*

PROOF. Let $\mathbf{1}$ be the final object in $\boldsymbol{Lset}$ (25.5). From the definition of k^r, it follows that $|k^r \mathbf{1}| = \{\langle x, p \rangle \in L \times L : x \leq p \text{ and } kx = kp\}$, where $\langle x, p \rangle$ is identified with $\langle y, p \rangle$ if $k(x) = k(y) = k(x \wedge y)$. Hence, $\langle x, p \rangle = \langle p, p \rangle$ in $k^r \mathbf{1}$ and so $k^r \mathbf{1} = \mathbf{1}$. If $A_1, \ldots, A_n$ are L-sets, the map

$$\langle \overline{x}, p \rangle \in |k^r(\textstyle\prod_{i=1}^n A_j)| \quad \longmapsto \langle \langle x_1, p \rangle, \ldots, \langle x_n, p \rangle \rangle \in |\textstyle\prod_{i=1}^n k^r A_j|$$

is easily seen to be an isomorphism. $\square$

35.7. **The regularization associated to double negation.** There is a special case of *regularization* [3] that is important in applications to logic : when H is a Heyting algebra and the semilattice morphism k is the canonical quotient map $\pi_D : H \longrightarrow H/D$, where D is the filter of dense elements in H (6.19, 6.20, 10.5). Note that even if H is not complete, π_D has a right adjoint,

$$g(p/D) = \neg\neg\, p.$$

In this case :

$[r\ 1]$: Write rA for $\pi_D^r A$. Thus,

$$|rA| = \{\langle x, p \rangle \in |A| \times H : Ex \leq p \ \text{ and } \neg\neg\, Ex = \neg\neg\, p\},$$

with the convention that

$$\langle x, p \rangle = \langle y, q \rangle \quad \text{iff} \quad p = q \ \text{ and } \ \neg\neg\, Ex = \neg\neg\, Ey = \neg\neg\, [\![x = y]\!].$$

Moreover, equality in rA is given by

$$[\![\langle x, p \rangle = \langle y, q \rangle]\!]_r = p \wedge q \wedge \neg\neg\, [\![x = y]\!].$$

$[r\ 2]$: Write $r : A \longrightarrow rA$ for the morphism $x \longmapsto rx =_{def} \langle x, Ex \rangle$;

$[r\ 3]$: If $f : A \longrightarrow B$ is a H-set morphism, $rf : rA \longrightarrow rB$ is given by

$$rf(\langle x, p \rangle) = \langle fx, p \rangle. \qquad \square$$

The results in this Chapter, together with 34.18, yield

COROLLARY 35.8. *Let H be a Heyting algebra and D the filter of dense elements in H. Let A have compatible structures of H-set and H-presheaf.*

a) The functor r : $\boldsymbol{Hset} \longrightarrow \boldsymbol{Hset}$ preserves all finite products.

b) $rA \approx \mathfrak{i}_{\pi_d}(A/D)$, $A/D = (rA)/D$ and $(rA)_D = A_D$. [4]

c) If F is a prime filter in the BA H/D, then $(A/D)_F \approx rA_{(\pi_D^{-1} F)}$, i.e. the stalk of A/D at F coincides with the stalk of rA at the ultrafilter $\pi_D^{-1}(F)$ of H. $\square$

When H is a frame, we have,

[3] Actually, the origin for the terminology.

[4] A_D is the fiber of A at D, as in 34.6.

PROPOSITION 35.9. *If H is a frame and A is a sheaf over H, then rA is a sheaf over H and A/D is a sheaf over H/D.*

PROOF. There are two ways to establish our result :
∗ Show that rA is a sheaf and conclude by 33.9.(e).(2) and 29.6;
∗ Show that A/D is a sheaf and obtain that rA is a sheaf from 33.3.(c).
The basic techniques are similar, and we take the first route.

Let $\{\langle x_i, p_i \rangle : i \in I\}$ be a compatible family in $|rA|$. Thus, $\forall\ i, j \in I$,

(A) $$[\![x_i = x_j]\!] \;\leq\; Ex_i \wedge Ex_j \;\leq\; p_i \wedge p_j \;\leq\; \neg\neg\,[\![x_i = x_j]\!].$$

By 8.25, there is a pairwise disjoint family $\{q_i : i \in I\}$, such that

(B) $$\forall\ i \in I,\;\; q_i \leq Ex_i \quad \text{and} \quad \bigvee_{i\in I} Ex_i \;\leq\; \neg\neg\bigvee_{i\in I} q_i.$$

Let $S = \{(x_i)_{|q_i} : i \in I\}$; since the q_i are pairwise disjoint, S is compatible in A. Let $x \in |A|$ be the gluing of S. Hence,

(C) $$\begin{cases} (i) \quad Ex = \bigvee_{i\in I} q_i; \\ (ii) \quad q_i = [\![x = (x_i)_{|q_i}]\!] = q_i \wedge [\![x = x_i]\!], \;\; \forall\ i \in I. \end{cases}$$

Thus, $Ex \;\leq\; p \;=_{def}\; \bigvee_{i\in I} p_i$; additionally, Lemma 8.16.(h), $\neg\neg\,Ex_i = \neg\neg\,p_i$ and (B) yield

$$\begin{aligned} \neg\neg\,Ex \;&=\; \neg\neg\bigvee_{i\in I} q_i \;=\; \neg\neg\bigvee_{i\in I} Ex_i \;=\; \neg\neg\bigvee_{i\in I} \neg\neg\,Ex_i \\ &=\; \neg\neg\bigvee_{i\in I} \neg\neg\,p_i \;=\; \neg\neg\bigvee_{i\in I} p_i \;=\; \neg\neg\,p, \end{aligned}$$

and $\langle x, p \rangle \in |rA|$. Then, $\langle x, p \rangle$ is the gluing of the original family in rA; clearly, it has the appropriate extent.

Fact. *For all $i \in I$, $[\![x = x_i]\!]$ is dense in p_i.*

Proof. Let $\alpha = \{j \in I : Ex_i \wedge q_j \neq \perp\}$; from (B) comes

(D) $$\begin{aligned} \neg\neg\,Ex_i \;&=\; \neg\neg\,Ex_i \wedge \neg\neg\bigvee_{j\in I} q_j \;=\; \neg\neg\,(Ex_i \wedge \bigvee_{j\in I} q_j) \\ &=\; \neg\neg\bigvee_{j\in A} Ex_i \wedge q_j. \end{aligned}$$

Consequently, for $j \in \alpha$, (A) and (C).(ii) entail

$$\begin{aligned} \neg\neg\,[\![x = x_i]\!] \;&\geq\; \neg\neg\Big([\![x = (x_j)_{|q_j}]\!] \wedge [\![(x_j)_{|q_j} = x_i]\!]\Big) \\ &=\; \neg\neg\,(q_j \wedge [\![x_i = x_j]\!]) \;=\; \neg\neg\,q_j \wedge \neg\neg\,[\![x = x_i]\!] \\ &=\; \neg\neg\,q_j \wedge \neg\neg\,(Ex_i \wedge Ex_j) \;=\; \neg\neg\,(q_j \wedge Ex_i \wedge Ex_j) \\ &=\; \neg\neg\,(q_j \wedge Ex_i). \end{aligned}$$

Whence, taking joins with respect to $j \in \alpha$, (D) and (A) yield

$$\begin{aligned} \neg\neg\,[\![x = x_i]\!] \;&\geq\; \neg\neg\bigvee_{j\in A} \neg\neg\,(q_j \wedge Ex_i) \;=\; \neg\neg\bigvee_{j\in A} Ex_i \wedge q_j \\ &=\; \neg\neg\,Ex_i \;=\; \neg\neg\,p_i, \end{aligned}$$

as claimed.

With the Fact, we then obtain

$$\begin{aligned} [\![\langle x, p \rangle = \langle x_i, p_i \rangle]\!]_r \;&=\; p \wedge p_i \wedge \neg\neg\,[\![x = x_i]\!] \;=\; p_i \wedge \neg\neg\,[\![x = x_i]\!] \\ &=\; p_i \;=\; E\langle x_i, p_i \rangle, \end{aligned}$$

and $\langle x, p \rangle$ is the gluing of the $\langle x_i, p_i \rangle$, as desired. □

Part 7

Characteristic Maps

Introduction

Characteristic maps are a means to construct an algebra of relations in the categories $\boldsymbol{\Omega set}$ and $\boldsymbol{pSh(\Omega)}$. The attentive reader will realize that many of our definitions and results are valid for general L-sets and L-presheaves. However, completeness of the base algebra guarantees that certain (possibly infinite) joins and meets, associated to quantification, exist in the algebra of characteristic functions and for this reason we concentrate on objects whose base algebra is a frame. The original idea for this development appeared in [**50**]; [**52**] describes how analogous methods may be applied to deal with subobjects in presheaves over idempotent quantales.

Although our main interest is establishing the basic structures needed to develop Model Theory in the category of Ω-presheaves, we shall also work with Ω-sets in order to guarantee a certain generality to our constructions, as well as to indicate the difficulties that arise when the domain of a first-order structure *is not* a presheaf. Since in first-order languages relation and function symbols, although finitary, might depend on arbitrarily large finite sets of variables, it seems appropriate to deal with arbitrary powers of a Ω-object. Moreover, if A and B are the domains of first-order structures, it is important to describe how relations in A transfer to B along a morphism. In this respect, inverse image is more important than direct image. Indeed, recall that if M, N are first-order structures, a map $f : M \longrightarrow N$ is a L-morphism if for each n-ary relation R in L and $\overline{a} \in M^n$

$$M \models R[\overline{a}] \quad \Rightarrow \quad N \models R[f\overline{a}].$$

This may be rephrased in terms of characteristic maps as $R^M \leq f^* R^N$, where $R^{(\cdot)}$ is the characteristic map of the interpretation of R. In a similar vein, f is a L-monic if $R^M = f^* R^N$ and an elementary embedding if a corresponding relation holds for all formulas.

The family of all characteristic maps on a Ω-set, as well as the collection of such maps that depend on a *fixed* subset of the index set, constitute frames. However, the family of all characteristic maps that depend on some finite subset of the index set *does not* have a natural structure of a frame. Nevertheless, it can be described as an inductive limit of an inductive system of frames of characteristic maps on finite powers of the base domain, together with open injections. Consequently, we have included material on the graded frame that arises by considering characteristic maps on finite powers of a Ω-set.

Even if we had adopted a completely finitistic approach, to establish the soundness of the Heyting Predicate Calculus for presheaves of structures over a frame, requires that characteristic maps defined on distinct finite powers, having as exponents subsets α, β of the natural numbers, be considered as characteristic maps on the power whose exponent is the union $\alpha \cup \beta$. This corresponds to a well-known identification : a formula in the variables indexed by α, is considered as formula in the variables indexed by $\alpha \cup \beta$. This, in turn, originates with the fact that for $n \leq m$, a subset of A^n can be identified with a subset of A^m, via inverse image by the projection that forgets coordinates outside n. This identification corresponds precisely to taking, as interpretations of formulas in n free variables, the inductive limit of its interpretations in A^m, where $n \leq m$.

Experience has shown that it is easier to handle the values of formulas if one does have to worry about the extent of the arguments. Hence, characteristic maps will be defined on a power of the domain of the base object (37.1). For Ω-presheaves – but not for Ω-sets –, this is equivalent to characteristic maps defined in the domain of the corresponding power of the base object (38.7). However, the generality and practical merits of the definition we have adopted justify the effort in developing its properties.

Unless otherwise explicitly stated, all objects are *extensional* and all semilattices have $\perp$ and $\top$.

CHAPTER 36

Closure

This Chapter discusses results of section 24.2 in the categories $\boldsymbol{\Omega set}$ and $\boldsymbol{pSh(\Omega)}$. The proofs for Ω-presheaves are so similar that we shall frequently omit them. The notion of *denseness* is defined in 25.32.

DEFINITION 36.1. *Let A be a Ω-set and $S \subseteq |A|$. The* **closure of S in A** *is*

$$\overline{S} \;=\; \{x \in |A| : Ex = \textstyle\bigvee_{s \in S} [\![x = s]\!]\}.$$

Clearly, $S \subseteq \overline{S}$. We say that S is **closed** *in A if $S = \overline{S}$. Write $\mathfrak{P}_{\top}(A)$ for the family of* closed *subsets of A. If the domain in which the closure operation is being performed has to be displayed, we write $\overline{S}^{A}$ for $\overline{S}$. The superscript will always be omitted whenever clear from context.*

REMARK 36.2. Recall our convention (25.1) that an Ω-set A has a *unique* section of extent $\perp$, written $*$. For all $S \subseteq |A|$

$$\perp \;=\; E* \;=\; \textstyle\bigvee_{s \in S} [\![* = s]\!],$$

and so $* \in \overline{S}$; $\{*\}$ is clearly closed and so it is *the least closed subset of A*. $\square$

Recall (25.3) that if A is a Ω-set, $S \subseteq |A|$ and $p \in \Omega$,

$*$ The *restriction* of S to p, $S_{|p}$, is the sub-Ω-set of A whose domain is

$$|S_{|p}| = \{x \in |A| : Ex \leq p\};$$

$*$ The *extent of* S is $ES \;=\; \bigvee_{x \in S} Ex$.

LEMMA 36.3. *Let A be a Ω-set and S, $T \subseteq |A|$.*

a) $ES = E\overline{S}$.

b) $T \subseteq \overline{S}$ iff S is dense in T.

c) If $S \subseteq T$, then $\overline{S} = \overline{T}$ iff S is dense in T.

d) The closure operation is increasing, inflationary and idempotent [1].

e) The intersection of any family of closed subsets of is closed.

f) If S is closed in A, then $S \cap T$ is closed in T.

g) If S is closed and $T \subseteq S$ is a compatible set in A, whose gluing t exists in A, then $t \in S$.

---

[1] That is, $S \subseteq T \Rightarrow \overline{S} \subseteq \overline{T}$, $S \subseteq \overline{S}$ and $\overline{\overline{S}} = \overline{S}$.

PROOF. Items (a), (b), (c) and (f) are clear, as is the fact that closure is increasing and inflationary in (d). For idempotency, we have

$$S \text{ dense in } \overline{S}, \quad \overline{S} \text{ dense in } \overline{\overline{S}},$$

and the conclusion follows from (c) and the transitivity of denseness (25.33.(a)). For (e), let S_i, $i \in I$, be closed subsets of $|A|$, and set $S \;=\; \bigcap_{i \in I} S_i$. By (c),

$$\overline{S} \;\subseteq\; \overline{S_i} \;=\; S_i, \text{ for all } i \in I,$$

and $\overline{S} \subseteq S$, i.e., S is closed in A. For (g), let $T \subseteq S$ be compatible in A, with gluing t. Then, [glu p] in 25.30, with $p = \top$, yields

$$Et \;=\; \bigvee_{u \in T} [\![t = u]\!] \;\leq\; \bigvee_{s \in S} [\![t = s]\!],$$

and so $t \in S$, as asserted. $\square$

For Ω-presheaves, closure has the following additional properties :

LEMMA 36.4. *Let A be a Ω-presheaf, S, T be subpresheaves of A and $p \in \Omega$.*

a) The following conditions are equivalent :

(1) $\overline{S} = S$;

(2) $\forall\, t \in |A|, \forall\, \alpha \subseteq \Omega, \;\; \bigvee \alpha = Et$ and $t_{|q} \in |S| \;\;\Rightarrow\;\; t \in |S|$.

b) $\overline{S \cap T} = \overline{S} \cap \overline{T}$.

c) All closed subsets of $|A|$ are subpresheaves of A.

d) Closure commutes with restriction, i.e., $\overline{(S_{|p})} = \overline{S}_{|p}$.

e) If A is a sheaf, then S is closed in A iff S is a subsheaf of A.

PROOF. Item (a) is straightforward. For (b), first note that 36.3.(b) entails $\overline{S \cap T} \;\subseteq\; \overline{S} \cap \overline{T}$. Let $x \in \overline{S} \cap T$; then

$$\text{(I)} \qquad Et \;=\; \bigvee_{s \in S} [\![x = s]\!] \;=\; \bigvee_{t \in T} [\![x = t]\!].$$

Let $D \;=\; \{s_{|([\![x=s]\!] \wedge [\![x=t]\!])} \in |A| : s \in S, t \in T\}$. Note that for all $\langle\, s, t\,\rangle \in S \times T$,

$$\text{(II)} \qquad s_{|[\![x=s]\!] \wedge [\![x=t]\!]} \;=\; x_{|[\![x=s]\!] \wedge [\![x=t]\!]} \;=\; t_{|[\![x=s]\!] \wedge [\![x=t]\!]}.$$

Since S, T are subpresheaves of A, (II) entails $D \subseteq (S \cap T)$. But then (I) yields

$$\begin{aligned} \bigvee_{d \in D} [\![x = d]\!] \;&=\; \bigvee_{\langle s,t \rangle \in S \times T} [\![x = s_{|[\![x=s]\!] \wedge [\![x=t]\!]}]\!] \\ &=\; \bigvee_{\langle s,t \rangle \in S \times T} [\![x = s]\!] \wedge [\![x = t]\!] \\ &=\; \bigvee_{s \in S} [\![x = s]\!] \wedge \bigvee_{t \in T} [\![x = t]\!] \;=\; Et \end{aligned}$$

showing that $x \in \overline{S \cap T}$.

c) It is enough to show that a closed subset is closed under restriction. Let C be a closed subset of A, $x \in C$ and $p \in \Omega$. Then

$$\bigvee_{c \in C} [\![x_{|p} = c]\!] \;\leq\; [\![x_{|p} = x]\!] \;=\; p \wedge Ex \;=\; Ex_{|p},$$

and so $x_{|p} \in \overline{S} = S$, as desired. For (d), since restriction to p is obviously increasing, it suffices to check that $|\overline{S}_{|p}| \subseteq |\overline{(S_{|p})}|$. If $x \in |\overline{S}_{|p}|$, then, because S is a subpresheaf, and $Ex \leq p$, we obtain

$$\begin{aligned} Ex &= \bigvee_{s \in |S|} [\![x = s]\!] = \bigvee_{s \in |S|} p \wedge [\![x = s]\!] = \bigvee_{s \in |S|} [\![x = s_{|p}]\!] \\ &\leq \bigvee_{d \in |S_{|p}|} [\![x = d]\!], \end{aligned}$$

and $x \in |\overline{(S_{|p})}|$, as needed. The proof of (e) is analogous to that of 24.13. □

Observe that

∗ If A is a Ω-presheaf, 36.4.(b) entails that $\mathfrak{P}_\top(A)$ is the family of *closed subpresheaves* of A;

∗ If A is a Ω-sheaf, 36.4.(d) implies that $\mathfrak{P}_\top(A)$ is the family of *subsheaves* of A.

COROLLARY 36.5. *If A is a Ω-set or a Ω-presheaf, $\langle \mathfrak{P}_\top(A), \subseteq \rangle$ is a complete lattice, with meets given by intersection.*

PROOF. Follows from 36.3.(d), 36.4.(b) and the fact that (set-theoretic) intersection of subpresheaves is a subpresheaf. □

If S_i, $i \in I$, are closed subsets of a Ω-set or a Ω-presheaf A, then [2]

$$(\bigvee) \qquad \bigvee_{i \in I} S_i = \overline{\bigcup_{i \in I} S_i}$$

is the join of the S_i in the complete lattice $\mathfrak{P}_\top(A)$.

REMARK 36.6. If A is a Ω-set, $\mathfrak{P}_\top(A)$ is *almost* a Ω-presheaf. By Exercise 36.14, closure is preserved by restriction. It is then straightforward that the maps

$$S \in \mathfrak{P}_\top(A) \longmapsto ES \in \Omega$$
$$\langle S, p \rangle \in \mathfrak{P}_\top(A) \times \Omega \longmapsto S_{|p} \in \mathfrak{P}_\top(A),$$

satisfy [rest 1] and [rest 3] in Definition 23.1, but, in general, *not* [rest 2] [3].

However, *if A is a Ω-presheaf*, then [rest 2] is verified and $\mathfrak{P}_\top(A)$ is a Ω-presheaf. In this case, Proposition 26.8 yields, for S, $T \in \mathfrak{P}_\top(A)$,

$$\begin{aligned} [\![S = T]\!] &= \bigvee \{p \in \Omega : p \leq ES \wedge ET \text{ and } S_{|p} = T_{|p}\} \\ &= \bigvee \{p \in \Omega : p \leq ES \wedge ET \text{ and } |S_{|p}| = |T_{|p}|\}. \end{aligned}$$

The reader is invited to show that, with the presheaf structure defined above, $\mathfrak{P}_\top(A)$ **is a sheaf** over Ω, whenever A is an extensional Ω-presheaf. □

EXAMPLE 36.7. Let $|A| = \{a, b, c, *\}$ be a set with four elements and let $B = \{\bot, p, \neg p, \top\}$ be the four-element BA. Define, for $x, y \in |A| - \{*\}$

[2] By 36.4.(b), in the case of presheaves.

[3] One may have $S_{|p} = \{*\}$, with $ES = \top$.

$$[\![x = y]\!] \;=\; [\![y = x]\!] \;=\; \begin{cases} \top & \text{if } x = y; \\ p & \text{if } x = a \text{ and } y = c; \\ \neg p & \text{if } x = b \text{ and } y = c; \\ \bot & \text{if } x = a \text{ and } y = b, \end{cases}$$

while $[\![x = *]\!] = [\![* = x]\!] = \bot$. It is straightforward that this defines a equality on $|A|$, with which it is an extensional B-set. Note that for all $x \in |A| - \{*\}$, $\{x, *\}$ is closed in A. Moreover, $\{a, b\}$ is dense in A, because

$$\top \;=\; Ec \;=\; [\![c = a]\!] \vee [\![c = b]\!] \;=\; p \vee \neg p.$$

Thus, $\{c, *\} \cap (\{a, *\} \vee \{b, *\}) = \{c, *\} \cap |A| = \{c, *\}$; on the other hand, $\{c, *\} \cap \{a, *\} = \{*\} = \{c, *\} \cap \{b, *\}$. Hence, $\mathfrak{P}_\top(A)$ *is not distributive and so cannot be a frame*. As we shall see shortly, this changes if A is a presheaf. $\square$

THEOREM 36.8. *Let A be a Ω-presheaf.*

a) $\mathfrak{P}_\top(A)$ is a frame, whose meets are (set-theoretic) intersections and whose joins are given by formula ($\bigvee$) on page 396.

b) If B is a dense subpresheaf of A, then the map

$$T \in \mathfrak{P}_\top(B) \;\longmapsto\; \overline{T}^A \in \mathfrak{P}_\top(A)$$

is a frame-isomorphism.

c) The map defined in (b) is a frame isomorphism between $\mathfrak{P}_\top(A)$ and $\mathfrak{P}_\top(cA)$, where cA is the completion of A over Ω (27.9).

PROOF. a) By 36.3.(a) and 36.4.(a), the map

$$\overline{\star} \;:\; 2^{|A|} \longrightarrow \mathfrak{P}_\top(A)$$

is a *nucleus* on the cBa $2^{|A|}$ (13.1). Hence, Theorem 13.2.(d) entails that $\mathfrak{P}_\top(A)$ is a frame. Item (a) in Theorem 7.5, together with the proof of 13.2, guarantee that meets and joins in $\mathfrak{P}_\top(A)$ are as asserted.

b) By 36.3.(d), the displayed map is increasing. Hence, to show that it is an isomorphism, it is enough to check that it is a bijection.

For T, $T' \in \mathfrak{P}_\top(B)$, assume that $\overline{T}^A = \overline{T'}^A$. By 36.3.(b), T, T' are dense in each other in A and so the same must be true in B. Since they are closed relative to B, we must have $T = T'$, verifying injectivity. If S is a closed subset of A, let T be the subpresheaf of B whose domain is $(|B| \cap |S|)$; T is closed in B (36.3.(f)) and dense in S, because B is dense in A. Thus, $\overline{T}^A = S$, establishing surjectivity. $\square$

The next result simplifies computations in non-empty powers of presheaves.

LEMMA 36.9. *Let A be a Ω-presheaf and $I \neq \emptyset$ be a set. For $S \subseteq |A|^I$, set* [4] *$pS = \{\overline{s}_{|E\overline{s}} \in A^I : \overline{s} \in S\}$.*

Then, $\overline{pS} = \{\overline{x} \in |A^I| : E\overline{x} = \bigvee_{\overline{s} \in S} [\![\overline{x} = \overline{s}]\!]\}$.

[4] Note that $S \subseteq |A^I|$ iff $pS = S$.

PROOF. Recall that A^I has a natural presheaf structure (26.12), compatible with its equality, given by (25.12) $[\![\overline{x} = \overline{y}]\!] = \bigwedge_{i \in I} [\![x_i = y_i]\!]$. For $\overline{x} \in |A^I|$ and $\overline{s} \in S$, 26.8.(b) yields $[\![\overline{x} = \overline{s}]\!] = [\![\overline{x} = \overline{s}_{|E\overline{s}}]\!]$, with $\overline{s}_{|E\overline{s}} \in pS$. Hence, $\bigvee_{\overline{s} \in S} [\![\overline{x} = \overline{s}]\!] = \bigvee_{\overline{c} \in pS} [\![\overline{x} = \overline{c}]\!]$, and the conclusion follows immediately. $\square$

REMARK 36.10. If A is an extensional Ω-presheaf, define a Ω-presheaf $\mathfrak{P}(A)$ by the following prescriptions :

$* \ |\mathfrak{P}(A)| = \{\langle S, p \rangle \in (\mathfrak{P}_\top(A) \times \Omega) : ES \leq p\}$; $\qquad * \ E\langle S, p \rangle = p$;

$* \ \langle S, p \rangle_{|q} = \langle S_{|q}, p \wedge q \rangle$.

As was the case for presheaves of sets, discussed in section 24.2 (24.38),

$\mathfrak{P}(A)$ is a Ω-sheaf, whose set of global sections is $\mathfrak{P}_\top(A)$.

Moreover, $[\![\langle S, p \rangle = \langle T, q \rangle]\!] = \bigvee \{u \leq p \wedge q : S_{|u} = T_{|u}\}$.

The sheaf $\mathfrak{P}(A)$ is the **sheaf of closed subpresheaves of A**. If A is a sheaf, $\mathfrak{P}(A)$ is the **sheaf of subsheaves of A**. $\square$

As for presheaves of sets (24.24, 24.26), we have

DEFINITION 36.11. *Let $\eta : A \longrightarrow B$ be a morphism of Ω-sets. If $S \subseteq |A|$,* **the image of S by η** *is*

$$\eta_* S = \overline{\{\eta x \ : \ x \in S\}}.$$

In particular, $Im\,\eta$ is the closure in B of the set-theoretical image of $|A|$ by η. If $T \subseteq |B|$, the **inverse image of T by η** *is*

$$\eta^* T = \{x \in |A| : \eta x \in T\}.$$

Exercises

36.12. Let A be a Ω-set and $\Delta = \{\langle a, a \rangle : a \in |A|\}$ be the diagonal of $A^2 = A \times A$.

a) $\overline{\Delta} = \{\langle x, y \rangle \in |A^2| : Ex = Ey = [\![x = y]\!]\}$.

b) A is extensional iff Δ is a closed subset of A^2. $\square$

36.13. Let $\mathbf{1}$ be the final object in $\boldsymbol{\Omega}\,\boldsymbol{set}$ (25.5, 26.18). For $S \subseteq |\mathbf{1}|$, as was the case in 24.21.(*b*), $S \in \mathfrak{P}_\top(\mathbf{1})$ iff $S = \mathbf{1}_{|ES}$. $\square$

36.14. Show that restriction takes closed sets to closed sets. $\square$

36.15. If A is a Ω-presheaf and S, $T \subseteq |A|$, define [5]

$$S \to T = \{x \in |A| : \text{For all } p \leq Ex, \ \ x_{|p} \in S \Rightarrow x_{|p} \in T\}.$$

a) $S, T \in \mathfrak{P}_\top(A) \ \Rightarrow \ S \to T \in \mathfrak{P}_\top(A)$ and $\overline{(\overline{S} \to \overline{T})} = \overline{S} \to \overline{T}$.

b) $S \to T$ is the implication operation in the frame $\mathfrak{P}_\top(A)$.

c) If S, $T \in \mathfrak{P}_\top(A)$, the following are equivalent, for $p \leq ES \wedge ET$:

[5] If S, T are subpresheaves, then $S \to T = \{x \in |A| : x \in |S| \Rightarrow x \in |T|\}$.

(1) $A_{|p} \subseteq (S \leftrightarrow T)$; (2) $S_{|p} = T_{|p}$,

where $\leftrightarrow$ is the equivalence operation in the frame $\mathfrak{P}_\top(A)$. □

36.16. If A is a Ω-presheaf, $\mathfrak{P}(A)$ is flabby (31.10.(e)). □

36.17. Let A be an extensional Ω-presheaf. Write $\mathcal{H}$ for the frame $\mathfrak{P}_\top(A)$.

a) The map $p \in \Omega \longmapsto \gamma(p) =_{def} A_{|p} \in \mathcal{H}$ is an open frame morphism (8.1.(b)).

b) If $A = \mathbf{1}$, the morphism in (a) is an isomorphism between $\mathfrak{P}_\top(\mathbf{1})$ and Ω [6].

c) Let $\widetilde{\mathcal{H}}$ be the sheaf over $\mathcal{H}$ in 27.7 and 25.4. Then, $\mathfrak{P}(A)$ is the inverse image of $\widetilde{\mathcal{H}}$ along the morphism γ in (a) [7]. □

36.18. Let A be an L-set and $S \subseteq |A|$. The **finite closure** of S in A is

$$fcl(S) = \{s \in |A| : \exists\ \alpha \subseteq_f S \text{ such that } Es = \textstyle\bigvee_{a\in\alpha} [\![s = a]\!]\ \}.$$

With 36.3 and 36.4 as models, discuss finite closure for L-sets and presheaves. □

36.19. With 24.25 and 24.27 as parameters, study image and inverse image by morphisms in $\boldsymbol{\Omega set}$ and $\boldsymbol{pSh(\Omega)}$, establishing their adjointness. □

36.20. Let A, B be Ω-presheaves.

a) If $S \in \mathfrak{P}_\top(A)$ and $T \in \mathfrak{P}_\top(B)$ then $S \times T \in \mathfrak{P}_\top(A \times B)$, where

$$S \times T \ = \ \{\langle a, b \rangle \in |A \times B| : a \in S \text{ and } b \in T\}.$$

b) $E(S \times T) \ = \ ES \wedge ET$ (ES is the *support* of S as in 26.1.(c)).

c) The map $\mathfrak{P}_\top(A) \times \mathfrak{P}_\top(B) \longrightarrow \mathfrak{P}_\top(A \times B)$, given by $\langle S, T \rangle \longmapsto S \times T$, is a frame-morphism, where $\mathfrak{P}_\top(A) \times \mathfrak{P}_\top(B)$ has the product frame structure [8].

d) If S, $S' \in \mathfrak{P}_\top(A)$ and $T \in \mathfrak{P}_\top(B)$, then

$$(S \to S') \times T \ = \ (A \times T) \ \cap \ \Big((S \times T) \to (S' \times T)\Big);$$

and, similarly if the first coordinate is kept fixed. Conclude that the maps

$$\begin{cases} \mathfrak{P}_\top(A) \longrightarrow \mathfrak{P}_\top(A \times B), & S \longmapsto S \times B \\ \mathfrak{P}_\top(B) \longrightarrow \mathfrak{P}_\top(A \times B), & T \longmapsto A \times T \end{cases}$$

are **open** injective frame-morphisms.

e) For $\langle S, T \rangle$, $\langle S', T' \rangle \in \mathfrak{P}_\top(A) \times \mathfrak{P}_\top(B)$, the following are equivalent:

(1) $S \times T = S' \times T'$;

$$(2) \begin{cases} (a) & ES \wedge ET = ES' \wedge ET'; \\ (b) & S_{|ES\wedge ET} = S'_{|ES\wedge ET} \quad \text{and} \quad T_{|ES\wedge ET} = T'_{|ES\wedge ET}. \end{cases}$$ □

36.21. State and prove the generalization of Theorem 24.15 to L-presheaves, assuming that L is a frame wherever necessary. □

[6] $\mathbf{1}$ is the final object in $\boldsymbol{\Omega\, set}$; this generalizes 24.21.(c).
[7] No wonder the constructions are so similar.
[8] Note : $S \times T = \pi_A^*(S) \cap \pi_B^*(T)$, where π_A, π_B are the natural projections.

CHAPTER 37

Characteristic Maps in Powers of a Ω-set

We shall constantly use the vector notation introduced in 25.13 and 26.11. The reader should keep in mind the distinction between the domain of a product of Ω-sets and the domain of their product. If A is a Ω-presheaf and I is a set,

$$\overline{x}_{|E\overline{x}} \text{ is always in } |A^I| \quad \text{and} \quad \overline{x}_{|E\overline{x}} = \overline{x} \ \textbf{ iff } \ \overline{x} \in |A^I|.$$

Moreover, if A has a compatible structures of Ω-set and presheaf (26.6), then for all $\overline{x}$, $\overline{y} \in |A|^I$ and p, $q \in L$,

$$\text{(Rest)} \qquad [\![\overline{x}_{|p} = \overline{y}_{|q}]\!] \ = \ p \wedge q \wedge [\![\overline{x} = \overline{y}]\!],$$

a very useful relation. Exercise 37.21 collects the properties that will be of constant use.

We begin by defining characteristic maps on powers of a Ω-set. However, most of the results can be readily extended to finite products of Ω-sets or Ω-presheaves, corresponding to relations whose variables range over distinct sorts. Recall (25.15.(a)) that if A is a Ω-set and $I \neq \emptyset$ is a set, then

$$\textstyle\bigvee_{\overline{a} \in |A|^I} E\overline{a} \ = \ EA.$$

Definition 37.1. *Let A be a Ω-set and I be a set. A* ***I-characteristic map*** *on A is a map, $|A|^I \xrightarrow{k} \Omega$, such that for $\overline{x}$, $\overline{y} \in |A|^I$,*

$$[\text{ch } 1]: \ k(\overline{x}) \ \leq \ E\overline{x}; \qquad [\text{ch } 2]: \ k(\overline{x}) \wedge [\![\overline{x} = \overline{y}]\!] \ \leq \ k(\overline{y}).$$

Let $\mathfrak{K}_I(A, \Omega)$ be the set of I-characteristic maps on A. Whenever context allows, write $\mathfrak{K}_I A$ for $\mathfrak{K}_I(A, \Omega)$. For $h \in \mathfrak{K}_I A$, set

$$Eh \ = \ \textstyle\bigvee_{\overline{a} \in |A|^I} h(\overline{a}),$$

called the **extent** *of h.*

If $I = \underline{n} = \{1, \dots, n\}$, a I-characteristic map is called a ***n-characteristic map on A****; write $\mathfrak{K}_n A$ for the set of n-characteristic maps on A.*

Remark 37.2. Let A be a Ω-set and I be a set.
a) It is straightforward that [ch 2] in 37.1 is equivalent to

$$[\text{ch } 2']: \quad k(\overline{x}) \wedge [\![\overline{x} = \overline{y}]\!] \ = \ k(\overline{y}) \wedge [\![\overline{x} = \overline{y}]\!],$$

or yet, using the equivalence in the frame Ω (6.9), [ch 2] and [ch 2′] can be written

$$[\text{ch } 2'']: \quad [\![\overline{x} = \overline{y}]\!] \ \leq \ k(\overline{x}) \leftrightarrow k(\overline{y}).$$

b) If D is a sub-Ω-set of A and $k \in \mathfrak{K}_I A$, write $k_{|D}$ for the restriction of k to $|D|^I$, called the **restriction of k to D**. Hence, for $\overline{d} \in |D|^I$, $k_{|D}(\overline{d}) = k(\overline{d})$. Clearly, $k_{|D}$ is I-characteristic map on D.

c) If $h \in \mathfrak{K}_I A$, it follows from 25.15.(a) and [ch 1] in 37.1 that

$$Eh = \bigvee_{\overline{a} \in |A|^I} E\overline{a} = EA.$$

Hence, $\mathfrak{K}_I A$ could be defined as the set of maps $h : |A|^I \longrightarrow EA$, satisfying [ch 1] and [ch 2] in 37.1. In our setting, the distinction between the definitions is slight, except when treating $\mathfrak{K}_\emptyset A$, discussed in 37.3, below.

These observations will be of constant use, frequently without explicit reference. $\square$

Remark 37.3. If $I = \emptyset$, a I-characteristic map $k : |A|^0 \longrightarrow \Omega$ is a function from $\{*\}$ to Ω. Hence, the statement of 37.1 implies that $\mathfrak{K}_\emptyset A$ is in bijective correspondence with Ω. However, taking 37.2.(c) into account, we shall instead

(!) **Identify $\mathfrak{K}_\emptyset A$ with the frame $(EA)^{\leftarrow}$.**

We shall also write $\mathfrak{K}_0 A$ for $\mathfrak{K}_\emptyset A$.

Convention **(!)** is adopted to prevent many of our central results to be falsified for trivial reasons. If characteristic maps on A are taken to have codomain EA, which, as noted in 37.2.(c), is equivalent to 37.1, then **(!)** is immediately forthcoming. On the other hand, one would have to constantly take into account the codomain of characteristic maps, burdening arguments unnecessarily. A structural reason for **(!)** will emerge in Chapter 40 of this Part (see 40.4).

As an additional argument for keeping 37.1, note that if I is *finite*, it can be stated *verbatim* for a L-set, where L is any semilattice. $\square$

The set of I-characteristic maps, $\mathfrak{K}_I A$, inherits the structure that is present in Ω. Care must be exercised in handling implication and negation. We summarize the pertinent results, omitting proofs, in

Proposition 37.4. *Let A be a L-set and I be a set.*

a) There is a partial order on $\mathfrak{K}_I A$, given by

$$h \le k \quad \textit{iff} \quad \forall\, \overline{x} \in |A|^I, \;\; h(\overline{x}) \le k(\overline{x}).$$

In this partial order, the maps $\bot_I$ and $\top_I$ [1] *are the bottom and top of $\mathfrak{K}_I A$, respectively.*

b) If $p \in L$ and $k \in \mathfrak{K}_I A$, then $p \wedge k \in \mathfrak{K}_I A$, where for all $\overline{x} \in |A|^I$

$$(p \wedge k)(\overline{x}) = p \wedge k(\overline{x}).$$

c) If L is a lattice, $\mathfrak{K}_I A$ is a lattice, with meets and joins computed pointwise, i.e.,

$$\left(\bigwedge_{\lambda\in\Lambda} h_\lambda\right)(\overline{x}) = \bigwedge_{\lambda\in\Lambda} h(\overline{x}) \quad \textit{and} \quad \left(\bigvee_{\lambda\in\Lambda} h_\lambda\right)(\overline{x}) = \bigvee_{\lambda\in\Lambda} h_\lambda(\overline{x}),$$

are the meet and join of the h_λ in the partial order of (a). Moreover, if L is complete or a frame, the same is true of $\mathfrak{K}_I A$.

d) If L is a HA, $\mathfrak{K}_I A$ is a HA in which implication and negation are given by

[1] In items (b) and (c) of 37.22.

$$\begin{cases} (h \to k)(\overline{x}) &= E\overline{x} \wedge (h(\overline{x}) \to k(\overline{x})); \\ (\neg\, h)(\overline{x}) &= E\overline{x} \wedge \neg\, h(\overline{x}), \end{cases}$$

where $\to$ and $\neg$ in the right-hand side of these formulas are implication and negation in L.

e) If L is a BA, then, with the operations described above, $\mathfrak{K}_I A$ is a Boolean algebra, which is complete whenever the same is true of L.

f) Let S_I be the set of permutations of I, that is, the set of bijective maps from I to I. If $\sigma \in S_I$ and $\overline{x} \in |A|^I$, let $\overline{x}^{\,\sigma} \in |A|^I$ be given by $\overline{x}_i^{\sigma} = \overline{x}_{\sigma(i)}$. For $h \in \mathfrak{K}_I A$, define $h^\sigma : |A|^I \longrightarrow \Omega$ by $h^\sigma(\overline{x}) = h(\overline{x}^{\,\sigma})$. Then

(1) *$h^\sigma \in \mathfrak{K}_I A$;* (2) *The map $h \mapsto h^\sigma$ is an automorphism of $\mathfrak{K}_I A$.* □

Example 37.5. Recall (25.5) that $\mathbf{1}$ is the final object in the category $\boldsymbol{\Omega set}$, where $|\mathbf{1}| = \Omega$ and $[\![p = q]\!] = p \wedge q$.

Fact 37.6. *Let $p \in \Omega$.*

a) If $k \in \mathfrak{K}_1 \mathbf{1}_{|p}$, then for all $q \in |\mathbf{1}_{|p}|$, $k(q) = k(p) \wedge q$.

b) The map $k \in \mathfrak{K}_1 \mathbf{1}_{|p} \longmapsto k(p) \in p^{\leftarrow}$ is an isomorphism.

Proof. Keep in mind that $|\mathbf{1}_{|p}| = p^{\leftarrow}$ and that if $q \in |\mathbf{1}_{|p}|$,

$$Eq = q = p \wedge q = [\![p = q]\!].$$

For $k \in \mathfrak{K}_1 \mathbf{1}_{|p}$ and $q \in |\mathbf{1}_{|p}|$,

$$k(q) = k(q) \wedge Eq = k(q) \wedge [\![p = q]\!] = k(p) \wedge [\![p = q]\!] = k(p) \wedge q,$$

establishing (a). Since $Ep = p$, it is clear that $k(p) \leq p$. It is immediate from (a) that the map in (b) is injective. Surjectivity comes from 37.4.(b) : if $u \leq p$, then $k = u \wedge \top_1 \in \mathfrak{K}_1 \mathbf{1}_{|p}$ and $k(p) = u$. △

If A is a Ω-set, then **(!)** in 37.3 and Fact 37.6 entail [2]

$$\mathfrak{K}_0 A \approx (EA)^{\leftarrow} \approx \mathfrak{K}_1 \mathbf{1}_{|EA}.$$

From this perspective, if I is any set, there is an isomorphic copy of $\mathfrak{K}_0 A$ inside $\mathfrak{K}_I A$, namely the image of the map

$$p \leq EA \longmapsto p \wedge \top_I \in \mathfrak{K}_I A$$

i.e., the I-characteristic map on A given by $\overline{a} \mapsto p \wedge E\overline{a}$. To see that this map is injective, note that if $p, q \leq EA$ are such that

$$\forall\, \overline{a} \in |A|^I, \ \ p \wedge E\overline{a} = q \wedge E\overline{a}$$

then taking joins on both sides over $\overline{a}$ yields $p \wedge EA = q \wedge EA$, and so $p = q$. We shall return to this theme in Chapter 40. □

Example 37.7. Let $f : A \longrightarrow B$ be a morphism of Ω-sets and let$I$ be a set. Fix $\xi = \overline{b} \in |B|^I$ and define

$$k^\xi : |A|^I \longrightarrow \Omega, \text{ by } \ k^\xi(\overline{a}) = [\![f(\overline{a}) = \overline{b}]\!].$$

[2] One is tempted by the idea that the empty power of A should be $\mathbf{1}_{|EA}$.

Then, k^ξ is a I-characteristic map on A. First note that since f is a morphism, we have $k^\xi(\overline{a}) \leq Ef\overline{a} = E\overline{a}$, for all $\overline{a} \in |A|^I$. If $\overline{x} \in |A|^I$, $[\![\overline{a} = \overline{x}]\!] \leq [\![f\overline{a} = f\overline{x}]\!]$ yields

$$\begin{aligned} k^\xi(\overline{a}) \wedge [\![\overline{a} = \overline{x}]\!] &= [\![f\overline{a} = \overline{b}]\!] \wedge [\![\overline{a} = \overline{x}]\!] \leq [\![f\overline{a} = \overline{b}]\!] \wedge [\![f\overline{a} = f\overline{x}]\!] \\ &\leq [\![f\overline{x} = \overline{b}]\!] = k^\xi(\overline{x}), \end{aligned}$$

verifying [ch 2] in 37.1. By 37.2.(a), for all $\overline{a}$, $\overline{x} \in |A|^I$

$$[\![f\overline{a} = \overline{b}]\!] \wedge [\![\overline{a} = \overline{x}]\!] = [\![f\overline{x} = \overline{b}]\!] \wedge [\![\overline{a} = \overline{x}]\!]. \qquad (*)$$

The reader has probably noticed that in case $I = \{1\}$, we have just shown that certain singletons are in $\mathfrak{K}_1(A)$. Similarly, one verifies that for $\overline{a} \in |A|^I$,

$$k_{\overline{a}}(\overline{b}) = [\![f\overline{a} = \overline{b}]\!]$$

is a I-characteristic map on B. Moreover, by 37.4.(c),

$$\Big(\bigvee\nolimits_{a\in|A|} k_{\overline{a}}\Big)(\overline{b}) = \bigvee\nolimits_{\overline{a}\in|A|^I} [\![f\overline{a} = \overline{b}]\!] \qquad (**)$$

is in $\mathfrak{K}_I B$, since it is a join of elements in $\mathfrak{K}_I B$. □

Our next result describes the basic properties of extension and restriction of characteristic maps; its item (c) is a *very important* property of characteristic maps on finite powers, guaranteeing that if D is dense in A, a characteristic map on D has a *unique* extension to a characteristic map on A.

THEOREM 37.8. *Let $D \subseteq A$ be Ω-sets and I be a set.*

a) For $h \in \mathfrak{K}_I D$, define $h^e : |A|^I \longrightarrow \Omega$ by

$$h^e(\overline{a}) = \bigvee\nolimits_{\overline{d}\in D^I} [\![\overline{a} = \overline{d}]\!] \wedge h(\overline{d}).$$

Then, $h^e \in \mathfrak{K}_I A$ and $h^e_{|D} = h$ [3].

b) The map $k \in \mathfrak{K}_I A \longmapsto k_{|D} \in \mathfrak{K}_I D$ is an open (8.1.(c)) *surjection.*

c) If I is finite and D is dense in A, then the restriction morphism in (b) *is an* isomorphism, *with inverse $h \mapsto h^e$.*

PROOF. a) Clearly, h^e verifies [ch 1] in 37.1. For $\overline{a}$, $\overline{b} \in |A|^I$,

$$\begin{aligned} h^e(\overline{a}) \wedge [\![\overline{a} = \overline{b}]\!] &= [\![\overline{a} = \overline{b}]\!] \wedge \bigvee\nolimits_{\overline{d}\in D^I} [\![\overline{a} = \overline{d}]\!] \wedge h(\overline{d}) \\ &= \bigvee\nolimits_{\overline{d}\in D^I} [\![\overline{a} = \overline{b}]\!] \wedge [\![\overline{a} = \overline{d}]\!] \wedge h(\overline{d}) \\ &\leq \bigvee\nolimits_{\overline{d}\in D^I} [\![\overline{b} = \overline{d}]\!] \wedge h(\overline{d}) = h^e(\overline{b}), \end{aligned}$$

showing that $h^e \in \mathfrak{K}_I A$. Now, if $\overline{t} \in |D|^I$, we get

$$\begin{aligned} h^e(\overline{t}) &= \bigvee\nolimits_{\overline{d}\in D^I} [\![\overline{t} = \overline{d}]\!] \wedge h(\overline{d}) = \bigvee\nolimits_{\overline{d}\in D^I} [\![\overline{t} = \overline{d}]\!] \wedge h(\overline{t}) \\ &= h(\overline{t}) \wedge \bigvee\nolimits_{\overline{d}\in D^I} [\![\overline{t} = \overline{d}]\!] = h(\overline{t}) \wedge E\overline{t} = h(\overline{t}), \end{aligned}$$

and $(h^e)_{|D} = h$, as desired.

b) By (a), restriction is a surjection from $\mathfrak{K}_I A$ onto $\mathfrak{K}_I D$; it is straightforward that it preserves all meets and joins, as well as implication, being therefore open.

[3] Restriction is defined in 37.2.(b).

c) We may assume that $I = \underline{n} = \{1, \ldots, n\}$. It is enough to check that $h_{|D} = k_{|D}$ implies $h = k$. Since D is dense in A, for $\overline{a} \in |A|^n$ we have $Ea_j = \bigvee_{d_j \in |D|} [\![a_j = d_j]\!]$, $1 \leq j \leq n$. Thus, distributivity of joins over finite meets (8.4) yields

$$E\overline{a} = \bigwedge_{j=1}^n Ea_j = \bigwedge_{j=1}^n \bigvee_{d_j \in |D|} [\![a_j = d_j]\!] = \bigvee_{\overline{d} \in D^n} [\![\overline{a} = \overline{d}]\!].$$

Hence,
$$\begin{aligned} h(\overline{a}) &= h(\overline{a}) \wedge E\overline{a} = h(\overline{a}) \wedge \bigvee_{\overline{d} \in D^n} [\![\overline{a} = \overline{d}]\!] \\ &= \bigvee_{\overline{d} \in D^n} h(\overline{a}) \wedge [\![\overline{a} = \overline{d}]\!] = \bigvee_{\overline{d} \in D^n} h(\overline{d}) \wedge [\![\overline{a} = \overline{d}]\!] \\ &= \bigvee_{\overline{d} \in D^n} k(\overline{d}) \wedge [\![\overline{a} = \overline{d}]\!] = \bigvee_{\overline{d} \in D^n} k(\overline{a}) \wedge [\![\overline{a} = \overline{d}]\!] \\ &= k(\overline{a}). \end{aligned}$$
□

Corollary 37.9. *If A is a Ω-set and $n \geq 0$ is an integer, there is a natural isomorphism between $\mathfrak{K}_n A$ and $\mathfrak{K}_n cA$, where cA is the completion of A* (27.9).

Proof. Notation as in Theorem 27.9, we may identify A with its image by c in cA. Then A is dense in cA and the conclusion follows from 37.8.(c). □

When A is a Ω-presheaf, characteristic maps on A have important additional properties. First, we introduce

37.10. **Notation.** Let A be a Ω-set and I be a set.

a) For $\overline{x} \in |A|^I$ and $\overline{u} \in \Omega^I$, set

$$\overline{x}_{|\overline{u}} = \text{the } I\text{-sequence in } |A| \text{ whose } i^{th}\text{-coordinate is } {x_i}_{|u_i}.$$

b) For $\overline{a} \in |A|^I$, $x \in |A|$ and $k \in I$, the **substitution of x at the k^{th} coordinate of $\overline{a}$** is the I-sequence in $|A|$ given by

$$\overline{a} \ulcorner x \mid k \urcorner (i) = \begin{cases} a_i & \text{if } i \neq k \\ x & \text{if } i = k. \end{cases}$$

Clearly, this definition applies to *any* product of sets. □

Lemma 37.11. *Let A be a Ω-set and $I \neq \emptyset$ be a set.*

a) For $\overline{a}, \overline{c} \in |A|^I$, $k \in I$ and $x, y \in |A|$,

(1) $[\![\overline{a} \ulcorner x \mid k \urcorner = \overline{c} \ulcorner y \mid k \urcorner]\!] = [\![x = y]\!] \wedge \bigwedge_{i \neq k} [\![a_i = c_i]\!]$.

(2) $[\![\overline{a} \ulcorner x \mid k \urcorner = \overline{a} \ulcorner y \mid k \urcorner]\!] = [\![x = y]\!] \wedge E\overline{a} \ulcorner x \mid k \urcorner$
$= [\![x = y]\!] \wedge E\overline{a} \ulcorner y \mid k \urcorner$.

(3) $E\overline{a} \wedge E\overline{a} \ulcorner x \mid k \urcorner = E\overline{a} \wedge Ex$.

b) If $h \in \mathfrak{K}_I A$, then

(1) $h(\overline{a} \ulcorner x \mid k \urcorner) \wedge [\![x = y]\!] \leq h(\overline{a} \ulcorner y \mid k \urcorner)$.

(2) $h(\overline{a} \ulcorner x \mid k \urcorner) \wedge [\![x = y]\!] = h(\overline{a} \ulcorner y \mid k \urcorner) \wedge [\![x = y]\!]$.

Proof. a) (1) and (3) are straightforward; for (2), we have

$$\begin{aligned} [\![\overline{a} \ulcorner x \mid k \urcorner = \overline{a} \ulcorner y \mid k \urcorner]\!] &= [\![x = y]\!] \wedge \bigwedge_{i \neq k} Ea_i \\ &= [\![x = y]\!] \wedge Ex \wedge \bigwedge_{i \neq k} Ea_i = [\![x = y]\!] \wedge E\overline{a} \ulcorner x \mid k \urcorner. \end{aligned}$$

A similar reasoning proves the other equality.

b) (2) follows from (1), as in 37.2.(a). For (1), (a).(2) yields

$$\begin{array}{rcl} h(\overline{a}\ulcorner y \mid k\urcorner) & \geq & h(\overline{a}\ulcorner x \mid k\urcorner) \wedge [\![\overline{a}\ulcorner x \mid k\urcorner = \overline{a}\ulcorner y \mid k\urcorner]\!] \\ & = & h(\overline{a}\ulcorner x \mid k\urcorner) \wedge [\![x = y]\!] \wedge E\overline{a}\ulcorner x \mid k\urcorner \\ & = & h(\overline{a}\ulcorner x \mid k\urcorner) \wedge [\![x = y]\!]. \end{array}$$

□

PROPOSITION 37.12. *Let A be a Ω-presheaf, I be a set and $k,\ h \in \mathfrak{K}_I A$.*

a) For $\overline{x} \in |A|^I$ and $\overline{u} \in \Omega^I$, $h(\overline{x}_{|\overline{u}}) = h(\overline{x}) \wedge \bigwedge_{i\in I} u_i$. In particular, $h(\overline{x}_{|E\overline{x}}) = h(\overline{x}_{|h(\overline{x})}) = h(\overline{x})$.

b) $h \leq k$ iff $\forall\, \overline{a} \in |A|^I$, $h(\overline{a}) = E\overline{a} \Rightarrow k(\overline{a}) = E\overline{a}$.

c) Assume that $I \neq \emptyset$. If $S \cup \{x\} \subseteq |A|$ and $p \in \Omega$ satisfy $p = \bigvee_{s\in S} [\![x = s]\!]$, then for all $\overline{y} \in |A|^I$ and $k \in I$

$$p \wedge h(\overline{y}\ulcorner x \mid k\urcorner) = \bigvee_{s\in S} h(\overline{y}\ulcorner s \mid k\urcorner) \wedge [\![s = x]\!].$$

In particular, if $p = \bigvee_{\lambda\in\Lambda} p_\lambda$, then $h(\overline{y}\ulcorner x_{|p} \mid k\urcorner) = \bigvee_{\lambda\in\Lambda} h(\overline{y}\ulcorner x_{|p_\lambda} \mid k\urcorner)$.

d) If D is a dense subset of A^I, then $h_{|D} = k_{|D} \Rightarrow h = k$ [4].

PROOF. a) For $\overline{x} \in |A|^I$, $\overline{u} \in \Omega^I$, [ch 2′] in 37.2.(a) yields

$$h(\overline{x}_{|\overline{u}}) \wedge [\![\overline{x}_{|\overline{u}} = \overline{x}]\!] = h(\overline{x}) \wedge [\![\overline{x}_{|\overline{u}} = \overline{x}]\!]. \tag{1}$$

Since $[\![\overline{x}_{|\overline{u}} = \overline{x}]\!] = \bigwedge_{i\in I} [\![x_{i_{|u_i}} = x_i]\!] = \bigwedge_{i\in I} Ex_i \wedge u_i = E\overline{x} \wedge \bigwedge_{i\in I} u_i = E\overline{x}_{|\overline{u}}$,

it follows from (1) that $h(\overline{x}_{|\overline{u}}) = h(\overline{x}_{|\overline{u}}) \wedge E\overline{x}_{|\overline{u}} = h(\overline{x}) \wedge E\overline{x} \wedge \bigwedge_{i\in I} u_i = h(\overline{x}) \wedge \bigwedge_{i\in I} u_i$,

as desired. The remaining assertions are clear.

b) Since the value of any characteristic map is less than or equal to the extent of its argument, it is clear that (1) $\Rightarrow$ (2). For the converse, let $\overline{x} \in |A|^I$; then

$$E\overline{x}_{|h(\overline{x})} = h(\overline{x})$$

and item (a) entails $h(\overline{x}_{|h(\overline{x})}) = h(\overline{x}) = E\overline{x}_{|h(\overline{x})}$. Hence, (2) and item (a) yield $h(\overline{x}) = E\overline{x}_{|h(\overline{x})} = k(\overline{x}_{|h(\overline{x})}) = h(\overline{x}) \wedge k(\overline{x})$, verifying that $h(\overline{x}) \leq k(\overline{x})$.

c) By 37.11.(b), if $s \in S$, then $h(\overline{y}\ulcorner k \mid x\urcorner) \wedge [\![x = s]\!] = h(\overline{y}\ulcorner k \mid s\urcorner) \wedge [\![x = s]\!]$.

Thus, $$\begin{array}{rcl} p \wedge h(\overline{a}\ulcorner x \mid k\urcorner) & = & h(\overline{a}\ulcorner x \mid k\urcorner) \wedge \bigvee_{s\in S} [\![x = s]\!] \\ & = & \bigvee_{s\in S} h(\overline{a}\ulcorner x \mid k\urcorner) \wedge [\![x = s]\!] \\ & = & \bigvee_{s\in S} h(\overline{a}\ulcorner s \mid k\urcorner) \wedge [\![x = s]\!]. \end{array}$$

The remaining statement is clear.

d) We start with

Fact. *If D is dense in A^I and $\overline{a} \in |A|^I$, then $E\overline{a} = \bigvee_{\overline{d}\in D} [\![\overline{a} = \overline{d}]\!]$.*

Proof. Since $\overline{a}_{|E\overline{a}} \in A^I$ and its extent is $E\overline{a}$, we obtain

[4] This is stronger than 37.8.(c).

$$E\overline{a} \;=\; \bigvee_{\overline{d}\in D} [\![\overline{a}_{|E\overline{a}} = \overline{d}]\!] \;=\; \bigvee_{\overline{d}\in D} [\![\overline{a} = \overline{d}]\!],$$

as asserted.

For $\overline{a} \in |A|^I$, the Fact above yields

$$\begin{aligned} k(\overline{a}) &= k(\overline{a}) \wedge E\overline{a} \;=\; \bigvee_{\overline{d}\in D} k(\overline{a}) \wedge [\![\overline{a} = \overline{d}]\!] \\ &= \bigvee_{\overline{d}\in D} k(\overline{d}) \wedge [\![\overline{a} = \overline{d}]\!] \;=\; \bigvee_{\overline{d}\in D} h(\overline{d}) \wedge [\![\overline{a} = \overline{d}]\!] \\ &= \bigvee_{\overline{d}\in D} h(\overline{a}) \wedge [\![\overline{a} = \overline{d}]\!] \;=\; h(\overline{a}), \end{aligned}$$

ending the proof. □

DEFINITION 37.13. *Let A be a Ω-set and $I \neq \emptyset$ be a set. For $S \subseteq |A|^I$, define*

$$k_S : |A|^I \longrightarrow \Omega \quad by \quad k_S(\overline{a}) \;=\; \bigvee_{\overline{s}\in S} [\![\overline{a} = \overline{s}]\!],$$

called the ***I*-characteristic map of *S* in *A***.

PROPOSITION 37.14. *Let A be a Ω-set and $I \neq \emptyset$ be a set. Let $k \in \mathfrak{K}_I A$ and $S,\, T \subseteq |A|^I$.*

a) $k_S \in \mathfrak{K}_I A$ and for all $\overline{s} \in S$, $k_S(\overline{s}) = E\overline{s}$. Moreover, $Ek_S = ES$.

b) $k_S \leq k$ iff $\forall\, \overline{s} \in S$, $k(\overline{s}) = E\overline{s}$.

c) $S \subseteq T \;\Rightarrow\; k_S \leq k_T$.

d) If $S \subseteq |A^I|$, then

(1) $\forall\, \overline{x} \in |A^I| \left(E\overline{x} \;=\; k_S(\overline{x}) \quad iff \quad \overline{x} \in \overline{S}\right)$;

(2) $k_S \;=\; k_{\overline{S}}$.

e) The following are equivalent, for $S,\, T \subseteq |A^I|$:

(1) $\forall\, \overline{s} \in S$, $k_T(\overline{s}) = E\overline{s}$;	(2) $k_S \leq k_T$;
(3) $S \subseteq \overline{T}$;	(4) $\overline{S} \subseteq \overline{T}$.

PROOF. Item (a) is straightforward. For (b), [ch 1] in 37.1 implies that the hypothesis $k_S \leq k$ forces the value of k at $\overline{s} \in S$ to be $E\overline{s}$. Conversely, if this condition holds and $\overline{a} \in |A|^I$, then

$$\begin{aligned} k_S(\overline{a}) &= \bigvee_{\overline{s}\in S} [\![\overline{a} = \overline{s}]\!] \;=\; \bigvee_{\overline{s}\in S} [\![\overline{a} = \overline{s}]\!] \wedge E\overline{s} \\ &= \bigvee_{\overline{s}\in S} [\![\overline{a} = \overline{s}]\!] \wedge k(\overline{s}) \;=\; \bigvee_{\overline{s}\in S} [\![\overline{a} = \overline{s}]\!] \wedge k(\overline{a}) \\ &= k(\overline{a}) \wedge \bigvee_{\overline{s}\in S} [\![\overline{a} = \overline{s}]\!] \;=\; k(\overline{a}) \wedge k_S(\overline{a}), \end{aligned}$$

and $k_S \leq k$. Item (c) follows immediately from (a) and (b).

d) (1) is immediate from the definition of closure (36.1). For (2), it suffices (by (c)) to check that $k_{\overline{S}} \leq k_S$, which by (b) is equivalent to

$$\forall\, \overline{y} \in \overline{S},\;\; k_S(\overline{y}) \;=\; \bigvee_{\overline{s}\in S} [\![\overline{y} = \overline{s}]\!] \;=\; E\overline{y}.$$

This clear, because S is dense in $\overline{S}$. Item (e) follows from the preceding ones. □

For Ω-presheaves, Lemma 36.9 and Proposition 37.14.(d).(2) yield

COROLLARY 37.15. *If A is a Ω-presheaf and $I \neq \emptyset$ is a set, then for all $S \subseteq |A|^I$, $k_S \;=\; k_{pS}$.* [5] □

[5]Recall that $pS = \{\overline{s}_{|E\overline{s}} \in |A^I| : \overline{s} \in S\}$.

We now state the fundamental

THEOREM 37.16. (The Representation Theorem) *Let A be a Ω-set and let $I \neq \emptyset$ be a set. Consider the maps $S_\star$ and $k_\star$ defined by :*

$$\begin{cases} S \in \mathfrak{P}_\top(A^I) & \longmapsto & k_S \in \mathfrak{K}_I A \\ k \in \mathfrak{K}_I A & \longmapsto & S_k \;=\; \{\overline{x} \in |A^I| : k(\overline{x}) = E\overline{x}\}. \end{cases}$$

Then,

a) For all $k \in \mathfrak{K}_I A$, S_k is a closed subset of A^I.

b) The map $k_\star$ is an injective $\bigvee$-morphism. Moreover,

(1) $\langle k_\star, S_\star \rangle$ is an adjoint pair between $\mathfrak{K}_I A$ and $\mathfrak{P}_\top(A)$;

(2) $S_\star$ is a surjective $\bigwedge$-morphism;

(3) For all $\langle S, k \rangle \in \mathfrak{P}_\top(A) \times \mathfrak{K}_I A$, $S_{k_S} = S$ and $k_{S_k} \leq k$.

*c) If A is a Ω-*presheaf*, the maps $S_\star$ and $k_\star$ are inverse* isomorphisms.

PROOF. a) Fix $\overline{t} \in |A^I|$, with $E\overline{t} = \bigvee_{\overline{s} \in S_k} [\![\overline{t} = \overline{s}]\!]$. If $\overline{s} \in |S_k|$, then

$$k(\overline{t}) \wedge [\![\overline{t} = \overline{s}]\!] \;=\; k(\overline{s}) \wedge [\![\overline{t} = \overline{s}]\!] \;=\; E\overline{s} \wedge [\![\overline{t} = \overline{s}]\!] \;=\; [\![\overline{t} = \overline{s}]\!].$$

Thus, $k(\overline{t}) \geq [\![\overline{t} = \overline{s}]\!]$, $\forall\, \overline{s} \in |S_k|$, and so $k(\overline{t}) \geq E\overline{t}$. Hence, $k(\overline{t}) = E\overline{t}$, and S_k is closed in A^I.

b) By 37.14.(d).(1), $S \longmapsto k_S$ is injective. To verify the preservation of joins, let S_λ, $\lambda \in \Lambda$, be closed subsets of A and set $T = \bigvee_{\lambda \in \Lambda} S_\lambda$. It is immediate from 37.14.(c) that $\bigvee_{\lambda \in \Lambda} k_\lambda \leq k_T$. Suppose $k \in \mathfrak{K}_I A$ satisfies $k_\lambda \leq k$, for all $\lambda \in \Lambda$. By 37.14.(b), to prove that $k_T \leq k$, it suffices to check that if $\overline{t} \in T$, then $k(\overline{t}) = E\overline{t}$. Recalling formula ($\bigvee$) in page 396, we have

$$E\overline{t} \;=\; \bigvee_{\overline{s} \in \bigcup S_\lambda} [\![\overline{t} = \overline{s}]\!]. \tag{+}$$

Hence, if $\overline{s} \in S_\lambda$, since $k_\lambda(\overline{s}) = E\overline{s}$, it follows that

$$k(\overline{t}) \;\geq\; k(\overline{s}) \wedge [\![\overline{t} = \overline{s}]\!] \;\geq\; k_\lambda(\overline{s}) \wedge [\![\overline{t} = \overline{s}]\!] \;=\; [\![\overline{t} = \overline{s}]\!],$$

which, together with (+), entails $k(\overline{t}) \geq E\overline{t}$. Hence, $k_T \;=\; \bigvee_{\lambda \in \Lambda} k_\lambda$, as desired. To finish the proof of (b), it suffices to see that $\langle k_\star, S_\star \rangle$ is an adjoint pair : the other assertions are then immediately forthcoming from Theorem 7.8 and Corollary 7.9. But adjointness means that for $T \in \mathfrak{P}_\top(A)$ and $k \in \mathfrak{K}_I A$

$$k_T \;\leq\; k \quad \text{iff} \quad T \;\leq\; S_k,$$

exactly the content of 37.14.(b).

c) We shall verify that for $k \in \mathfrak{K}_I A$, $k_{S_k} = k$. By (b).(3), it is enough to see that $k \leq k_{S_k}$. Observe that (b) and 36.4.(b) imply S_k is a *closed subpresheaf* of A^I.

If $\overline{x} \in |A|^I$, then, $k(\overline{x}) \leq E\overline{x}$ and 37.14.(a) yield :

$$\overline{x}_{|k(\overline{x})} \in |A^I| \quad \text{and} \quad k(\overline{x}) \;=\; k(\overline{x}_{|k(\overline{x})}) \;=\; E\overline{x}_{|k(\overline{x})}.$$

Hence, $\overline{x}_{|k(\overline{x})} \in S_k$ and so $k(\overline{x}) \;=\; k(\overline{x}_{|k(\overline{x})}) \;=\; k_{S_k}(\overline{x}_{|k(\overline{x})}) \;=\; k(\overline{x}) \wedge k_{S_k}(\overline{x})$. Thus, $k \;\leq\; k_{S_k}$, and the adjointness in (b).(1) guarantees that $k_\star$ is an isomorphism, with inverse $S_\star$. $\square$

REMARK 37.17. If $I = \emptyset$, Example 37.5 and Exercise 36.13 imply that $k \longmapsto Ek$ is a natural isomorphism between $\mathfrak{K}_0 A$ and $(EA)^{\leftarrow}$, for all Ω-sets A. This will be understood as the natural extension of the representation Theorem 37.16 to the empty power of a Ω-set. $\square$

Theorem 37.16 allows the identification of closed subsets of A^I by I-characteristic maps. Further, if A is a presheaf, this identification is an isomorphism, through which operations on subpresheaves are reflected in the corresponding operations on characteristic maps. This will be the way we shall henceforth deal with I-ary relations on Ω-sets and presheaves. We have had the opportunity of using, in Chapter 27, a special class of characteristic maps, namely *singletons*, to construct the completion of an Ω-set. The root that idea is described in Exercise 37.26.

REMARK 37.18. For Ω-sets in general, the map $k_\star$ of 37.16 is not surjective and does not preserve even *finite* meets. To see this, let A be a B-set in Example 36.7 (B is a cBa). The map

$$k(a) = p \wedge Ea,$$

is in $\mathfrak{K}_1 A$, in fact, $k = p \wedge \top_1$. Since $Ek = p \notin \{\bot, \top\}$, 37.14.(a) guarantees that k cannot be equal to k_S, for any $S \subseteq |A|$, because all such subsets have extent either equal to $\top$ or to $\bot$.

For $x \in A(\top)$, let k_x be the characteristic map of $\{x, *\}$. The reader can check that

$$k_a \wedge k_c \quad \neq \quad k_{\{*\}} = \bot_1,$$

although $\{a, *\} \cap \{c, *\} = \{*\}$ [6]. In fact, $k_a \wedge k_c$ and $k_b \wedge k_c$, the first of extent p and the latter of extent $\neg\, p$, are not characteristic maps of any subset of $|A|$.

This example shows that, in the case of Ω-sets, treating relations via their characteristic maps has shortcomings. To begin with, an element of $\mathfrak{K}_1(A^I)$ might not correspond to a I-relation on A. Moreover, the correspondence $S \longmapsto k_S$ does not preserve intersections. Nevertheless, there are important situations, as shown by 37.26 and the very definition of Ω-set $-$ $[\![\cdot = \cdot]\!]$ is an element of $\mathfrak{K}_2 A$, not of $\mathfrak{K}_1(A^2)$ $-$, wherein characteristic maps on a Ω-set provide a means to arrive at significant constructions. In this respect, see also 37.19, below.

It would be interesting to determine, for Ω-sets in general, conditions for an element of $\mathfrak{K}_I A$ to belong to the image of $k_\star$. $\square$

LEMMA 37.19. *Let A be a Ω-set and $n \geq 2$ be an integer. If Δ_n is the diagonal of A^n, then for all $\overline{x} \in |A|^n$,* $[\![\, \Delta_n(\overline{x}) \,]\!] \;=\; \bigwedge_{i=1}^{n-1} [\![x_i = x_{i+1}]\!]$.

PROOF. Fix $\overline{x} \in |A|^n$; the exchange rule in 25.35.(b) yields, for $a \in |A|$, $[\![a = x_1]\!] \wedge [\![a = x_2]\!] \;=\; [\![x_1 = x_2]\!] \wedge [\![a = x_2]\!]$. Thus,

$$\bigwedge_{i=1}^{n} [\![a = x_i]\!] \;=\; [\![x_1 = x_2]\!] \wedge \bigwedge_{i=2}^{n} [\![a = x_i]\!],$$

and induction then implies

$$(\sharp) \qquad \bigwedge_{i=1}^{n} [\![a = x_i]\!] \;=\; [\![a = x_n]\!] \wedge \bigwedge_{i=1}^{n-1} [\![x_i = x_{i+1}]\!].$$

[6] This is to be expected : $\mathfrak{K}_1(A)$ is a frame, while $\mathfrak{P}_\top(A)$ is not.

Hence, if $\widehat{a}$ is the constant n-sequence with entries equal to a,

$$\begin{aligned}[\![\,\Delta_n(\overline{x})\,]\!] &= \bigvee_{a\in|A|} [\![\overline{x} = \widehat{a}]\!] = \bigvee_{a\in|A|} \bigwedge_{i=1}^{n} [\![a = x_i]\!] \\ &= \bigwedge_{i=1}^{n-1} [\![x_i = x_{i+1}]\!] \wedge \bigvee_{a\in|A|} [\![a = x_n]\!] \\ &= \bigwedge_{i=1}^{n-1} [\![x_i = x_{i+1}]\!] \wedge Ex_n = \bigwedge_{i=1}^{n-1} [\![x_i = x_{i+1}]\!],\end{aligned}$$

as desired. □

Whenever convenient, our notation for the characteristic map of $R \subseteq |A|^I$ will be $[\![\,R(\cdot)\,]\!] : |A|^I \longrightarrow \Omega$, instead k_R. Thus, for $\overline{x} \in |A|^I$,

$$[\![\,R(\overline{x})\,]\!] = \bigvee \{[\![\overline{x} = \overline{s}]\!] : \overline{s} \in |R|\}.$$

If it is necessary to register that this characteristic map is being computed relative to A, we shall write $[\![\,R(\overline{x})\,]\!]_A$.

In this notation, Proposition 37.14, Corollary 37.15 and Theorem 37.16 yield

COROLLARY 37.20. *If A is a Ω-presheaf and I is a set, the following are equivalent, where $S \subseteq |A|^I$ and $R \in \mathfrak{P}_\top(A^I)$:* [7]

(1) $pS \subseteq |R|$; (2) $\overline{pS} \subseteq R$;

(3) $[\![\,S(\cdot)\,]\!] \le [\![\,R(\cdot)\,]\!]$; (4) *For all* $\overline{s} \in S$, $[\![\,R(\overline{s})\,]\!] = E\overline{s}$. □

Exercises

37.21. Let A be a Ω-presheaf and I be a set. Let $\overline{x}$, $\overline{y}$, $\overline{z}$ be elements in $|A|^I$.

a) $[\![\overline{x} = \overline{y}]\!] \wedge [\![\overline{y} = \overline{z}]\!] \le [\![\overline{x} = \overline{z}]\!]$.

b) If A is extensional, then $\overline{x}_{|[\![\overline{x}=\overline{y}]\!]} = \overline{y}_{|[\![\overline{x}=\overline{y}]\!]}$.

c) In the presheaf A^I, extensionality is equivalent to

$$\forall\, \overline{x}, \overline{y} \in |A|^I,\quad E\overline{x} = E\overline{y} = [\![\overline{x} = \overline{y}]\!] \quad\Rightarrow\quad \overline{x}_{|E\overline{x}} = \overline{y}_{|E\overline{y}}.$$

d) A family $\overline{x}_\lambda$ in $|A^I|$, $\lambda \in \Lambda$, is compatible iff for all λ, $\mu \in \Lambda$,

$$[\![\overline{x}_\lambda = \overline{x}_\mu]\!] = E\overline{x}_\lambda \wedge E\overline{x}_\mu.$$

e) If $S \subseteq |A^I|$, then the domain of closure of S in A^I is given by

$$|\overline{S}| = \{\overline{t} \in |A^I| : E\overline{t} = \bigvee_{\overline{s}\in S} [\![\overline{t} = \overline{s}]\!].$$

f) A^I is a sheaf iff for all compatible families $\overline{x}_\lambda$ in $|A^I|$, $\lambda \in \Lambda$, there is $\overline{t} \in |A^I|$ such that

$$(i)\quad E\overline{t} = \bigvee_{\lambda\in\Lambda} E\overline{x}_\lambda \qquad \text{and} \qquad (ii)\quad E\overline{x}_\lambda = [\![\overline{t} = \overline{x}_\lambda]\!].$$

g) The above laws, with the appropriate modifications, hold in **any** product of presheaves over Ω. □

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[7] $pS = \{\overline{s}_{|E\overline{s}} \in |A^I| : \overline{s} \in S\}$; see 36.9.

37.22. Let A be a Ω-set and I a set.

a) Equality is a 2-characteristic map on A.

b) The map $\perp_I(\overline{x}) = \perp$ is a I-characteristic map on A.

c) The map $\top_I(\overline{x}) = E\overline{x}$ is a I-characteristic map on A.

37.23. Let A be a Ω-set and I be a set. The following are equivalent, for a map $h : |A|^I \longrightarrow \Omega$:

(1) h is a I-characteristic map on A;

(2) h verifies [ch 1] in 37.1 and for all $\overline{x}$, $\overline{y} \in |A|^I$ [8]

$$h(\overline{x}) \wedge [\overline{x} \equiv \overline{y}] \quad \leq \quad h(\overline{y}). \qquad \Box$$

37.24. (Fixing Coordinates) Let A be a Ω-set and $J \subseteq I$ be sets. Set $K = I - J$. Define

$$\langle \cdot\, ;\, \cdot \rangle : |A|^J \times |A|^K \longrightarrow |A|^I, \text{ by } \langle \overline{a}; \overline{c} \rangle(i) = \begin{cases} a_i & \text{if } i \in J; \\ c_i & \text{if } i \in K. \end{cases}$$

a) If $\overline{a} \in |A|^I$ and $\overline{b}, \overline{c} \in |A|^K$, then

(1) $E\langle \overline{a}; \overline{b} \rangle \;=\; E\overline{a} \wedge E\overline{b}$.

(2) $[\![\langle \overline{a}; \overline{b} \rangle = \langle \overline{a}; \overline{c} \rangle]\!] \;=\; E\overline{a} \wedge [\![\overline{b} = \overline{c}]\!]$.

b) For $\overline{a} \in |A|^I$ and $h \in \mathfrak{K}_I(A)$, define $k : |A|^K \longrightarrow \Omega$ by $k(\overline{c}) \;=\; h(\langle \overline{a}; \overline{c} \rangle)$. Then, $k \in \mathfrak{K}_K A$, obtained by *fixing the coordinates in* J. $\Box$

37.25. Generalizing 37.4.(f), if A is a Ω-set and $I \xrightarrow{\sigma} J$ is a bijection, the map

$$h \in \mathfrak{K}_I A \;\longmapsto\; h^\sigma \in \mathfrak{K}_J A,$$

where $h^\sigma(\overline{a}) \;=\; h(\overline{a}^{\,\sigma})$, is a frame-isomorphism [9]. $\Box$

37.26. If A is a Ω-set, there is a natural bijective correspondence between singletons in A and the closure of compatible sets of sections in A. $\Box$

37.27. Let A be a Ω-presheaf and n, $m \geq 0$ be integers. With notation as in 36.20, if $S \in \mathfrak{P}_\top(A^n)$ and $T \in \mathfrak{P}_\top(A^m)$, then for all $\overline{x} \in |A|^{n+m}$

$$[\![\, S \times T(\overline{x}) \,]\!] \;=\; [\![\, S(x_1, \ldots, x_n) \,]\!] \wedge [\![\, T(x_{n+1}, \ldots, x_{n+m}) \,]\!]. \qquad \Box$$

[8] $[\cdot \equiv \cdot]$ is strict equality in A, discussed in Chapter 28.

[9] Hence, $\mathfrak{K}_I A$ is independent, up to isomorphism, of the "presentation" of I.

CHAPTER 38

Exterior Products and Q-morphisms

Recall that the canonical Ω-set structure on a product is described in 25.12. Lemma 25.14 suggests the following construction.

38.1. **Exterior Product** Let A_i, $i \in I$, be Ω-sets, with $I \neq \emptyset$. Define a Ω-set, $\bigotimes_{i\in I} A_i$, by the following prescriptions :

$*$ $|\bigotimes_{i\in I} A_i| \;=\; \prod_{i\in I} |A_i|$;

$*$ For $\overline{a}$, $\overline{c} \in \prod_{i\in I} |A_i|$, $[\![\overline{a} = \overline{c}]\!] \;=\; \bigwedge_{i\in I} [\![a_i = b_i]\!]$.

By 25.14.(a).(2), $\bigotimes_{i\in I} A_i$ is a Ω-set, called the **exterior product** of the A_i.

If L is a semilattice, this construction makes sense for finite families of L-sets; and arbitrary ones, when L is complete. Clearly, the canonical product, $\prod_{i\in I} A_i$, is a sub-Ω-set of $\bigotimes_{i\in I} A_i$.

The exterior product of I copies of a Ω-set A, written $\bigotimes^I A$, is the **exterior power** of A by I. *When* $I = \emptyset$, we set

$$\bigotimes^{\emptyset} A \;=_{def}\; \bigotimes^{0} A \;=_{def}\; \mathbf{1}_{|EA},$$

where $\mathbf{1}$ is the final object in $\boldsymbol{\Omega set}$, as in 25.5 [1]. $\square$

REMARK 38.2. Let A_i, $i \in I$, be Ω-sets, with $I \neq \emptyset$. Care must be exercised in treating exterior products, as shown by the following comments :

a) Even if all A_i are extensional, $\bigotimes_{i\in I} A_i$ **will not** be extensional. As an example, let a, b be compatible sections of distinct extent in a Ω-set A. Then, in $A \bigotimes A$

$$E\langle a, b\rangle \;=\; E\langle b, a\rangle \;=\; Ea \wedge Eb \;=\; [\![a = b]\!] \;=\; [\![\langle a, b\rangle = \langle b, a\rangle]\!],$$

but $\langle a, b\rangle \neq \langle b, a\rangle$.

b) The canonical projections π_i no longer are morphisms, because although they verify [mor 2] in 25.10, [mor 1] is, in general, violated.

c) If each A_i is a presheaf, then we may define restriction and extent in $\bigotimes_{i\in I} A_i$ using 26.11, that is, for $\overline{a} \in \prod_{i\in I} |A_i|$ and $p \in \Omega$,

$$E\overline{a} = \bigwedge_{i\in I} Ea_i \quad\text{and}\quad \overline{a}_{|p} \;=\; \langle {a_i}_{|p}\rangle.$$

However, $\bigotimes_{i\in I} A_i$ **is not** a Ω-presheaf, since, in general, $\overline{a}_{|E\overline{a}} \neq \overline{a}$, violating [rest 1] in 26.1. Nevertheless, [rest 2] and [rest 3] are satisfied, that is,

$$E\overline{a}_{|p} = p \wedge E\overline{a}; \qquad\qquad (\overline{a}_{|p})_{|q} \;=\; \overline{a}_{|p\wedge q},$$

[1] But the empty product and power are still equal to $\mathbf{1}$.

as well as the important relation in 26.8.(b) :

$$[\![\overline{a}_{|p} = \overline{c}_{|q}]\!] \;=\; p \wedge q \wedge [\![\overline{a} = \overline{c}]\!].$$

In the case of presheaves, if each component is extensional, then

$$E\overline{a} \;=\; E\overline{c} \;=\; [\![\overline{a} = \overline{c}]\!] \quad\Rightarrow\quad \overline{a}_{|E\overline{a}} \;=\; \overline{c}_{|E\overline{c}},$$

simply because $\overline{a}_{|E\overline{a}} \in |\prod_{i\in I} A_i|$.

Even with all these shortcomings, the exterior product will be useful in unifying the presentation and proof of some of our basic results. □

Recall (25.1) that the *support* of a Ω-set A is $EA \;=\; \bigvee_{a\in|A|} Ea$. From 25.15 and 37.3 we get

Corollary 38.3. *Let A be a Ω-set.*

a) If $I \neq \emptyset$ is a set, then $E\bigotimes^I A \;=\; EA^I \;=\; EA$.

b) $E\bigotimes^0 A \;=\; E\mathbf{1}_{|EA} \;=\; EA$. □

Proposition 38.4. *Let $K \neq \emptyset$ be a set, A_k, $k \in K$, be Ω-presheaves and B be a Ω-set.*

a) The map

$$\rho : |\bigotimes_{k\in K} A_K| \longrightarrow |\prod_{k\in K} A_k|, \quad \rho(\overline{a}) \;=\; \overline{a}_{|E\overline{a}}$$

is a Ω-set morphism, satisfying the following properties :

(1) *ρ is surjective and a regular monic;*

(2) *If B is a presheaf and $p \in \Omega$, then* $\rho(\overline{a}_{|p}) \;=\; \rho(\overline{a})_{|p}$ [2].

b) The map

$$f \in [\prod_{k\in K} A_k, B] \longmapsto f \circ \rho \in [\bigotimes_{k\in K} A_k,\; B]$$

is injective. If B is a presheaf, then it is a bijection [3].

c) If A is a Ω-set and I is a set, there is a natural bijective correspondence between $\mathfrak{K}_I A$ *and* $[\bigotimes^I A,\; \widetilde{\Omega}]$ [4].

d) If A is a Ω-presheaf and $I \neq \emptyset$ is a set, there is a natural bijective correspondence between $[A^I, \widetilde{\Omega}]$ *and* $\mathfrak{K}_I A$.

Proof. a) First, it is clear that $\overline{a}_{|E\overline{a}}$ is in the domain of $\prod_{k\in K} A_k$. It is also clear that ρ is surjective. Next, for $\overline{a}, \overline{c} \in \prod_{k\in K} |A_k|$, $E\rho(\overline{a}) \;=\; E\overline{a}_{|E\overline{a}} \;=\; E\overline{a}$, and

$$\begin{aligned}[\![\rho(\overline{a}) = \rho(\overline{c})]\!] &= [\![\overline{a}_{|E\overline{a}} = \overline{c}_{|E\overline{c}}]\!] \;=\; E\overline{a} \wedge E\overline{c} \wedge [\![\overline{a} = \overline{c}]\!] \\ &= [\![\overline{a} = \overline{c}]\!],\end{aligned}$$

[2]But ρ *is not* a presheaf morphism.
[3]Recall that $[C, D]$ is the set of morphisms from C to D.
[4]$\widetilde{\Omega}$ is defined in 25.4; see also 27.7.

and so ρ is a Ω-set morphism and a regular monic [5]. If B is a presheaf, then recalling 38.2.(c) and that $E\overline{a}_{|p} = p \wedge E\overline{a}$, we get

$$\rho(\overline{a}_{|p}) \;=\; (\overline{a}_{|p})_{|p\wedge E\overline{a}} \;=\; \overline{a}_{|p\wedge E\overline{a}} \;=\; (\overline{a}_{|E\overline{a}})_{|p} \;=\; \rho(\overline{a})_{|p},$$

ending the proof of (a).

b) Since ρ is surjective (and therefore, epic), the map $f \mapsto \rho \circ f$ is injective. If B is a presheaf and g is a Ω-set morphism in $[\bigotimes_{k\in K} A_k,\ B]$, let f be the restriction of g to $|\prod_{k\in K} A_k|$; since the product is a sub-Ω-set of the exterior product, f is a Ω-set morphism. For $\overline{a} \in \prod_{k\in K} |A_k|$, the fact that B is a presheaf and (a).(2) yield $f(\rho(\overline{a})) = f(\overline{a}_{|E\overline{a}}) = g(\overline{a}_{|E\overline{a}}) = g(\overline{a})_{|E\overline{a}} = g(\overline{a})_{|Eg(\overline{a})} = g(\overline{a})$, verifying that $g = f \circ \rho$.

c) Let $f \in [\bigotimes^I A,\ \widetilde{\Omega}]$; for $\overline{a} \in |A|^I$, we may write

$$f(\overline{a}) \;=\; \langle\, h_f(\overline{a}),\, E\overline{a}\,\rangle,$$

where $h_f : |A|^I \longrightarrow \Omega$; the definition of $\widetilde{\Omega}$ guarantees that $h_f(\overline{a}) \leq E\overline{a}$. Since f is a Ω-set morphism, we have

$$\begin{aligned}[\![\overline{a} = \overline{c}]\!] \;&\leq\; [\![f(\overline{a}) = f(\overline{c})]\!] \;=\; [\![\langle\, h_f(\overline{a}), E\overline{a}\,\rangle = \langle\, h_f(\overline{c}), E\overline{c}\,\rangle]\!] \\ &=\; E\overline{a} \wedge E\overline{x} \wedge (h_f(\overline{a}) \leftrightarrow h_f(\overline{c})),\end{aligned}$$

a relation that is equivalent to [ch 2′] in 37.2.(a). Hence, $h_f \in \mathfrak{K}_I A$. Conversely, given $h \in \mathfrak{K}_I A$, define a morphism $f_h \in [\bigotimes^I A,\ \widetilde{\Omega}]$ by

$$f_h(\overline{a}) \;=\; \langle\, h(\overline{a}),\, E\overline{a}\,\rangle.$$

It is left to the reader to check that

∗ f_h is a Ω-set morphism and $f \mapsto h_f$ and $h \mapsto f_h$ are inverse bijective correspondences;

∗ The above computations hold true even for $I = \emptyset$, because $\bigotimes^0 A = \mathbf{1}_{|EA}$.

Item (d) is immediate from (b) and (c). □

Remark 38.5. In the setting of 38.4, if $I = \emptyset$, then

$$\bigotimes\nolimits^0 A \;=\; \mathbf{1}_{|EA} \quad \text{and} \quad A^I = \mathbf{1}.$$

The morphism ρ in 38.4.(b) is the canonical injection of $\mathbf{1}_{|EA}$ into $\mathbf{1}$. It is a regular monic, but will not be epic unless $EA = \top$. In a similar vein, in item (c) we have, recalling 37.5 and 37.6

$$\mathfrak{K}_0 A \;=\; (EA)^{\leftarrow} \quad \text{and} \quad \mathfrak{K}_0 \mathbf{1} \;=\; \Omega,$$

which are distinct, unless $EA = \top$. □

Corollary 38.6. *If A_i, $i \in I$, are extensional Ω-presheaves, the extensionalization [6] of $\bigotimes_{i\in I} A_i$ is $\prod_{i\in I} A_i$.* □

Corollary 38.7. *If A is a Ω-presheaf and I, J are sets, $\mathfrak{K}_{I\times J} A$ and $\mathfrak{K}_J(A^I)$ are naturally isomorphic. In particular, $\mathfrak{K}_I A \;\approx\; \mathfrak{K}_1(A^I)$.*

[5] But not injective : $\rho(\overline{a}) = \rho(\overline{a}_{|E\overline{a}})$.

[6] As in Remark 30.7 and Corollary 30.8.

PROOF. It is straightforward that $A^{I\times J}$ is naturally isomorphic to $(A^I)^J$ [7]. Then, 38.4.(d) yields

$$\mathfrak{K}_{I\times J}A \;\approx\; [A^{I\times J}, \widetilde{\Omega}] \;\approx\; [(A^I)^J, \widetilde{\Omega}] \;\approx\; \mathfrak{K}_J(A^I)$$

as claimed. Since $I \approx I \times \{1\}$, 37.25 entails $\mathfrak{K}_I A \approx \mathfrak{K}_1(A^I)$. □

The statement of 38.7 is false for Ω-sets in general. Hence, if A, B are Ω-sets, a *morphism* $f : A^I \longrightarrow B^J$ will induce an adjoint pair [8]

$$f_* : \mathfrak{K}_1(A^I) \longrightarrow \mathfrak{K}_1(B^J) \quad \text{and} \quad f^* : \mathfrak{K}_1(B^J) \longrightarrow \mathfrak{K}_1(A^I)$$

but not the image and inverse image connecting $\mathfrak{K}_I A$ and $\mathfrak{K}_J B$. To obtain these, one has to consider maps between the exterior powers of A and B. But it has already been remarked (38.2.(b)) that the projection that forgets coordinates *is not* a morphism when defined in exterior powers. Since we wish to discuss the classical quantifiers for relations defined in Ω-sets − not just in Ω-presheaves −, there is need of a method to treat uniformly morphisms and projections that forget certain coordinates. To this end, we introduce the following

DEFINITION 38.8. *Let A, B be Ω-sets. A map $f : |A| \longrightarrow |B|$ is a* **Q-morphism** *if for all $x, y \in |A|$*

$$[\![x = y]\!] \;\leq\; [\![fx = fy]\!].$$

Hence a Q-morphism is a map that verifies [mor 2] in 25.10*. Note that a Q-morphism satisfies $Ex \leq Efx$.*

It is clear that

∗ The composition of Q-morphisms is a Q-morphism.

∗ Every Ω-set morphism is a Q-morphism.

We shall be mostly interested in Q-morphisms defined and with values in exterior *powers* of Ω-sets (38.1). Generalizing 38.2.(b), we have

EXAMPLE 38.9. Let A_i, $i \in I$, be Ω-sets and $\emptyset \neq J \subseteq I$. There are natural projections

$$\begin{cases} \prod_{i\in I} A_i & \longrightarrow & \prod_{j\in J} A_j \\ \bigotimes_{i\in I} A_i & \longrightarrow & \bigotimes_{j\in J} A_j, \end{cases}$$

both be written π_J, that forget the coordinates outside J. Hence, for $\overline{a} \in \prod_{i\in I} |A_i|$, and $j \in J$ [9] $\pi_J(\overline{a})(j) \;=\; a_j$. It is straightforward that if $\overline{a}$, $\overline{c} \in \prod_{i\in I} |A_i|$, then

$[=_J]$: $[\![\overline{a} = \overline{c}]\!] \;=\; [\![\pi_J(\overline{a}) = \pi_J(\overline{c})]\!] \wedge [\![\pi_{I-J}(\overline{a}) = \pi_{I-J}(\overline{c})]\!]$;

$[E_J]$: $E\overline{a} \;=\; E\pi_J(\overline{a}) \wedge E\pi_{I-J}(\overline{a})$.

$[ch_J]$: If $R \in \mathfrak{K}_I A$ and $\overline{u} \in |A|^{I-J}$, then, with notation as in 37.24,

$$[\![\pi_J\overline{a} = \pi_J\overline{c}]\!] \;\wedge\; [\![\, R(\pi_J\overline{a}; \overline{u}) \,]\!] \;\leq\; [\![\, R(\pi_J\overline{c}; \overline{u}) \,]\!].$$

Note that $[ch_J]$ follows from [ch 2] in 37.1 once it is observed that

[7] True even in $\boldsymbol{Sh(\Omega)}$, see 24.44.

[8] Image and inverse image; see Theorem 38.11 and Corollary 38.15.

[9] We refrain from writing $\overline{a}_{|J}$, in place of $\pi_J(\overline{a})$, for obvious motives.

$$\begin{cases} [\![\, R(\pi_J\overline{a};\, \overline{u})\,]\!] \ \le \ E\overline{u}; \\ [\![\langle\, \pi_J\overline{a};, \overline{u}\,\rangle \ = \ \langle\, \pi_J\overline{c};\, \overline{u}\,\rangle]\!] \ = \ [\![\pi_J\overline{a} \ = \ \pi_J\overline{c}]\!] \wedge E\overline{u}. \end{cases}$$

Hence,

∗ Considered as a map $\bigotimes_{i\in I} A_i \longrightarrow \bigotimes_{j\in J} A_j$, π_J is a Q-morphism [10];

∗ Considered a map $\prod_{i\in I} A_i \longrightarrow \prod_{j\in J} A_j$, π_J is a morphism of Ω-sets.

These projections shall be used frequently in the sequel and context will establish which is the pertinent meaning. Note that

∗ If $I = J$, π_J is the identity of $\bigotimes_{i\in I} A_i$ or $\prod_{i\in I} A_i$;

∗ If $J = \emptyset$ and $A_i = A$, $i \in I$, then $\pi_J \ : \ \begin{cases} A^I & \longrightarrow & \mathbf{1} \\ \bigotimes^I A & \longrightarrow & \mathbf{1}_{|EA}, \end{cases}$

in both cases given by $\overline{a} \longmapsto E\overline{a}$. □

EXAMPLE 38.10. Let $f : A \longrightarrow B$ be a morphism of Ω-sets. Then f induces Ω-set morphisms, indicated by the same symbol

$$f^I \ : \ \begin{cases} A^I & \longrightarrow & B^I \\ \bigotimes^I A & \longrightarrow & \bigotimes^I B, \end{cases}$$

where $f^I(\overline{a}) = \langle\, f(a_i)\,\rangle$ [11]. Note that in this case, if $\overline{a} \in |A|^I$,

$$Ef^I(\overline{a}) \ = \ \bigwedge_{i\in I} Ef(a_i) \ = \ \bigwedge_{i\in I} Ea_i \ = \ E\overline{a},$$

and f^I is a Ω-set morphism, whether with domain the power or the exterior power of A. We have agreed to indicate $f^I(\overline{a})$ by $f(\overline{a})$ (1.4). Again, context will make clear to which map we are referring. If $I = \emptyset$, then

$$f^{\emptyset} \ : \ \begin{cases} \mathbf{1} & \longrightarrow & \mathbf{1} \\ \mathbf{1}_{|EA} & \longrightarrow & \mathbf{1}_{|EB}, \end{cases}$$

in both cases being the map $p \mapsto p$ [12].

If $J \subseteq I$, we may compose f^I with π_J to obtain a Ω-set and a Q-morphism, again denoted by the same symbol

$$\pi_J \circ f^I \ : \ \begin{cases} A^I & \longrightarrow & B^J \\ \bigotimes^I A & \longrightarrow & \bigotimes^J B. \end{cases}$$

The above construction may be generalized by considering a family of Q-morphisms, $f_i : A_i \longrightarrow B_i$, $i \in I$. Then

$$f = \prod_{i\in I} f_i, \ \ \overline{a} \in |\bigotimes_{i\in I} A_i| \ \longmapsto \langle\, f_i a_i\,\rangle \in |\bigotimes_{i\in I} B_i|,$$

is a Q-morphism. Indeed, for $\overline{a}$, $\overline{c} \in \prod_{i\in I} |A_i|$,

$$[\![f(\overline{a}) \ = \ f(\overline{c})]\!] \ = \ \bigwedge_{i\in I} [\![f_i(a_i) \ = \ f_i(c_i)]\!] \ \ge \ \bigwedge_{i\in I} [\![a_i \ = \ b_i]\!] \ = \ [\![\overline{a} \ = \ \overline{c}]\!],$$

as needed. Similarly, f can be understood as a Q-morphism from $\prod_{i\in I} A_i$ to $\prod_{i\in I} B_i$. The reader can check that if each f_i is a Ω-set morphism, then $\prod_{i\in I} f_i$

[10] But *not* a Ω-set morphism.

[11] Both for $\overline{a} \in |A|^I$ and in $|A^I|$.

[12] By Exercise 25.36, $EA \le EB$.

defines Ω-set morphisms between the product and the exterior product of the A_i, respectively. $\square$

The main result of this Chapter is

THEOREM 38.11. *Let A, B be Ω-sets and let I, J be sets. A Q-morphism, $f : \bigotimes^I A \longrightarrow \bigotimes^J B$, induces maps*

$$f_* : \mathfrak{K}_I A \longrightarrow \mathfrak{K}_J B \quad and \quad f^* : \mathfrak{K}_J B \longrightarrow \mathfrak{K}_I A,$$

defined for $R \in \mathfrak{K}_I A$, $S \in \mathfrak{K}_J B$, $\overline{a} \in |A|^I$ and $\overline{s} \in |B|^J$, by

$$\begin{cases} f_* R(\overline{s}) &= \bigvee_{\overline{a} \in |A|^I} [\![f(\overline{a}) = \overline{s}]\!] \wedge [\![R(\overline{a})]\!]; \\ f^* S(\overline{a}) &= [\![S(f(\overline{a}))]\!] \wedge E\overline{a}, \end{cases}$$

satisfying the following conditions [13] *:*

a) f_* is a $\bigvee$-morphism such that

$$f_* \bot_I \;=\; \bot_J \quad and \quad f_* \top_I \;\leq\; [\![f(|A|^I)(\cdot)]\!].$$

If f is a morphism of Ω-sets, then $f_* \top_I \;=\; [\![f(|A|^I)(\cdot)]\!]$.

b) f^* is a open morphism and $f^* \bot_J \;=\; \bot_I$ and $f^* \top_J \;=\; \top_I$.

c) If $EA = EB$ or if f is a Ω-set morphism, then $\langle f_*, f^* \rangle$ is an adjoint pair (7.8).

d) If f is a Ω-set morphism, the following are equivalent :

(1) For all $\overline{a}$, $\overline{c} \in |A|^I$, $[\![f(\overline{a}) = f(\overline{c})]\!] \;=\; [\![\overline{a} = \overline{c}]\!]$ [14]*;*

(2) f_* is injective; *(3) f^* is surjective.*

e) Consider the following conditions :

(1) f^* is injective; *(2) For all $\overline{s} \in |B|^J$, $E\overline{s} \;=\; \bigvee_{\overline{a} \in |A|^I} [\![f(\overline{a}) = \overline{s}]\!]$;*

(3) f_* is surjective.

Then, (1) $\Rightarrow$ (2)*; if f is a Ω-set morphism, all three conditions are equivalent.*

PROOF. a) Clearly, $f_* h(\overline{s}) \leq E\overline{s}$. For $R \in \mathfrak{K}_I A$ and $\overline{s}$, $\overline{u} \in |B|^J$, we have

$$\begin{aligned} [\![\overline{s} = \overline{u}]\!] \wedge f_* R(\overline{s}) &= [\![\overline{s} = \overline{u}]\!] \wedge \bigvee_{\overline{a} \in |A|^I} [\![f\overline{a} = \overline{s}]\!] \wedge [\![R(\overline{a})]\!] \\ &= \bigvee_{\overline{a} \in |A|^I} [\![\overline{s} = \overline{u}]\!] \wedge [\![f\overline{a} = \overline{s}]\!] \wedge [\![R(\overline{a})]\!] \\ &\leq \bigvee_{\overline{a} \in |A|^I} [\![f\overline{a} = \overline{u}]\!] \wedge [\![R(\overline{a})]\!] \;=\; f_* R(\overline{u}), \end{aligned}$$

establishing [ch 2] in 37.1. Let R_λ, $\lambda \in \Lambda$, be a family in $\mathfrak{K}_I A$. Recall (37.4.(c)) that joins in $\mathfrak{K}_I A$ are computed pointwise. Hence,

$$\begin{aligned} f_*\big(\bigvee_{\lambda \in \Lambda} R_\lambda\big)(\overline{s}) &= \bigvee_{\overline{a} \in |A|^I} [\![f\overline{a} = \overline{s}]\!] \wedge \bigvee_{\lambda \in \Lambda} [\![R_\lambda(\overline{a})]\!] \\ &= \bigvee_{\lambda \in \Lambda} \bigvee_{\overline{a} \in |A|^I} [\![f\overline{a} = \overline{s}]\!] \wedge [\![R_\lambda(\overline{a})]\!] \\ &= \bigvee_{\lambda \in \Lambda} f_* R_\lambda(\overline{s}) \;=\; [\bigvee_{\lambda \in \Lambda} f_* R_\lambda](\overline{s}), \end{aligned}$$

and f_* is a $\bigvee$-morphism. Since $\bot_I(\overline{a}) = \bot$ (37.22.(b)), it is obvious that $f_* \bot_I = \bot_J$. Recalling that $\top_I(\overline{a}) = E\overline{a} \;\leq\; Ef(\overline{a})$, we get

[13] $\bot_I$ and $\top_I$ are the bottom and top of $\mathfrak{K}_I A$, as in 37.22.(b) and (c).

[14] Hence, f is a regular monic.

$$\begin{aligned} f_*\top_I(\overline{s}) &= \textstyle\bigvee_{\overline{a}\in|A|^I} [\![f\overline{a} = \overline{s}]\!] \wedge E\overline{a} \;\leq\; \bigvee_{\overline{a}\in|A|^I} [\![f\overline{a} = \overline{s}]\!] \wedge Ef(\overline{a}) \\ &= \textstyle\bigvee_{\overline{a}\in|A|^I} [\![f\overline{a} = \overline{s}]\!] \;=\; [\![\, f(|A|^I)(\overline{s}) \,]\!]. \end{aligned}$$

If f is a morphism, then $Ef(\overline{a}) = E\overline{a}$ and the only inequality in the preceding computation is an equality; hence, $f_*\top_I = [\![\, f(|A|^I)(\cdot) \,]\!]$.

b) For $S \in \mathfrak{K}_J B$, it is clear that $f^*S(\overline{a}) \leq E\overline{a}$; if $\overline{c} \in |A|^I$, then

$$\begin{aligned} [\![\overline{a} = \overline{c}]\!] \wedge f^*S(\overline{a}) &= [\![\overline{a} = \overline{c}]\!] \wedge [\![\, S(f(\overline{a})) \,]\!] \wedge E\overline{a} \;=\; [\![\overline{a} = \overline{c}]\!] \wedge [\![\, S(f(\overline{a})) \,]\!] \wedge E\overline{c} \\ &\leq [\![f(\overline{c}) = f(\overline{a})]\!] \wedge [\![\, S(f(\overline{a})) \,]\!] \wedge E\overline{c} \\ &\leq [\![\, S(f(\overline{c})) \,]\!] \wedge E\overline{c} \;=\; f^*S(\overline{c}), \end{aligned}$$

and $f^*k \in \mathfrak{K}_I A$. It is straightforward that f^* is a $[\bigwedge, \bigvee]$-morphism. To check that it is open, let $S_1, S_2 \in \mathfrak{K}_I B$; then

$$\begin{aligned} f^*(S_1 \to S_2)(\overline{a}) &= [\![\, (S_1 \to S_2)(f\overline{a}) \,]\!] \wedge E\overline{a} \\ &= E\overline{a} \wedge Ef(\overline{a}) \wedge \Big([\![\, S_1(f\overline{a}) \,]\!] \to [\![\, S_2(f\overline{a}) \,]\!] \Big) \\ &= E\overline{a} \wedge \Big([\![\, S_1(f\overline{a}) \,]\!] \to [\![\, S_2(f\overline{a}) \,]\!] \Big) \\ &= E\overline{a} \wedge \Big((E\overline{a} \wedge [\![\, S_1(f\overline{a}) \,]\!]) \to (E\overline{a} \wedge [\![\, S_2(f\overline{a}) \,]\!]) \Big) \\ &= [f^*S_1 \to f^*S_2](\overline{a}), \end{aligned}$$

as needed. The remaining statement is clear.

c) For $R \in \mathfrak{K}_I A$ and $\overline{a} \in |A|^I$, we have, recalling [ch $2'$] in 37.2.(a)

$$\begin{aligned} f^*(f_*R)(\overline{a}) &= E\overline{a} \wedge f_*R(f(\overline{a})) \;=\; E\overline{a} \wedge \textstyle\bigvee_{\overline{c}\in|A|^I} [\![f(\overline{c}) = f(\overline{a})]\!] \wedge [\![\, R(\overline{c}) \,]\!] \\ &\geq \textstyle\bigvee_{\overline{c}\in|A|^I} [\![\overline{a} = \overline{c}]\!] \wedge [\![\, R(\overline{c}) \,]\!] \;=\; \bigvee_{\overline{c}\in|A|^I} [\![\overline{a} = \overline{c}]\!] \wedge [\![\, R(\overline{a}) \,]\!] \\ &= [\![\, R(\overline{a}) \,]\!] \wedge \textstyle\bigvee_{\overline{c}\in|A|^I} [\![\overline{a} = \overline{c}]\!] \;=\; [\![\, R(\overline{a}) \,]\!] \wedge E\overline{a} \;=\; [\![\, R(\overline{a}) \,]\!]. \end{aligned}$$

We have just verified that $f^* \circ f_* \geq Id_{\mathfrak{K}_I A}$. Now, if $S \in \mathfrak{K}_I B$, then

$$(*) \qquad \begin{aligned} f_*(f^*S)(\overline{s}) &= \textstyle\bigvee_{\overline{a}\in|A|^I} [\![f(\overline{a}) = \overline{s}]\!] \wedge f^*S(\overline{a}) \\ &= \textstyle\bigvee_{\overline{a}\in|A|^I} [\![f(\overline{a}) = \overline{s}]\!] \wedge [\![\, S(f(\overline{a})) \,]\!] \wedge E\overline{a}. \end{aligned}$$

We now discuss two cases :

<u>f is a morphism :</u> Then, $Ef(\overline{a}) = E\overline{a}$ and (*) yields

$$\begin{aligned} f_*(f^*S)(\overline{s}) &= \textstyle\bigvee_{\overline{a}\in|A|^I} [\![f(\overline{a}) = \overline{s}]\!] \wedge [\![\, S(f(\overline{a})) \,]\!] \wedge Ef(\overline{a}) \\ &= \textstyle\bigvee_{\overline{a}\in|A|^I} [\![f(\overline{a}) = \overline{s}]\!] \wedge [\![\, S(f(\overline{a})) \,]\!] \;\leq\; [\![\, S(\overline{s}) \,]\!]. \end{aligned}$$

<u>$EA = EB$:</u> From (*), 38.3 and $[\![\, S(\overline{s}) \,]\!] \leq E\overline{s} \leq EB$ we get

$$\begin{aligned} f_*(f^*S)(\overline{s}) &= \textstyle\bigvee_{\overline{a}\in|A|^I} [\![f(\overline{a}) = \overline{s}]\!] \wedge [\![\, S(f(\overline{a})) \,]\!] \wedge E(\overline{a}) \\ &\leq \textstyle\bigvee_{\overline{a}\in|A|^I} [\![\, S(f(\overline{s})) \,]\!] \wedge E\overline{a} \\ &= [\![\, S(\overline{s}) \,]\!] \wedge E\textstyle\bigotimes^I A \;=\; [\![\, S(\overline{s}) \,]\!] \wedge EB \;=\; [\![\, S(\overline{s}) \,]\!]. \end{aligned}$$

In both of the above cases $f_* \circ f^* \leq Id_{\mathfrak{K}_J B}$; since also have $f^* \circ f_* \geq Id_{\mathfrak{K}_I A}$, Exercise 7.12 implies that $\langle f_*, f^* \rangle$ is an adjoint pair.

d) <u>(1) $\Rightarrow$ (2) :</u> If $P, Q \in \mathfrak{K}_I A$ verify $f_*P = f_*Q$, then for $\overline{s} \in |B|^I$, we have

$$\textstyle\bigvee_{\overline{a}\in|A|^I} [\![f\overline{a} = \overline{s}]\!] \wedge [\![\, P(\overline{a}) \,]\!] \;=\; \bigvee_{\overline{a}\in|A|^I} [\![f\overline{a} = \overline{s}]\!] \wedge [\![\, Q(\overline{a}) \,]\!]. \qquad \text{(I)}$$

Fix $\overline{c} \in |A|^n$; (I) then yields, with $\overline{s} = f\overline{c}$,

$$\bigvee_{\overline{a}\in|A|^I} [\![f\overline{a} = f\overline{c}]\!] \wedge [\![\, P(\overline{a})\,]\!] \;=\; \bigvee_{\overline{a}\in|A|^I} [\![f\overline{a} = f\overline{c}]\!] \wedge [\![\, Q(\overline{a})\,]\!]. \qquad \text{(II)}$$

From (1) we get

$$[\![f\overline{a} = f\overline{c}]\!] \wedge [\![\, P(\overline{a})\,]\!] \;=\; [\![\overline{a} = \overline{c}]\!] \wedge [\![\, P(\overline{a})\,]\!] \;=\; [\![\overline{a} = \overline{c}]\!] \wedge [\![\, P(\overline{c})\,]\!],$$

with a similar relation holding for Q. But then

$$\begin{aligned} \bigvee_{\overline{a}\in|A|^I} [\![f\overline{a} = f\overline{c}]\!] \wedge [\![\, P(\overline{a})\,]\!] &= \bigvee_{\overline{a}\in|A|^I} [\![\overline{a} = \overline{c}]\!] \wedge [\![\, P(\overline{c})\,]\!] \\ &= [\![\, P(\overline{c})\,]\!] \wedge \bigvee_{\overline{a}\in|A|^I} [\![\overline{a} = \overline{c}]\!] \\ &= [\![\, P(\overline{c})\,]\!] \wedge E\overline{c} \;=\; [\![\, P(\overline{c})\,]\!], \end{aligned}$$

with an analogous result for Q. Thus, (II) yields $[\![\, P(\overline{c})\,]\!] = [\![\, Q(\overline{c})\,]\!]$, verifying that f_* is injective.

<u>(2) ⇒ (1) :</u> Exactly as in 37.7, for all $\overline{s} \in |B|^J$, $h(\overline{x}) = [\![f\overline{x} = \overline{s}]\!]$ is a I-characteristic map on A. Hence, [ch 2′] in 37.2.(a) applied to h yields

(*) $\qquad \forall\, \overline{a}, \overline{c} \in |A|^I,\;\; [\![f\overline{a} = \overline{s}]\!] \wedge [\![\overline{a} = \overline{c}]\!] \;=\; [\![f\overline{c} = \overline{s}]\!] \wedge [\![\overline{a} = \overline{c}]\!].$

Fix $\overline{c} \in |A|^I$; the following are I-characteristic maps on A : $h_1(\overline{x}) = [\![\overline{x} = \overline{c}]\!]$ and $h_2(\overline{x}) = [\![f\overline{x} = f\overline{c}]\!]$ [15]. For $\overline{s} \in |B|^J$, (*) then yields

$$\begin{aligned} f_*(h_1)(\overline{s}) &= \bigvee_{\overline{a}\in|A|^I} [\![f\overline{a} = \overline{s}]\!] \wedge [\![\overline{a} = \overline{c}]\!] \;=\; \bigvee_{\overline{a}\in|A|^I} [\![f\overline{c} = \overline{s}]\!] \wedge [\![\overline{a} = \overline{c}]\!] \\ &= [\![f\overline{c} = \overline{s}]\!] \wedge \bigvee_{\overline{a}\in|A|^I} [\![\overline{a} = \overline{c}]\!] \;=\; [\![f\overline{c} = \overline{s}]\!] \wedge E\overline{c} \\ &= [\![f\overline{c} = \overline{s}]\!] \wedge Ef\overline{c} \;=\; [\![f\overline{c} = \overline{s}]\!]. \end{aligned}$$

On the other hand, the exchange rule in 25.35.(b) entails

$$\begin{aligned} f_*(h_2)(\overline{s}) &= \bigvee_{\overline{a}\in|A|^I} [\![f\overline{a} = \overline{s}]\!] \wedge [\![f\overline{a} = f\overline{c}]\!] \;=\; [\![f\overline{c} = \overline{s}]\!] \wedge \bigvee_{\overline{a}\in|A|^I} [\![f\overline{a} = f\overline{c}]\!] \\ &= [\![f\overline{c} = \overline{s}]\!] \wedge Ef\overline{c} \;=\; [\![f\overline{c} = \overline{s}]\!]. \end{aligned}$$

We have shown that $f_*h_1 = f_*h_2$, and so $h_1 = h_2$ in $\mathfrak{K}_I A$. But this means that for all $\overline{x} \in |A|^I$, $[\![\overline{x} = \overline{s}]\!] = [\![f\overline{x} = f\overline{s}]\!]$. i.e., f is a regular monic.

Since f is a morphism, it follows from the adjointness in (c) and 7.9.(b) that (2) and (3) are equivalent.

e) <u>(1) ⇒ (2) :</u> By item (b), $f^*\top_J = \top_I$. Let h be the J-characteristic map of $f(|A|^I)$ in B, that is, $h(\overline{s}) = \bigvee_{\overline{a}\in|A|^I} [\![f\overline{a} = \overline{s}]\!]$, $\overline{s} \in |B|^I$. Then, if $\overline{c} \in |A|^I$

$$f^*(h)(\overline{c}) \;=\; h(f\overline{c}) \wedge E\overline{c} \;=\; E\overline{c} \wedge \bigvee_{\overline{a}\in|A|^I} [\![f\overline{a} = f\overline{c}]\!] \;=\; E\overline{c} \wedge Efc \;=\; Ec,$$

and so $f^*h = f^*\top_J$ in $\mathfrak{K}_I A$. Thus, $h = \top_J$, that is, for $\overline{s} \in |B|^I$,

$$E\overline{s} \;=\; \bigvee_{\overline{a}\in|A|^I} [\![f\overline{a} = \overline{s}]\!],$$

establishing (2). If f is a Ω-set morphism, we prove

<u>(2) ⇒ (1) :</u> Let $S, T \in \mathfrak{K}_J B$ verify $f^*S = f^*T$. Since for all $\overline{a} \in |A|^I$, $Ef\overline{a} = E\overline{a}$, the hypothesis and the definition of f^* entail

$$(\#) \qquad \begin{aligned} [\![\, S(f(\overline{a})\,]\!] &= [\![\, S(f(\overline{a})\,]\!] \wedge Ef\overline{a} \;=\; [\![\, S(f\overline{a})\,]\!] \wedge E\overline{a} \\ &= [\![\, T(f\overline{a})\,]\!] \wedge E\overline{a} \;=\; [\![\, T(f\overline{a})\,]\!] \wedge Ef\overline{a} \;=\; [\![\, T(f\overline{a})\,]\!]. \end{aligned}$$

[15] For h_1 this is clear; h_2 is a special case of the preceding.

Hence, if $\overline{s} \in |B|^J$, (2) and (#) yield

$$\begin{array}{rcl}
[\![S(\overline{s})]\!] & = & [\![S(\overline{s})]\!] \wedge E\overline{s} \;=\; [\![S(\overline{s})]\!] \wedge \bigvee_{\overline{a} \in |A|^I} [\![f\overline{a} = \overline{s}]\!] \\
& = & \bigvee_{\overline{a} \in |A|^I} [\![S(\overline{s})]\!] \wedge [\![f\overline{a} = \overline{s}]\!] \\
& = & \bigvee_{\overline{a} \in |A|^I} [\![S(f\overline{a})]\!] \wedge [\![f\overline{a} = \overline{s}]\!] \;=\; \bigvee_{\overline{a} \in |A|^I} [\![T(f\overline{a})]\!] \wedge [\![f\overline{a} = \overline{s}]\!] \\
& = & \bigvee_{a \in |A|^I} [\![T(\overline{s})]\!] \wedge [\![f\overline{a} = \overline{s}]\!] \;=\; [\![T(\overline{s})]\!] \wedge E\overline{s} \;=\; [\![T(\overline{s})]\!],
\end{array}$$

verifying that f^* is injective. As in (d), (2) $\Leftrightarrow$ (3) is a consequence of the adjunction in (c) and 7.9.(a), ending the proof. $\square$

DEFINITION 38.12. *Let A, B be Ω-sets and I, J be sets. If*
$$f : \bigotimes^I A \longrightarrow \bigotimes^J B$$
is a Q-morphism, $R \in \mathfrak{K}_I A$ and $S \in \mathfrak{K}_J B$,

a) $f_* R$ is the **image of R by f**.

b) $f^* S$ is the **inverse image of S by f**.

COROLLARY 38.13. *Let A be a Ω-set and J, I be sets. Let*
$$\Delta \;:\; \begin{cases} A & \longrightarrow \quad A^J \\ A & \longrightarrow \quad \bigotimes^J A \end{cases}$$
be the diagonal embeddings and $\eta : A^J \longrightarrow \bigotimes^J A$ be the natural embedding. The diagrams below are commutative

$$\begin{tikzcd}
\mathfrak{K}_I A \arrow[rr, "\Delta_*"] \arrow[dr, "\Delta_*"'] & & \mathfrak{K}_I(A^J) \arrow[dl, "\eta_*"] \\
& \mathfrak{K}_{I\times J}(A) &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{K}_{I\times J}(A) \arrow[rr, "\eta^*"] \arrow[dr, "\Delta^*"'] & & \mathfrak{K}_I(A^J) \arrow[dl, "\Delta^*"] \\
& \mathfrak{K}_I A &
\end{tikzcd}$$

and have the following properties :

a) All arrows above-left are injective $\bigvee$-morphisms, while those above-right are surjective open morphisms, with corresponding to adjoint pairs $\cdot_*$ and $\cdot^*$.

b) If A is a presheaf, η_* and η^* are inverse isomorphisms.

PROOF. a) The commutative diagram of regular Ω-set embeddings below-left, induces, by 38.10, the commutative diagram of regular Ω-set embeddings below-right

$$\begin{tikzcd}
A \arrow[rr, "\Delta"] \arrow[dr, "\Delta"'] & & A^J \arrow[dl, "\eta"] \\
& \bigotimes^J A &
\end{tikzcd}
\qquad
\begin{tikzcd}
\bigotimes^I A \arrow[rr, "\Delta^I"] \arrow[dr, "\Delta^I"'] & & \bigotimes^I (A^J) \arrow[dl, "\eta^I"] \\
& \bigotimes^{I\times J} A &
\end{tikzcd}$$

recalling that $\bigotimes^I(\bigotimes^J A)$ is naturally isomorphic to $\bigotimes^{I\times J} A$ [16]. Theorem 38.11 then yields the commutative diagrams in the statement.

b) This is an explicit rendering of the isomorphism in 38.7. For S in $\mathfrak{K}_{I\times J}A$, let $T = S_{|\,|A^J|^I}$; then $T \in \mathfrak{K}_I(A^J)$ (37.2.(b)); we show that $\eta_* T = S$, completing the proof. For $\bar{t} \in |A|^{I\times J}$ we have

$$\eta_* T(\bar{t}) \;=\; \bigvee_{\bar{s}\in|A^J|^I} [\![\eta\bar{s} = \bar{t}]\!] \wedge [\![\, T(\bar{s}) \,]\!] \;=\; \bigvee_{\bar{s}\in|A^J|^I} [\![\eta\bar{s} = \bar{t}]\!] \wedge [\![\, T(\bar{t}) \,]\!] \;\leq\; [\![\, T(\bar{t}) \,]\!].$$

On the other hand, noting that $\bar{t}_{|E\bar{t}} \in |A^{I\times J}| \subseteq |A^J|^I$ and $[\![\bar{t} = \bar{t}_{|E\bar{t}}]\!] = E\bar{t}$, 37.4.(a) yields

$$[\![\, S(\bar{t}) \,]\!] \;=\; [\![\, S(\bar{t}_{|E\bar{t}}) \,]\!] \;\leq\; \bigvee_{\bar{s}\in|A^J|^I} [\![\eta\bar{s} = \bar{t}]\!] \wedge [\![\, T(\bar{t}_{|E\bar{t}}) \,]\!] \;=\; \eta_* T(\bar{t}),$$

as needed. □

REMARK 38.14. If A is a Ω-set and I, J are sets, 38.13 implies that

∗ $\mathfrak{K}_I(A^J)$ is a retract of $\mathfrak{K}_{I\times J}A$ and $\mathfrak{K}_I A$ is a retract of $\mathfrak{K}_I(A^J)$;

∗ $\mathfrak{K}_I A$ is a quotient of $\mathfrak{K}_I(A^J)$, which in turn is a quotient of $\mathfrak{K}_{I\times J}A$. Since the surjections that characterize these quotients are open, it follows from Remark 10.7 that there are *principal* filters $\mathcal{F} \subseteq \mathcal{G}$ in $\mathfrak{K}_{I\times J}A$, such that

$$\mathfrak{K}_I(A^J) \;\approx\; (\mathfrak{K}_{I\times J}A)/\mathcal{F} \quad \text{and} \quad \mathfrak{K}_I A \;\approx\; (\mathfrak{K}_{I\times J}A)/\mathcal{G}.$$

The reader can check that $\mathcal{F}$ is the principal filter generated by $\eta_* \top_I$, i.e., the characteristic map of $|A^J|^I \subseteq |A|^{I\times J}$. Similarly, one obtains $\mathcal{G}$.

Similar comments apply, by the same argument and 38.11.(d), to the inverse image by any regular embedding. □

COROLLARY 38.15. *Let $f : A \longrightarrow B$ be a Ω-set morphism. For each set I, f induces an adjoint pair $\langle f_*, f^* \rangle$, satisfying the following conditions :*

a) $f_* : \mathfrak{K}_I A \longrightarrow \mathfrak{K}_I B$, given by $f_* R(\bar{b}) \;=\; \bigvee_{\bar{a}\in|A|^I} [\![f\bar{a} = \bar{b}]\!] \wedge [\![\, R(\bar{a}) \,]\!]$, is a $\bigvee$-morphism that takes $\top_I$ to $[\![\, f(|A|^I)(\cdot) \,]\!]$.

b) $f^* : \mathfrak{K}_I B \longrightarrow \mathfrak{K}_I A$, given by $f^* S(\bar{a}) \;=\; [\![\, S(f\bar{a}) \,]\!]$ is an open frame morphism.

c) The following conditions are equivalent :

(1) *f is a regular monic;* (2) *f_* is injective;* (3) *f^* is surjective.*

d) If I is finite and B is extensional, the following are equivalent

(1) *f is epic;* (2) *f_* is surjective;* (3) *f^* is injective.*

PROOF. By Example 38.10, $f^I : \bigotimes^I A \longrightarrow \bigotimes^I B$ is a Ω-set morphism and items (a), (b) and (c) follow from 38.11. For (d), it is clear that f^I is a regular monic if the same is true of f; the stated equivalence follows from 38.11.(d).

e) By Lemma 25.24, f is epic iff for all $b \in |B|$

$$Eb \;=\; \bigvee_{a\in|A|} [\![fa = b]\!].$$

If I is finite, then for $\bar{b} \in |B|^I$, distributivity of joins over *finite* meets (8.4) yields

[16] By exponential adjunction.

$$E\overline{b} \;=\; \bigwedge_{i \in I} Eb_i \;=\; \bigwedge_{i \in I} \bigvee_{a_i \in |A|} [\![fa_i = b_i]\!]$$
$$= \bigvee_{\overline{a} \in |A|^I} \bigwedge_{i \in I} [\![fa_i = b_i]\!] \;=\; \bigvee_{\overline{a} \in A^I} [\![f\overline{a} = \overline{b}]\!],$$

and f^I verifies condition (2) in 38.11.(e), ending the proof. $\square$

Corollary 38.16. *Let A be a Ω-set and I be a set. For $J \subseteq I$, the projection*

$$\pi_J : \bigotimes^I A \longrightarrow \bigotimes^J A$$

induces an adjoint pair $\langle \pi_J^*, \pi_{J*} \rangle$ satisfying the following conditions :

a) $\pi_J^* : \mathfrak{K}_J A \longrightarrow \mathfrak{K}_I A$, given, for $R \in \mathfrak{K}_J A$ and $\overline{a} \in |A|^I$ by

$$\pi_J^* R(\overline{a}) \;=\; [\![R(\pi_J \overline{a})]\!] \wedge E\overline{a}$$

is an open frame embedding.

b) $\pi_{J*} : \mathfrak{K}_I A \longrightarrow \mathfrak{K}_J A$, given, for $S \in \mathfrak{K}_I A$ and $\overline{c} \in |A|^J$, by

$$\pi_{J*} S(\overline{c}) \;=\; \bigvee_{\overline{a} \in |A|^I} [\![\pi_J(\overline{a}) = \overline{c}]\!] \wedge [\![S(\overline{a})]\!]$$

is a surjective *$\bigvee$-morphism that preserves $\bot_I$ and $\top_I$.*

Proof. Since $\pi_J : \bigotimes^I A \longrightarrow \bigotimes^J A$ is a Q-morphism (38.9) and $E\otimes^I A = E\otimes^J A = EA$ (38.3), Theorem 38.11 entails all conclusions, except

(1) $\quad \pi_J^*$ is injective.

Once (1) is proven, 7.9 implies that π_{J*} is surjective and so must preserve $\top_I$ because, being a $\bigvee$-morphism, it is increasing.

Proof of (1) : First assume that $J = \emptyset$. Then, $\bigotimes^J A = \mathbf{1}_{|EA}$; by the isomorphism of $\mathfrak{K}_1 \mathbf{1}_{|EA}$ with $(EA)^{\leftarrow}$ in 37.6, the inverse image by π_J of $p \leq EA$ is the I-characteristic map $p \wedge \top_I$. The argument that $p \leq EA \longmapsto p \wedge \top_I \in \mathfrak{K}_I A$ is injective was presented at the end of Example 37.5.

Now suppose that $J \neq \emptyset$ and fix k in J. We may assume that $J \neq I$, otherwise there is nothing to prove, since π_J is the identity. Let $R, T \in \mathfrak{K}_J A$ satisfy

(*) $\quad \forall\, \overline{a} \in |A|^I, \;\; [\![R(\pi_J \overline{a})]\!] \wedge E\overline{a} \;=\; [\![T(\pi_J \overline{a})]\!] \wedge E\overline{a},$

and fix $\overline{c} \in |A|^J$. Let $\widehat{c_k}$ be the constant $I - J$ sequence with value c_k and consider $\overline{a} = \langle \overline{c}; \widehat{c_k} \rangle \in |A|^I$ [17]. Observe that

$$\pi_J \overline{a} = \overline{c} \quad \text{and} \quad E\overline{a} = \bigwedge_{i \in I} Ea_i = E\overline{c} \wedge Ec_k = E\overline{c}.$$

Hence, (*) yields

$$[\![R(\overline{c})]\!] \;=\; [\![R(\overline{c})]\!] \wedge E\overline{c} \;=\; [\![R(\pi_J \overline{a})]\!] \wedge E\overline{a} \;=\; [\![T(\pi_J \overline{a})]\!] \wedge E\overline{a} \;=\; [\![T(\overline{c})]\!],$$

as needed. $\square$

Corollary 38.17. *Let A be a Ω-set and $n \geq 1$ be an integer. Let*

$$\pi_n : \bigotimes^{n+1} A \longrightarrow \bigotimes^n A$$

be the projection that forgets the $(n+1)^{th}$-coordinate. If $R \in \mathfrak{K}_n A$, then for all $\overline{a} \in |A|^{n+1}$

$$\pi_n^* R(\overline{a}) \;=\; [\![R(a_1, \ldots, a_n)]\!] \wedge Ea_{n+1}.$$

[17] Notation as in 37.24.

PROOF. Immediate from 38.16.(a), since for $\overline{a} \in |A|^{n+1}$

$$[\![R(a_1, \ldots, a_n)]\!] \wedge E\overline{a} \;=\; [\![R(a_1, \ldots, a_n)]\!] \wedge Ea_{n+1}$$

because $[\![R(a_1, \ldots, a_n)]\!] \;\leq\; E\langle a_1, \ldots, a_n \rangle \;=\; \bigwedge_{i=1}^{n} Ea_i$. □

Exercises

38.18. If A, B, C are Ω-sets and I, J, K are sets, then

a) $Id_{(\bigotimes^I A)*} \;=\; Id^*_{\bigotimes^I A} \;=\; Id_{\mathfrak{K}_I A}$.

b) If $\bigotimes^I A \xrightarrow{f} \bigotimes^J B \xrightarrow{g} \bigotimes^K C$ are Q-morphisms, then

$$(g \circ f)_* = g_* \circ f_* \quad \text{and} \quad (g \circ f)^* = f^* \circ g^*.$$ □

38.19. In the setting 38.11, assume that A is a Ω-*presheaf*. If $R \in \mathfrak{K}_I A$, then

$$f_* R(\overline{s}) \;=\; \bigvee_{\overline{a} \in |A^I|} [\![f(\overline{a}) \;=\; \overline{s}]\!] \wedge [\![R(\overline{a})]\!].$$

Thus, the parameters $\overline{a}$ in $|A|^I$ in the definition, may be substituted for sections in A^I. Verify that in this case $f_* \top_I \;=\; [\![f(A^I)]\!]$. □

38.20. With notation as in 38.13, assume that J is finite. For $t \in |A|^{I \times J}$ and $j \in J$, let

$$t_j : I \longrightarrow |A| \text{ be given by } t_j(i) = t(i, j).$$

a) If $R \in \mathfrak{K}_I A$, then for all $t \in |A|^{I \times J}$,

$$\Delta^* R(t) \;=\; \bigwedge_{j \in J} [\![R(t_j)]\!],$$

with $\Delta : \bigotimes^I A \longrightarrow \bigotimes^{I \times J} A$ as in 38.13. A similar relation holds for image along the diagonal embedding $\Delta : \bigotimes^I A \longrightarrow \bigotimes^I A^J$.

b) Both maps Δ^* from $\mathfrak{K}_I A$ to $\mathfrak{K}_{I \times J} A$ and $\mathfrak{K}_I(A^J)$ preserve arbitrary meets being therefore complete (or regular) embeddings. □

The next Exercise furnishes yet another argument in favor of dealing with extensional objects.

38.21. Let A be a Ω-set and $I \neq \emptyset$ be a set. Let $\boldsymbol{\varepsilon} A$ be the extensionalization of A (30.7, 30.8). Then, $\boldsymbol{\varepsilon}^* : \mathfrak{K}_I A \longrightarrow \mathfrak{K}_I \boldsymbol{\varepsilon} A$ is a frame isomorphism. □

CHAPTER 39

Image and Inverse Image under Change of Base

In this Chapter we present a generalization of image and inverse image for morphisms between objects with (possibly) distinct basis (29.1), as well as several applications of these set of ideas. We shall deal mainly with presheaves, but many of our results hold true, with the same proofs, in the category $\boldsymbol{\Omega set}$.

THEOREM 39.1. *Let L, Ω be frames and A, B be presheaves over L and Ω, respectively. Let*

$$\mathfrak{f} = \langle f, \lambda \rangle : A \longrightarrow B$$

be a morphism in $\boldsymbol{pSh}$ [1], *with λ a frame-morphism. Let $\rho : \Omega \longrightarrow L$ be the right adjoint of λ. Then,* **for each set $\boldsymbol{I}$**, *$\mathfrak{f}$ induces an adjoint pair $\langle \mathfrak{f}_*, \mathfrak{f}^* \rangle$, as follows :*

a) $\mathfrak{f}_* : \mathfrak{K}_I A \longrightarrow \mathfrak{K}_I B$ is a $\bigvee$-morphism, called **image by $\mathfrak{f}$**, *given, for $\overline{b} \in |B|^I$ and $R \in \mathfrak{K}_I A$ by*

$$\mathfrak{f}_* R(\overline{b}) \;=\; \textstyle\bigvee_{\overline{a} \in |A|^I} [\![f\overline{a} = \overline{b}]\!] \wedge \lambda \Big([\![R(\overline{a})]\!] \Big).$$

b) $\mathfrak{f}^* : \mathfrak{K}_I B \longrightarrow \mathfrak{K}_I A$ is a $\bigwedge$-morphism, called **inverse image by $\mathfrak{f}$**, *defined for $\overline{a} \in |A|^I$ and $S \in \mathfrak{K}_I B$ by*

$$\mathfrak{f}^* S(\overline{a}) \;=\; E\overline{a} \wedge \rho \Big([\![S(f\overline{a})]\!] \Big).$$

Moreover, the pair $\langle \mathfrak{f}_*, \mathfrak{f}^* \rangle$ has the following properties :

c) If $I = \emptyset$, then $\mathfrak{f}_* = \lambda$ and $\mathfrak{f}^* = g$.

$$d) \begin{cases} (1) \quad \mathfrak{f}_* \bot_I = \bot_I \quad \text{and} \quad \mathfrak{f}_* \top_I = [\![f(A^I)]\!]. \\ (2) \quad \mathfrak{f}^* \bot_I = \bot_I \quad \text{and} \quad \mathfrak{f}^* \top_I = \top_I. \end{cases}$$

e) If ρ preserves implication or joins, the same is true of $\mathfrak{f}^*$.

f) Suppose λ is injective and consider the following conditions :

(1) *For all $a, b \in |A|$,* $[\![fa = fb]\!] \;=\; \lambda \Big([\![a = b]\!] \Big)$;

(2) *For all finite I, $\mathfrak{f}_*$ is injective;*

(3) *For all finite I, $\mathfrak{f}^*$ is surjective;*

(4) *For all $a, b \in |A|$,* $[\![a = b]\!] = \rho \Big([\![fa = fb]\!] \Big)$.

Then, (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) $\Rightarrow$ (4). If ρ is a $\bigvee$-morphism, then (2), (3) and (4) are equivalent.

---

[1] As in 29.1.

g) If λ is surjective, the following conditions are equivalent :

(1) *For all $b \in |B|$,* $\;Eb = \bigvee_{a \in |A|} [\![fa = b]\!]$*;*

(2) *For all finite I,* $\;\mathfrak{f}_*$ *is surjective;*

(3) *For all finite I,* $\;\mathfrak{f}^*$ *is injective.*

h) If $\langle g, \mu \rangle : B \longrightarrow C$ is a morphism in $\boldsymbol{pSh}$, with μ a frame-morphism, then

$$\begin{array}{lrcl} (1) & (\langle g,\mu\rangle \circ \langle f,\lambda\rangle)_* & = & \langle g,\mu\rangle_* \circ \langle f,\lambda\rangle_* \\ (2) & (\langle g,\mu\rangle \circ \langle f,\lambda\rangle)^* & = & \langle f,\lambda\rangle^* \circ \langle g,\mu\rangle^*. \end{array}$$

Proof. Analogous to the corresponding statements in 38.11. The only significant difference is the presence of the right adjoint ρ; and we shall verify that $\mathfrak{f}^*S$ is in $\mathfrak{K}_I A$, adjointness, (f) and (g), sampling the techniques to handle its presence.

From the definition of $\mathfrak{f}^*$ in (b), it is enough to check [ch 2] in 37.1. If $S \in \mathfrak{K}_I B$ and $\overline{a} \in |A|^I$, since $\rho \circ \lambda \geq Id_L$ (7.8.(a)), we get

$$\begin{array}{rcl} \mathfrak{f}^*S(\overline{a}) \wedge [\![\overline{a} = \overline{c}]\!] & = & E\overline{a} \wedge \rho\Big([\![S(f\overline{a})]\!]\Big) \wedge [\![\overline{a} = \overline{c}]\!] \\ & \leq & E\overline{c} \wedge \rho\Big(\lambda([\![\overline{a} = \overline{c}]\!])\Big) \wedge \rho\Big([\![S(f\overline{a})]\!]\Big) \\ & \leq & E\overline{c} \wedge \rho\Big([\![f\overline{a} = f\overline{c}]\!] \wedge [\![S(f\overline{a})]\!]\Big) \\ & \leq & E\overline{c} \wedge \rho\Big([\![S(f\overline{c})]\!]\Big) \;\; = \;\; \mathfrak{f}^*S(\overline{c}), \end{array}$$

and $\mathfrak{f}^*S \in \mathfrak{K}_I A$. Adjointness means that for $R \in \mathfrak{K}_I A$ and $S \in \mathfrak{K}_I B$, $\mathfrak{f}_*R \leq S \;\Leftrightarrow\; R \leq \mathfrak{f}^*S$. Assume that $\mathfrak{f}_*R(\overline{b}) \leq [\![S(\overline{b})]\!]$, for all $\overline{b} \in |B|^I$. Then, for all $\overline{a} \in |A|^I$, $\;[\![f\overline{a} = f\overline{a}]\!] \wedge \lambda\Big([\![R(f\overline{a})]\!]\Big) \;\leq\; [\![S(f\overline{a})]\!]$. Since $Ef\overline{a} = \lambda(E\overline{a})$, the preceding inequality yields

$$\lambda\Big(E\overline{a} \wedge [\![R(\overline{a})]\!]\Big) \;=\; \lambda\Big([\![R(\overline{a})]\!]\Big) \;\leq\; [\![S(f\overline{a})]\!],$$

that, by adjointness, implies $\;[\![R(\overline{a})]\!] \;\leq\; E\overline{a} \wedge \rho\Big([\![S(f\overline{a})]\!]\Big) \;=\; \mathfrak{f}^*S(\overline{a})$, establishing ($\Rightarrow$). The argument for ($\Leftarrow$) is similar.

f) By 7.9, (2) and (3) are equivalent. Moreover, since λ preserves finite meets, if $\overline{a}$, $\overline{c} \in |A|^I$, then (1) entails

(λ) $\;[\![f\overline{a} = f\overline{c}]\!] \;=\; \bigwedge_{i \in I} [\![fa_i = fc_i]\!] \;=\; \bigwedge_{i\in I} \lambda([\![a_i = c_i]\!]) \;=\; \lambda([\![\overline{a} = \overline{c}]\!])$.

A similar argument shows that (4) implies

(ρ) $$\rho([\![f\overline{a} = f\overline{c}]\!]) \;=\; [\![\overline{a} = \overline{c}]\!].$$

<u>(1) $\Rightarrow$ (2)</u> : For $R \in \mathfrak{K}_I A$ and $\overline{a} \in |A|^I$, (λ) yields

$$\begin{array}{rcl} \mathfrak{f}_*R(f\overline{a}) & = & \bigvee_{\overline{c} \in |A|^I} [\![f\overline{c} = f\overline{a}]\!] \wedge \lambda\Big([\![R(\overline{c})]\!]\Big) \\ & = & \bigvee_{\overline{c} \in |A|^I} \lambda\Big([\![\overline{c} = \overline{a}]\!]\Big) \wedge \lambda\Big([\![R(\overline{c})]\!]\Big) \\ & = & \bigvee_{\overline{c} \in |A|^I} \lambda\Big([\![\overline{c} = \overline{a}]\!] \wedge [\![R(\overline{c})]\!]\Big) \\ & = & \bigvee_{\overline{c} \in |A|^I} \lambda\Big([\![\overline{c} = \overline{a}]\!] \wedge [\![R(\overline{a})]\!]\Big) \;\; = \;\; \lambda\Big([\![R(\overline{a})]\!]\Big). \end{array}$$

If $R' \in \mathfrak{K}_I A$ is such that $\mathfrak{f}_* R = \mathfrak{f}_* R'$, the above computation shows that for all $\overline{a} \in |A|^I$,

$$\lambda\Big([\![\, R(\overline{a})\,]\!]\Big) \;=\; \lambda\Big([\![\, R'(\overline{a})\,]\!]\Big),$$

whence $[\![\, R(\overline{a})\,]\!] \;=\; [\![\, R'(\overline{a})\,]\!]$, establishing (2). Since λ is injective, 7.9 yields $\rho \circ \lambda = Id_L$ and $\lambda \circ \rho \leq Id_R$.

(2) $\Rightarrow$ (4) : Fix $c \in |A|$. Consider the elements of $\mathfrak{K}_1 A$ given by

$$h_1(x) \;=\; [\![x \,=\, c]\!] \quad\text{and}\quad h_2(x)\ 2\ \rho\Big([\![fc \,=\, fx]\!]\Big).$$

To see that $h_2 \in \mathfrak{K}_1 A$, let $x,\, y \in |A|$. Then,

$$* \; h_2(x) \;=\; \rho\Big([\![fc \,=\, fx]\!]\Big) \;\leq\; \rho(Efx) \;=\; \rho(\lambda(Ex)) \;=\; Ex;$$

$$\begin{aligned} * \; h_2(x) \wedge [\![x \,=\, y]\!] \;&=\; \rho\Big([\![fc \,=\, fx]\!]\Big) \wedge [\![x \,=\, y]\!] \;=\; \rho\Big([\![fc \,=\, fx]\!] \wedge \lambda([\![x \,=\, y]\!])\Big) \\ &\leq\; \rho\Big([\![fc \,=\, fx]\!] \wedge [\![fx \,=\, fy]\!]\Big) \;\leq\; \rho\Big([\![fx \,=\, fy]\!]\Big) \\ &=\; h_2(y). \end{aligned}$$

We shall now verify that $\mathfrak{f}_* h_1 = \mathfrak{f}_* h_2$. For $b \in |B|$,

$$\begin{aligned} \mathfrak{f}_* h_1(b) \;&=\; \textstyle\bigvee_{u \in |A|} [\![fu \,=\, b]\!] \wedge \lambda([\![u \,=\, c]\!]) \\ &=\; \textstyle\bigvee_{u \in |A|} [\![fc \,=\, b]\!] \wedge \lambda([\![u \,=\, c]\!]) \\ &=\; [\![fc \,=\, b]\!] \wedge \lambda\Big(\textstyle\bigvee_{u \in |A|} [\![u \,=\, c]\!]\Big) \\ &=\; [\![fc \,=\, b]\!] \wedge \lambda(Ec) \;=\; [\![fc \,=\, b]\!] \wedge Efc \;=\; [\![fc \,=\, b]\!]. \end{aligned}$$

For h_2, first note that

$$\begin{aligned} \mathfrak{f}_* h_2(b) \;&=\; \textstyle\bigvee_{u \in |A|} [\![fu \,=\, b]\!] \wedge \lambda(\rho([\![fu \,=\, fc]\!])) \\ &\leq\; \textstyle\bigvee_{u \in |A|} [\![fu \,=\, b]\!] \wedge [\![fu \,=\, fc]\!] \;\leq\; [\![fc \,=\, b]\!]. \end{aligned}$$

On the other hand,

$$\begin{aligned} [\![fc \,=\, b]\!] \wedge \lambda(\rho([\![fc \,=\, fc]\!])) \;&=\; [\![fc \,=\, b]\!] \wedge \lambda(\rho(\lambda(Ec)) \\ &=\; [\![fc \,=\, b]\!] \wedge \lambda(Ec) \\ &=\; [\![fc \,=\, b]\!] \wedge Efc \\ &=\; [\![fc \,=\, b]\!]. \end{aligned}$$

Hence, $\mathfrak{f}_* h_2(b) \;=\; [\![fc \,=\, b]\!]$ and so $\mathfrak{f}_* h_1 = \mathfrak{f}_* h_2$, as desired. Thus, the injectivity of $\mathfrak{f}_*$ entails $h_1 = h_2$, proving (4).

Now assume that ρ is a $\bigvee$-morphism to show

(4) $\Rightarrow$ (2) : If $R \in \mathfrak{K}_I A$ and $\overline{a} \in |A|^I$, we get, recalling (ρ) above,

$$\begin{aligned} \rho\Big(\mathfrak{f}_* R(f\overline{a})\Big) \;&=\; \textstyle\bigvee_{\overline{c} \in |A|^I} \rho\Big([\![f\overline{c} \,=\, f\overline{a}]\!]\Big) \wedge \rho(\lambda([\![\, R(\overline{c})\,]\!])) \\ &=\; \textstyle\bigvee_{\overline{c} \in |A|^I} [\![\overline{c} \,=\, \overline{a}]\!] \wedge [\![\, R(\overline{c})\,]\!] \\ &=\; \textstyle\bigvee_{\overline{c} \in |A|^I} [\![\overline{c} \,=\, \overline{a}]\!] \wedge [\![\, R(\overline{a})\,]\!] \;=\; [\![\, R(\overline{a})\,]\!]. \end{aligned}$$

Hence, if $R,\, R' \in \mathfrak{K}_I A$ are such that $\mathfrak{f}_* R = \mathfrak{f}_* R'$, the preceding computation immediately implies $R = R'$, as needed.

g) By 7.9, (2) $\Leftrightarrow$ (3) and $\lambda \circ \rho = Id_R$. Hence, it suffices to see that (1) $\Leftrightarrow$ (3). The proof of (3) $\Rightarrow$ (1) is similar to that of (1) $\Rightarrow$ (2) in 38.11.(e). For (1) $\Rightarrow$ (3), let $S \in \mathfrak{K}_I B$ and $\overline{a} \in |A|^I$; then

$$\begin{aligned}\lambda(\mathfrak{f}^* S(\overline{a})) &= \lambda\left(E\overline{a} \wedge \rho([\![S(f\overline{a})]\!])\right) = \lambda(E\overline{a}) \wedge [\![S(f\overline{a})]\!] \\ &= Ef\overline{a} \wedge [\![S(f\overline{a})]\!] = [\![S(f\overline{a})]\!].\end{aligned}$$

Hence, if $\mathfrak{f}^* S = \mathfrak{f}^* S'$, the preceding computation shows that S and S' coincide in $(fA)^I$. Since (1) implies that fA is dense in B, 37.8.(c) entails $S = S'$, as needed. $\square$

Recall that if L is a frame, D_L is the filter of dense elements in L and $Reg(L)$ is the cBa of regular elements in L, naturally isomorphic to the quotient L/D [2].

LEMMA 39.2. *Let A be a Ω-set and I be a set.*

a) $Reg(\mathfrak{K}_I A) = \{R \in \mathfrak{K}_I A : \forall\, \overline{a} \in |A|^I,\ \ E\overline{a} \wedge \neg\neg [\![R(\overline{a})]\!] = [\![R(\overline{a})]\!]\}$.

b) $$\begin{aligned} D_{\mathfrak{K}_I A} &= \{R \in \mathfrak{K}_I A : \forall\, \overline{a} \in |A|^I,\ \ E\overline{a} \leq \neg\neg [\![R(\overline{a})]\!]\} \\ &= \{R \in \mathfrak{K}_I A : \forall\, \overline{a} \in |A|^I,\ \ \neg\neg E\overline{a} = \neg\neg [\![R(\overline{a})]\!]\}.\end{aligned}$$

PROOF. Item (a) is immediate from 37.4.(d). For (b), since the top of $\mathfrak{K}_I A$ is $\top_I$, R is dense iff $\neg\neg R = \top_I$. Hence, for all $\overline{a} \in |A|^I$,

$$[\![\neg\neg R(\overline{a})]\!] = E\overline{a} \wedge \neg\neg [\![R(\overline{a})]\!] = \top_I(\overline{a}) = E\overline{a}$$

which is equivalent to $\neg\neg E\overline{a} = \neg\neg [\![R(\overline{a})]\!]$. $\square$

One might inquire as to whether the cBa $\mathfrak{K}_I A / D_{\mathfrak{K}_I A}$ corresponds to characteristic maps on some object. For *finite* I, we have

THEOREM 39.3. *Let A be a Ω-presheaf and I a finite set. Let D, D_k be the filters of dense elements in Ω and $\mathfrak{K}_I A$, respectively. Let* [3]

$$\mathfrak{h} = \langle \varepsilon_D, \pi_D \rangle : A \longrightarrow A/D$$

be the localization morphism of A at the filter D. Then,

a) $\mathfrak{h}^* : \mathfrak{K}_I(A/D) \longrightarrow Reg(\mathfrak{K}_I A)$ *is an isomorphism of complete Boolean algebras.*

b) $\mathfrak{K}_I A / D_k$ *is naturally isomorphic to* $\mathfrak{K}_I(A/D)$.

PROOF. Let $\rho : \Omega/D \longrightarrow \Omega$ be the right adjoint of π_D, $\rho(p/D) = \neg\neg p$. Hence,

(+) $$Im\,\rho = Reg(\Omega) \quad \text{and} \quad \rho(\pi_D(p)) = \neg\neg p.$$

Furthermore, since I *is finite*, if $\overline{a}, \overline{c} \in |A|^I$, then (see 34.2)

(&) $$\begin{cases} (i) & \varepsilon_D \overline{a} = \langle a_i/D \rangle_{i \in I} =_{def} \overline{a}/D; \\ (ii) & [\![\overline{a}/D = \overline{c}/D]\!] = \bigwedge_{i \in I} \pi_D([\![a_i = c_i]\!]) = \pi_D([\![\overline{a} = \overline{c}]\!]). \end{cases}$$

Since ε_D and π_D are surjective, 39.1.(g) guarantees that $\mathfrak{h}^*$ is injective. Recalling that in Ω/D

[2] See 6.19, 6.21 and 10.5.

[3] Notation as in 34.1.

$$\neg(p/D) \;=\; \neg p/D \quad \text{and} \quad (p \to q)/D \;=\; p/D \to q/D$$

items (d) and (j) in 6.8 imply that ρ preserves implication and negation. Hence, $\mathfrak{h}^*$ is a $\bigwedge$-morphism, preserving implication and negation (39.1.(e)). Because $\mathfrak{K}_I(A/D)$ is a cBa (37.4.(e)), the de Morgan laws (8.16.(h)) entail that $\mathfrak{h}^*$ is an open injection, i.e., it preserves all meets and joins, as well as implication. The same argument shows that ρ is open, in fact, an isomorphism between Ω/D and $Reg(\Omega)$.

For $S \in \mathfrak{K}_I(A/D)$ and $\overline{a} \in |A|^I$, 37.4.(d) and 6.8.(g) yield, since $Im\,\rho = Reg(\Omega)$

$$\begin{aligned}[\neg\neg\,\mathfrak{h}^* S](\overline{a}) \;&=\; E\overline{a} \wedge \neg\neg\,\mathfrak{h}^* R(\overline{a}) \;=\; E\overline{a} \wedge \neg\neg\, E\overline{a} \wedge \neg\neg\,\rho\Big(\llbracket\, S(\overline{a}/D)\,\rrbracket\Big) \\ &=\; E\overline{a} \wedge \rho\Big(\llbracket\, S(\overline{a}/D)\,\rrbracket\Big) \;=\; \mathfrak{h}^* S(\overline{a})\end{aligned}$$

and $Im\,\mathfrak{h}^* \subseteq Reg(\mathfrak{K}_I A)$. To end the proof of (a) we need :

FACT 39.4. *If $\overline{a}$, $\overline{u} \in |A|^I$, then*

a) $\rho\Big(\llbracket \overline{a}/D = \overline{u}/D \rrbracket\Big) \;=\; \neg\neg\,\llbracket \overline{a} = \overline{u} \rrbracket.$

b) $\rho\Big(\bigvee_{\overline{c} \in |A|^I} \llbracket \overline{c}/D = \overline{a}/D \rrbracket\Big) \;=\; \neg\neg\, E\overline{a}.$

c) For $R \in \mathfrak{K}_I A$, $\mathfrak{h}^*\mathfrak{h}_* R \;=\; \neg\neg\, R$.

Proof. a) Item (ii) in (&) yields

$$\rho\Big(\llbracket \overline{a}/D = \overline{u}/D \rrbracket\Big) \;=\; \rho(\pi_D(\llbracket \overline{a} = \overline{u} \rrbracket)) \;=\; \neg\neg\,\llbracket \overline{a} = \overline{u} \rrbracket.$$

b) With (a) and the fact that ρ is open we obtain

$$\begin{aligned}\rho\Big(\bigvee_{\overline{c} \in |A|^I} \llbracket \overline{c}/D = \overline{a}/D \rrbracket\Big) \;&=\; \bigvee_{\overline{c} \in |A|^I} \rho(\pi_D(\llbracket \overline{c} = \overline{a} \rrbracket)) \\ &=\; \bigvee_{\overline{c} \in |A|^I} \neg\neg\,\llbracket \overline{c} = \overline{a} \rrbracket \;=\; \neg\neg\, E\overline{a},\end{aligned}$$

as needed.

c) Since ρ is open, (a), (b) and 8.16.(g) yield, for $\overline{a} \in |A|^I$,

$$\begin{aligned}\mathfrak{h}^*[\mathfrak{h}_* R](\overline{a}) \;&=\; E\overline{a} \;\wedge\; \rho\Big(\mathfrak{h}_* R(\overline{a}/D)\Big) \\ &=\; E\overline{a} \;\wedge\; \rho\Big(\bigvee_{\overline{c} \in |A|^I} \llbracket \overline{c}/D = \overline{a}/D \rrbracket \wedge \pi_D(\llbracket\, R(\overline{c})\,\rrbracket)\Big) \\ &=\; E\overline{a} \;\wedge\; \bigvee_{\overline{c} \in |A|^I} \rho\Big(\llbracket \overline{c}/D = \overline{a}/D \rrbracket\Big) \wedge \rho(\pi_D(\llbracket\, R(\overline{c})\,\rrbracket)) \\ &=\; E\overline{a} \;\wedge\; \bigvee_{\overline{c} \in |A|^I} \neg\neg\,\llbracket \overline{c} = \overline{a} \rrbracket \wedge \neg\neg\,\llbracket\, R(\overline{c})\,\rrbracket \\ &=\; E\overline{a} \;\wedge\; \bigvee_{\overline{c} \in |A|^I} \neg\neg\,\llbracket \overline{c} = \overline{a} \rrbracket \wedge \neg\neg\,\llbracket\, R(\overline{a})\,\rrbracket \\ &=\; E\overline{a} \wedge \neg\neg\,\llbracket\, R(\overline{a})\,\rrbracket \wedge \bigvee_{\overline{c} \in |A|^I} \neg\neg\,\llbracket \overline{c} = \overline{a} \rrbracket \\ &=\; E\overline{a} \wedge \neg\neg\,\llbracket\, R(\overline{a})\,\rrbracket \wedge \neg\neg\, E\overline{a} \;=\; \llbracket\, \neg\neg\, R(\overline{a})\,\rrbracket,\end{aligned}$$

as desired. □

It follows immediately from 39.4 that $\mathfrak{h}^*$ is onto $Reg(\mathfrak{K}_I A)$, ending the proof of (a). Item (b) follows from (a) and 10.5. □

As long as we are on the subject, there is also an isomorphism between $Reg(\mathfrak{K}_I A)$ and $Reg(\mathfrak{K}_I rA)$, where rA is the regularization of A associated to double negation, as in 35.7 :

THEOREM 39.5. *Let A be a Ω-presheaf and $A \xrightarrow{r} rA$ be the regularization of A associated to double negation* (35.7). *For each finite set I, the pair $\langle r_*, r^* \rangle$ induces inverse isomorphisms between $Reg(\mathfrak{K}_I A)$ and $Reg(\mathfrak{K}_I(rA))$.*

PROOF. We assume the reader is familiar with Chapter 35, in particular with the notational conventions in 35.7. By 30.6, we may assume I to be $\underline{n} = \{1, 2, \ldots, n\}$. The fact that double negation distributes over finite meets (6.8.(g)) will be used repeatedly, without further comment. Write

$$\overline{\langle x, p \rangle} \;=\; \langle \langle x_1, p_1 \rangle, \ldots, \langle x_n, p_n \rangle \rangle$$

for a typical element of $|rA|^n$. We shall also write $\mathfrak{K}_n(*)$ for $\mathfrak{K}_I(*)$.

By 39.1.(e) [4], $r^* : \mathfrak{K}_n(rA) \longrightarrow \mathfrak{K}_n A$ is an open morphism.

FACT 39.6. *a) If $T \in \mathfrak{K}_n(rA)$ and $\overline{\langle x, p \rangle} \in |rA|^n$, then*

$$[\![T(\overline{\langle x, p \rangle})]\!] \wedge E\overline{x} \;=\; [\![T(r\overline{x})]\!].$$

b) The restriction of r^* to $Reg(\mathfrak{K}_n(rA))$ is an open embedding into $Reg(\mathfrak{K}_n A)$.

c) If $S \in \mathfrak{K}_n A$ and $\overline{\langle x, p \rangle} \in |rA|^n$, define

$$rS(\overline{\langle x, p \rangle}) \;=\; \textstyle\bigwedge_{i=1}^{n} p_j \;\wedge\; \neg\neg [\![S(\overline{x})]\!].$$

Then, $rS \in Reg(\mathfrak{K}_n(rA))$ and $r^*(rS) \;=\; \neg\neg S$.

Proof a) By [ch $2'$] in 37.2, we have

$$(+) \qquad [\![T(\overline{\langle x, p \rangle})]\!] \wedge [\![\overline{\langle x, p \rangle} = r\overline{x}]\!]_r \;=\; [\![T(r\overline{x})]\!] \wedge [\![\overline{\langle x, p \rangle} = r\overline{x}]\!]_r.$$

Since $Ex_i \leq p_i \leq \neg\neg Ex_i$, $1 \leq i \leq n$, it follows [r 1] in 35.7 that

$$[\![\overline{\langle x, p \rangle} = r\overline{x}]\!]_r \;=\; \textstyle\bigwedge_{i=1}^{n} p_i \wedge E\overline{x} \wedge \neg\neg E\overline{x} \;=\; E\overline{x} \;=\; Er\overline{x},$$

which substituted into (+) yields the stated equality.

b) If $T \in Reg(\mathfrak{K}_n(rA))$ and $\overline{a} \in |A|^n$, since r is a morphism [5], we get

$$\begin{aligned} \neg\neg T^*(\overline{a}) \;&=\; E\overline{a} \wedge \neg\neg [\![T(r\overline{a})]\!] \;=\; Er\overline{a} \wedge \neg\neg [\![T(r\overline{a})]\!] \\ &=\; [\![\neg\neg T(r\overline{a})]\!] \;=\; [\![T(r\overline{a})]\!] \;=\; r^*T(\overline{a}), \end{aligned}$$

and $T^* \in Reg(\mathfrak{K}_n A)$. To verify injectivity, suppose T, $T' \in Reg(\mathfrak{K}_n(rA))$ satisfy $r^*T = r^*T'$. Then, $[\![T(r\overline{a})]\!] = [\![T'(r\overline{a})]\!]$, for all $\overline{a} \in |A|^n$, and item (a) yields, for $\overline{\langle x, p \rangle} \in |rA|^n$,

$$(\&) \qquad [\![T(\overline{\langle x, p \rangle})]\!] \wedge E\overline{x} \;=\; E\overline{x} \wedge [\![T'(\overline{\langle x, p \rangle})]\!].$$

Taking double negation on both sides of (&) and recalling that

$$(*) \qquad E\overline{\langle x, p \rangle} \;=\; \textstyle\bigwedge_{i=1}^{n} p_i \;\leq\; \neg\neg E\overline{x},$$

we arrive at $\neg\neg [\![T(\overline{\langle x, p \rangle})]\!] \;=\; \neg\neg [\![T'(\overline{\langle x, p \rangle})]\!]$ and so $T = T'$, as desired. Since r^* is an open morphism and $Reg(\mathfrak{K}_n(rA))$ is both a Boolean algebra and a $[\bigwedge, \neg]$-sublattice of $\mathfrak{K}_n(rA)$, it follows that the restriction of r^* to the regular elements in $\mathfrak{K}_n(rA)$ is still an open morphism.

[4] Applied to the pair $\langle Id_\Omega, Id_\Omega \rangle$.

[5] And so $r^*T(\overline{a}) = E\overline{a} \wedge [\![T(r\overline{a})]\!] = Er\overline{a} \wedge [\![T(r\overline{a})]\!] = [\![T(\overline{a})]\!]$.

c) It is clear from (*) above that rS satisfies [ch 1] in 37.1. For the Leibniz rule we have

$$\begin{aligned}
[\![\, rS(\overline{\langle\, x, p\,\rangle})\,]\!] \wedge [\![\overline{\langle\, x, p\,\rangle} = \overline{\langle\, y, q\,\rangle}]\!]_r \ = & \\
&= \ \textstyle\bigwedge_{i=1}^{n} p_i \wedge \neg\neg\,[\![\, S(\overline{x})\,]\!] \wedge \bigwedge_{i=1}^{n} q_i \wedge \neg\neg\,[\![\overline{x} = \overline{y}]\!] \\
&\leq \ \textstyle\bigwedge_{i=1}^{n} q_i \wedge \neg\neg\,([\![\, S(\overline{x})\,]\!] \wedge [\![\overline{x} = \overline{y}]\!]) \\
&\leq \ \textstyle\bigwedge_{i=1}^{n} q_i \wedge \neg\neg\,[\![\, S(\overline{y})\,]\!] \ = \ [\![\, rS(\overline{\langle\, y, q\,\rangle})\,]\!],
\end{aligned}$$

and $rS \in \mathfrak{K}_n(rA)$. It is straightforward that $r^*(rS) = \neg\neg\, S$.

It is immediate from (b) and (c) in 39.6 that r^* is an isomorphism from $Reg(\mathfrak{K}_n(rA))$ onto $Reg(\mathfrak{K}_n A)$, whose inverse is, in fact, r_* [6]. □

Remark 39.7. Theorem 39.5 can be generalized to other regularization functors. Let $k : L \longrightarrow \Omega$ be a frame-morphism with right adjoint g. If A is a L-presheaf, $n \geq 1$ is a positive integer, $R \in \mathfrak{K}_n$A and $\overline{a} \in |A|^n$, define

$$[\![\, gkR(\overline{a})\,]\!] \ = \ E\overline{a} \wedge gk([\![\, R(\overline{a})\,]\!]).$$

Then, $gkR \in \mathfrak{K}_n A$. Let

$$\mathcal{R}_k(\mathfrak{K}_n A) \ = \ \{R \in \mathfrak{K}_n A : gkR = R\}.$$

If $k^r : A \longrightarrow k^r A$ is the regularization morphism in 35.1, the adjoint pair $\langle\, k^r_*, k^{r*}\,\rangle$ establishes algebraic connections between $\mathcal{R}_k(\mathfrak{K}_n A)$ and $\mathcal{R}_k(\mathfrak{K}_n(k^r A))$, that depend on the properties of k and g. □

[6] Hint : $rS = r_*(\neg\neg\, S)$.

CHAPTER 40

Dependence on Coordinates

This Chapter generalizes to Ω-sets the content of Definition 24.28.

DEFINITION 40.1. *Let A_i, $i \in I$, be Ω-sets. For $\overline{x}$, $\overline{y} \in \prod_{i\in I} |A_i|$, define*

$$\mathfrak{c}(\overline{x},\overline{y}) \;=\; \{i \in I : [\![x_i \,=\, y_i]\!] = Ex_i \wedge Ey_i\}$$

i.e., $\mathfrak{c}(\overline{x},\overline{y})$ is the set of $i \in I$ wherein x_i and y_i are compatible in A_i. It is clear that $\mathfrak{c}(\overline{x},\overline{y}) = \mathfrak{c}(\overline{y},\overline{x})$ [1].

DEFINITION 40.2. *Let B, A be Ω-sets, I be a set and $J \subseteq I$.*

a) A characteristic map $R \in \mathfrak{K}_I A$ **depends on J** *if for all $\overline{x}$, $\overline{y} \in |A|^I$*

$$J \subseteq \mathfrak{c}(\overline{x},\overline{y}) \quad \Rightarrow \quad E\overline{x} \wedge [\![\, R(\overline{y}) \,]\!] \;=\; E\overline{y} \wedge [\![\, R(\overline{x}) \,]\!].$$

Write $\mathfrak{K}_I(A,J)$ for the set of $R \in \mathfrak{K}_I A$ that depend on J.

b) A subset of A^I **depends on J** *if its characteristic map* (37.13) *depends on J.*

c) Let A_i, $i \in I$, be Ω-sets. A morphism $f : \prod_{i\in I} A_i \longrightarrow B$ **depends on J** *if for all $\overline{a}$, $\overline{c} \in |\prod_{i\in I} A_i|$*

$$J \subseteq \mathfrak{c}(\overline{a},\overline{c}) \quad \Rightarrow \quad [\![f(\overline{a}) \,=\, f(\overline{c})]\!] \;=\; E\overline{a} \wedge E\overline{c},$$

that is, $f(\overline{a})$ and $f(\overline{c})$ are compatible in B. In particular, if B is extensional and $E\overline{a} = E\overline{c}$, then $f(\overline{a}) = f(\overline{c})$. Write $[A^I,\, B]_J$ for the collection of morphisms from A^I to B that depend on J.

LEMMA 40.3. *Let $J \subseteq I$ be sets and A, B, A_i, $i \in I$, be Ω-sets and let $P = \prod_{i\in I} A_i$. Let $f : P \longrightarrow B$ be a Ω-set morphism.*

a) If $R \in \mathfrak{K}_I A$ and $\overline{a}$, $\overline{c} \in |P|$ are such that $\mathfrak{c}(\overline{a},\overline{c}) = I$, then

$$E\overline{a} \wedge [\![\, R(\overline{c}) \,]\!] \;=\; E\overline{c} \wedge [\![\, R(\overline{a}) \,]\!].$$

b) For $R \in \mathfrak{K}_I A$, consider the following conditions :

(1) *R depends on J;*

(2) *For all $\overline{a}$, $\overline{c} \in |P|$, $\pi_J(\overline{a}) = \pi_J(\overline{c}) \;\Rightarrow\; E\overline{a} \wedge [\![\, R(\overline{c}) \,]\!] = E\overline{c} \wedge [\![\, R(\overline{a}) \,]\!]$.*

(3) *For all $\overline{a}$, $\overline{c} \in |P|$,*

$$\pi_J(\overline{a}) = \pi_J(\overline{c}) \;\Rightarrow\; E\pi_{J^c}(\overline{a}) \wedge [\![\, R(\overline{c}) \,]\!] = E\pi_{J^c}(\overline{c}) \wedge [\![\, R(\overline{a}) \,]\!].$$

(4) *For all $\overline{a}$, $\overline{c} \in P$, $\;E\overline{a} = E\overline{c}$ and $\pi_J(\overline{a}) = \pi_J(\overline{c}) \;\Rightarrow\; [\![\, R(\overline{c}) \,]\!] = [\![\, R(\overline{a}) \,]\!]$.*

Then, (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Rightarrow$ (4)*. If A is a presheaf, these conditions are equivalent.*

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[1]However, it is not true in general that $\mathfrak{c}(\overline{x},\overline{y}) \cap \mathfrak{c}(\overline{y},\overline{z}) \subseteq \mathfrak{c}(\overline{x},\overline{z})$.

c) Consider the following conditions :

(1) *f depends on J;*

(2) *For all $\overline{x}$, $\overline{y} \in |P|$, $\quad \pi_J(\overline{x}) = \pi_J(\overline{y}) \quad \Rightarrow \quad [\![f\overline{x} = f\overline{y}]\!] = E\overline{x} \wedge E\overline{y}$;*

(3) *For all $\overline{x}$, $\overline{y} \in |P|$, $\quad E\overline{x} = E\overline{y}$ and $\pi_J(\overline{x}) = \pi_J(\overline{y}) \quad \Rightarrow \quad f\overline{x} = f\overline{y}$.*

Then, (1) $\Rightarrow$ (2) $\Rightarrow$ (3). *If B, A_i, $i \in I$, are presheaves, these conditions are equivalent.*

Proof. a) If the conditions in the statement are met, then

$$[\![\overline{a} = \overline{c}]\!] \;=\; \textstyle\bigwedge_{i\in I} [\![a_i = c_i]\!] \;=\; \textstyle\bigwedge_{i\in I} Ea_i \wedge Ec_i \;=\; E\overline{a} \wedge E\overline{c},$$

and the definition of characteristic map (37.1) entails $E\overline{a} \wedge [\![\, R(\overline{c})\,]\!] = E\overline{c} \wedge [\![\, R(\overline{a})\,]\!]$.

b) Note that

$$\pi_J(\overline{a}) = \pi_J(\overline{c}) \quad \Rightarrow \quad J \subseteq \mathfrak{c}(\overline{a}, \overline{c}),$$

to obtain (1) $\Rightarrow$ (2). Since $[\![\, R(\overline{u})\,]\!] \leq E\pi_J\overline{u}$ for all $\overline{u} \in |A|^I$, we have (2) $\Rightarrow$ (3). For the converse take the meet on both sides of the equality in (3) with $E\pi_J\overline{a} = E\psi_J\overline{c}$ to obtain (2). Clearly, (2) $\Rightarrow$ (4). To complete the first batch of equivalences, it will be verified that (2) $\Rightarrow$ (1).

If $\overline{u}$, $\overline{v} \in |A|^I$ verify $J \subseteq \mathfrak{c}(\overline{u}, \overline{v})$, consider $\overline{x} \in |A|^I$, defined by

$$x_i \;=\; \begin{cases} v_i & \text{if } i \notin J \\ u_i & \text{if } i \in J. \end{cases}$$

Then,

(i) $\pi_J\overline{x} = \pi_J\overline{u}$; $\qquad (ii)$ $E\overline{x} \;=\; E\pi_{J^c}\overline{v} \wedge E\pi_J\overline{u}$;

$$\begin{aligned} (iii)\ [\![\overline{v} = \overline{x}]\!] \;&=\; E\pi_{J^c}\overline{v} \wedge [\![\pi_J\overline{v} = \pi_J\overline{u}]\!] \;=\; E\pi_{J^c}\overline{v} \wedge \textstyle\bigwedge_{i\in J} [\![v_i = u_i]\!] \\ &=\; E\pi_{J^c}\overline{v} \wedge \textstyle\bigwedge_{i\in J} Ev_i \wedge Eu_i \;=\; E\pi_{J^c}\overline{v} \wedge E\pi_J\overline{v} \wedge E\pi_J\overline{u} \\ &=\; E\overline{v} \wedge E\pi_J\overline{u}. \end{aligned}$$

Hence, (2), (i) and (iii) entail

$$\text{(A)} \qquad E\overline{u} \wedge [\![\, R(\overline{x})\,]\!] \;=\; E\overline{x} \wedge [\![\, R(\overline{u})\,]\!] \;=\; E\pi_{J^c}\overline{v} \wedge [\![\, R(\overline{u})\,]\!].$$

Taking the meet with $E\overline{v}$ on both sides of (A) yields

$$\text{(B)} \qquad E\overline{u} \wedge E\overline{v} \wedge [\![\, R(\overline{x})\,]\!] \;=\; E\overline{v} \wedge [\![\, R(\overline{u})\,]\!].$$

Now, (ii) implies
$$\begin{aligned} E\overline{u} \wedge E\overline{v} \wedge [\![\, R(\overline{x})\,]\!] \;&=\; E\overline{u} \wedge E\pi_J\overline{u} \wedge E\overline{v} \wedge [\![\, R(\overline{x})\,]\!] \\ &=\; E\overline{u} \wedge [\![\overline{v} = \overline{x}]\!] \wedge [\![\, R(\overline{x})\,]\!] \\ &\leq\; E\overline{u} \wedge [\![\, R(\overline{v})\,]\!], \end{aligned}$$

that, together with (B), entails $E\overline{v} \wedge [\![\, R(\overline{u})\,]\!] \;\leq\; E\overline{u} \wedge [\![\, R(\overline{v})\,]\!]$. Since the argument is symmetrical in $\overline{u}$, $\overline{v}$, we conclude the equality needed to establish (1).

To show that (4) $\Rightarrow$ (1), assume that A is a Ω-presheaf. If $J \subseteq \mathfrak{c}(\overline{a}, \overline{c})$, set

$$p = E\overline{a} \wedge E\overline{c}, \quad \overline{x} = \overline{a}_{|p} \quad \text{and} \quad \overline{y} = \overline{c}_{|p}.$$

Then, $E\overline{a}_{|p} = E\overline{c}_{|p} = p$; if $i \in \mathfrak{c}(\overline{a}, \overline{c})$, then $Ex_i = Ey_i = p$, with

$$[\![x_i = y_i]\!] = [\![{a_i}_{|p} = {c_i}_{|p}]\!] = p \wedge [\![a_i = c_i]\!] = p \wedge Ea_i \wedge Ec_i = p,$$

and so the extensionality of A_i entails $x_i = y_i$. Since $J \subseteq \mathfrak{c}(\overline{a}, \overline{c})$, we conclude that $\pi_J(\overline{x}) = \pi_J(\overline{y})$. Now (3) and 37.12.(a) yield

$$\begin{aligned} E\overline{a} \wedge [\![R(\overline{c})]\!] \;&=\; E\overline{a} \wedge E\overline{c} \wedge [\![R(\overline{c})]\!] \;=\; [\![R(\overline{c}_{|p})]\!] \;=\; [\![R(\overline{a}_{|p})]\!] \\ &=\; E\overline{c} \wedge E\overline{a} \wedge [\![R(\overline{a})]\!] \;=\; E\overline{c} \wedge [\![R(\overline{a})]\!], \end{aligned}$$

as needed. The arguments for item (b) are similar. $\square$

Lemma 40.4. *Let A be a Ω-set and I be a set.*
a) For $R \in \mathfrak{K}_I A$, the following conditions are equivalent :

(1) *R depends on $\emptyset$;* (2) *For all $\overline{a} \in |A|^I$, $[\![R(\overline{a})]\!] \;=\; ER \wedge E\overline{a}$* [2].

b) The map $R \in \mathfrak{K}_I(A, \emptyset) \;\longmapsto\; ER \in (EA)^{\leftarrow}$ is a regular embedding and an isomorphism whenever A is a Ω-presheaf.

Proof. a) It is clear that (2) $\Rightarrow$ (1); for the converse, if (1) holds, then R satisfies $\forall\, \overline{a}, \overline{c} \in |A|^I$, $E\overline{a} \wedge [\![R(\overline{c})]\!] = E\overline{c} \wedge [\![R(\overline{a})]\!]$. Taking joins with respect to $\overline{c}$ on both sides yields, $E\overline{a} \wedge ER \;=\; [\![R(\overline{a})]\!] \wedge \bigvee_{\overline{c} \in |A|^I} E\overline{c} \;=\; [\![R(\overline{a})]\!]$, as needed.

b) It is straightforward that the displayed map is a regular embedding. The isomorphism in case A is a presheaf is left to the reader. $\square$

Proposition 40.5. *Let A be a Ω-set and I be a set. Let J, K and J_λ, $\lambda \in \Lambda$, be subsets of I.*

a) $J \subseteq K \subseteq I \;\Rightarrow\; \mathfrak{K}_I(A, J) \subseteq \mathfrak{K}_I(A, K)$.

b) $R_\lambda \in \mathfrak{K}_I(A, J_\lambda)$, $\lambda \in \Lambda \;\Rightarrow\; \bigvee_{\lambda \in \Lambda} R_\lambda, \bigwedge_{\lambda \in \Lambda} R_\lambda \in \mathfrak{K}_I(A, \bigcup J_\lambda)$.

c) $R \in \mathfrak{K}_I(A, J)$, $S \in \mathfrak{K}_I(A, K) \;\Rightarrow\; \begin{cases} \neg R \;\in\; \mathfrak{K}_I(A, J) \\ (R \to S) \;\in\; \mathfrak{K}_I(A, J \cup K). \end{cases}$

d) With the operations induced by $\mathfrak{K}_I A$, $\mathfrak{K}_I(A, J)$ is a frame and the canonical inclusion into $\mathfrak{K}_I A$ is an open embedding.

e) If Λ is finite and $J = \bigcap_{\lambda \in \Lambda} J_\lambda$, then

(1) *$\bigcap_{\lambda \in \Lambda} \mathfrak{K}_I(A, J_\lambda) \;=\; \mathfrak{K}_I(A, J)$.*

(2) *If B, A_i, $i \in I$, are Ω-presheaves, then*

$$\bigcap_{\lambda \in \Lambda} \left[\prod_{i \in I} A_i,\, B\right]_{J_\lambda} \;=\; \left[\prod_{i \in I} A_i,\, B\right]_J.$$

Proof. Item (a) is clear. For (b), suppose $\overline{a}, \overline{c} \in |A|^I$ satisfy $J_\lambda \subseteq \mathfrak{c}(\overline{a}, \overline{c})$, for all $\lambda \in \Lambda$.

Then,
$$\begin{aligned} E\overline{c} \wedge [\![\bigvee_{\lambda \in \Lambda} R_\lambda(\overline{a})]\!] \;&=\; \bigvee_{\lambda \in \Lambda} E\overline{c} \wedge [\![R_\lambda(\overline{a})]\!] \;=\; \bigvee_{\lambda \in \Lambda} E\overline{a} \wedge [\![R_\lambda(\overline{c})]\!] \\ &=\; E\overline{a} \wedge [\![\bigvee_{\lambda \in \Lambda} R_\lambda(\overline{c})]\!], \end{aligned}$$

and $\bigvee_{\lambda \in \Lambda} R_\lambda$ depends on $\bigcup_{\lambda \in \Lambda} J_\lambda$. The case of meets is analogous.

c) If $J \cup K \subseteq \mathfrak{c}(\overline{a}, \overline{c})$, then 6.4.(i) entails

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[2] Recall from 37.1 that $ER = \bigvee_{\overline{a} \in |A|^I} [\![R(\overline{a})]\!]$ is the extent of R.

$$\begin{aligned}
E\overline{c} \wedge [\![(R \to S)(\overline{a})]\!] &= E\overline{c} \wedge E\overline{a} \wedge ([\![R(\overline{a})]\!] \to [\![S(\overline{a})]\!]) \\
&= E\overline{c} \wedge E\overline{a} \wedge [(E\overline{c} \wedge [\![R(\overline{a})]\!]) \to (E\overline{c} \wedge [\![S(\overline{a})]\!])] \\
&= E\overline{c} \wedge E\overline{a} \wedge [(E\overline{a} \wedge [\![R(\overline{c})]\!]) \to (E\overline{a} \wedge [\![S(\overline{c})]\!])] \\
&= E\overline{a} \wedge [\![(R \to S)(\overline{c})]\!],
\end{aligned}$$

as needed. The case of negation is similar. Item (d) is an immediate consequence of the preceding.

e) Proof of (1) : By (a), $\mathfrak{K}_I(A, J) \subseteq \bigcap_{\lambda \in \Lambda} \mathfrak{K}_I(A, J_\lambda)$. For the reverse inclusion, it is enough to verify the statement for J_1, $J_2 \subseteq I$ and use induction. If $J_1 \subseteq J_2$ or vice-versa there is nothing to prove. Assume then that $R \in \mathfrak{K}_I A$ depends on J_1 and J_2 and that $\overline{a}$, $\overline{c} \in |A|^I$ verify $J \subseteq \mathfrak{c}(\overline{a}, \overline{c})$. Consider $\overline{x} \in |A|^I$, defined by

$$x_i = \begin{cases} a_i & \text{if } i \in J_1 \\ c_i & \text{if } i \notin J_1. \end{cases}$$

Then : (i) $\overline{x}$ coincides with $\overline{a}$ in J_1 and $J_2 \subseteq \mathfrak{c}(\overline{x}, \overline{c})$ [3].

(ii) $E\overline{x} = E\pi_{J_1^c}\overline{c} \wedge E\pi_{J_1}\overline{a} \geq E\overline{c} \wedge E\overline{a}$ and so

$$E\overline{a} \wedge [\![R(\overline{c})]\!] \leq E\overline{x} \quad \text{and} \quad E\overline{c} \wedge [\![R(\overline{a})]\!] \leq E\overline{x}.$$

Hence, 40.3.(b) and (i) imply

$$(*) \qquad \begin{cases} E\overline{x} \wedge [\![R(\overline{c})]\!] = E\overline{c} \wedge [\![R(\overline{x})]\!] \\ E\overline{x} \wedge [\![R(\overline{a})]\!] = E\overline{a} \wedge [\![R(\overline{x})]\!]. \end{cases}$$

Now, (ii) and (*) yield

$$\begin{aligned}
E\overline{a} \wedge [\![R(\overline{c})]\!] &= E\overline{a} \wedge E\overline{x} \wedge [\![R(\overline{c})]\!] = E\overline{a} \wedge E\overline{c} \wedge [\![R(\overline{x})]\!] \\
&= E\overline{c} \wedge E\overline{x} \wedge [\![R(\overline{a})]\!] = E\overline{c} \wedge [\![R(\overline{a})]\!],
\end{aligned}$$

as needed.

For (2), the above argument shows that if f is a morphism from $\prod_{i \in I} A_i$ to B depending on J_1 and J_2, and $\overline{a}$, $\overline{c} \in |A|^I$ verify $E\overline{a} = E\overline{c}$ and $J \subseteq \mathfrak{c}(\overline{a}, \overline{c})$, then $f(\overline{a}) = f(\overline{c})$. Since domain and codomain are presheaves, the desired conclusion follows from 40.3.(c). □

Example 40.6. Item (e) in Proposition 40.5 is false for infinite intersections, even in the classical, set-theoretic, case. As an example, let $\mathcal{F}$ be the filter of cofinite subsets in $\mathbb{N}$ and consider the canonical projection

$$\pi_{\mathcal{F}} : \mathbb{N}^{\mathbb{N}} \longrightarrow \mathbb{N}^{\mathbb{N}}/\mathcal{F}$$

from $\mathbb{N}^{\mathbb{N}}$ to the reduced power modulo $\mathcal{F}$. Since

$$\pi_{\mathcal{F}}(s) = \pi_{\mathcal{F}}(t) \quad \text{iff} \quad \{k \in \mathbb{N} : s(k) = t(k)\} \in \mathcal{F},$$

it is clear that $\pi_{\mathcal{F}}$ depends on every cofinite set. However, since it *is not* constant, it does not depend on $\emptyset$, the intersection of all cofinite sets. To obtain an example of a characteristic map violating 40.5.(e).(1) for infinite intersections, it suffices to consider the characteristic map of the equivalence relation that originates $\pi_{\mathcal{F}}$. □

Remark 40.7. If A is a Ω-set and I is a set, by Proposition 40.5 the functor

[3] In J, $\overline{a}$ is compatible with $\overline{c}$; in $J_2 - J$, $\overline{x}$ *coincides* with $\overline{c}$.

$$\mathfrak{D} \;:\; \begin{cases} J \in 2^I & \longmapsto & \mathfrak{K}_I(A, J) \\ J \subseteq K & \longmapsto & \iota_{JK} : \mathfrak{K}_I(A, J) \longrightarrow \mathfrak{K}_I(A, K), \end{cases}$$

where ι_{JK} is the canonical inclusion, is an inductive system of frames and open embeddings over the poset $\langle 2^I, \subseteq \rangle$. It is clear that $\mathfrak{K}_I A$ is the inductive limit of this system, i.e., $\mathfrak{K}_I A \;=\; \varinjlim \;\mathfrak{D}$. $\square$

We now show that $\mathfrak{K}_I(A, J)$ is naturally isomorphic to $\mathfrak{K}_J A$.

THEOREM 40.8. *Let A be a Ω-set and I be a set. For $J \subseteq I$, let*

$$\pi_J : \bigotimes^I A \longrightarrow \bigotimes^J A$$

be the projection that forgets the coordinates outside J [4]. *Then, π_J^* is an isomorphism from $\mathfrak{K}_J A$ onto $\mathfrak{K}_I(A, J)$.*

PROOF. By 38.16, $\pi_J^* : \mathfrak{K}_J A \longrightarrow \mathfrak{K}_I A$ is an open embedding; thus, it is enough to show that its image is $\mathfrak{K}_I(A, J)$. For $R \in \mathfrak{K}_J A$ and $\overline{a} \in |A|^I$, we have

$$[\![\, \pi_J^* R(\overline{a}) \,]\!] \;=\; E\overline{a} \wedge [\![\, R(\pi_J \overline{a}) \,]\!].$$

Hence, if $\overline{a}$, $\overline{c} \in |A|^I$ verify $J \subseteq \mathfrak{c}(\overline{a}, \overline{c})$, then, $\mathfrak{c}(\pi_J(\overline{a}),\; \pi_J(\overline{c})) = J$ and so $[E_J]$ in 38.9 and 40.3.(a) yield

$$\begin{aligned} E\overline{c} \wedge [\![\, \pi_J^* R(\overline{a}) \,]\!] \;&=\; E\overline{c} \wedge E\overline{a} \wedge [\![\, R(\pi_J \overline{a}) \,]\!] \\ &=\; E\overline{a} \wedge E\pi_{J^c}\overline{c} \wedge E\pi_J\overline{c} \wedge [\![\, R(\pi_J \overline{a}) \,]\!] \\ &=\; E\overline{a} \wedge E\pi_{J^c}\overline{c} \wedge E\pi_J\overline{a} \wedge [\![\, R(\pi_J \overline{c} \,]\!] \\ &=\; E\overline{a} \wedge E\pi_{J^c}\overline{c} \wedge [\![\, R(\pi_J \overline{c}) \,]\!] \\ &=\; E\overline{a} \wedge E\pi_{J^c}\overline{c} \wedge E\pi_J\overline{c} \wedge [\![\, R(\pi_J \overline{c}) \,]\!] \\ &=\; E\overline{a} \wedge E\overline{c} \wedge [\![\, R(\pi_J \overline{c}) \,]\!] \;=\; E\overline{a} \wedge [\![\, \pi_J^* R(\overline{c}) \,]\!], \end{aligned}$$

and $\pi_J^* R \in \mathfrak{K}_I(A, J)$. To verify that π_J^* is onto $\mathfrak{K}_I(A, J)$, let S be an element of $\mathfrak{K}_I(A, J)$. For $\overline{z} \in |A|^J$, define [5]

$$[\![\, R(\overline{z}) \,]\!] \;=\; \textstyle\bigvee_{\overline{e} \in |A|^{I-J}} [\![\, S(\langle\, \overline{z}; \overline{e}\,\rangle) \,]\!].$$

Then, $R \in \mathfrak{K}_J A$; clearly, it satisfies [ch 1] in 37.1. If $\overline{x}$, $\overline{y} \in |A|^J$ and $\overline{e} \in |A|^{I-J}$, note that $[\![\langle\, \overline{x}; \overline{e}\,\rangle = \langle\, \overline{y}; \overline{e}\,\rangle]\!] \;=\; [\![\overline{x} = \overline{y}]\!] \wedge E\overline{e}$, and so, since $[\![\, S(\langle\, \overline{x}; \overline{e}\,\rangle) \,]\!] \leq E\overline{e}$,

$$\begin{aligned} [\![\overline{x} = \overline{y}]\!] \wedge [\![\, R(\overline{x}) \,]\!] \;&=\; [\![\overline{x} = \overline{y}]\!] \wedge \textstyle\bigvee_{\overline{e} \in |A|^{I-J}} [\![\, S(\langle\, \overline{x}; \overline{e}\,\rangle) \,]\!] \\ &=\; \textstyle\bigvee_{\overline{e} \in |A|^{I-J}} [\![\overline{x} = \overline{y}]\!] \wedge E\overline{e} \wedge [\![\, S(\langle\, \overline{x}; \overline{e}\,\rangle) \,]\!] \\ &=\; \textstyle\bigvee_{\overline{e} \in |A|^{I-J}} [\![\langle\, \overline{x}; \overline{e}\,\rangle = \langle\, \overline{y}; \overline{e}\,\rangle]\!] \wedge [\![\, S(\langle\, \overline{x}; \overline{e}\,\rangle) \,]\!] \\ &\leq\; \textstyle\bigvee_{\overline{e} \in |A|^{I-J}} [\![\, S(\langle\, \overline{y}; \overline{e}\,\rangle) \,]\!] \;=\; [\![\, R(\overline{y}) \,]\!], \end{aligned}$$

verifying [ch 2]. Note that the above holds true for *any* S in $\mathfrak{K}_I A$. We shall now verify that $\pi_J^* R = S$. For $\overline{a} \in |A|^I$,

$$(1) \qquad [\![\, \pi_J^* R(\overline{a}) \,]\!] \;=\; [\![\, R(\pi_J \overline{a}) \,]\!] \wedge E\overline{a} \;=\; \textstyle\bigvee_{\overline{e} \in |A|^{I-J}} E\overline{a} \wedge [\![\, S(\langle\, \pi_J(\overline{a}); \overline{e}\,\rangle) \,]\!].$$

Since $S \in \mathfrak{K}_I(A, J)$ and $\pi_J \overline{a} = \pi_J(\langle\, \pi_J \overline{a}; \overline{e}\,\rangle)$, 40.3.(b) entails

$$E\overline{a} \wedge [\![\, S(\langle\, \pi_J \overline{a};\, \overline{e}\,\rangle) \,]\!] \;=\; E\langle\, \pi_J \overline{a};\, \overline{e}\,\rangle \wedge [\![\, S(\overline{a}) \,]\!],$$

[4] As in 38.9 and 38.16.

[5] Notation as in 37.24.

which substituted into (1) yields, recalling 25.15.(a),

$$\begin{aligned}[\![\, \pi_J^* R(\overline{a}) \,]\!] &= \bigvee_{\overline{e} \in |A|^{I-J}} E\langle\, \pi_J \overline{a};\, \overline{e}\,\rangle \wedge [\![\, S(\overline{a}) \,]\!] \;=\; [\![\, S(\overline{a}) \,]\!] \wedge \bigvee_{\overline{e} \in |A|^{I-J}} E\langle\, \pi_J \overline{a};\, \overline{e}\,\rangle \\ &= [\![\, S(\overline{a}) \,]\!] \wedge E\pi_J \overline{a} \wedge \bigvee_{\overline{e} \in |A|^{I-J}} E\overline{e} \;=\; [\![\, S(\overline{a}) \,]\!] \wedge EA \;=\; [\![\, S(\overline{a}) \,]\!],\end{aligned}$$

ending the proof. $\square$

REMARK 40.9. Direct image by a morphism *does not* preserve dependence on coordinates, even in the classical, set-theoretic, setting. If $f : A \longrightarrow B$ is a *non-surjective* map and I is a set, the characteristic map of A^I depends on the empty set, but its image under f, that is the characteristic map of $f(A)^I$ (38.15.(a)), depends on I and on no smaller subset. In particular, the image of a characteristic map that depends on a finite set may turn out to be dependent on an infinite set of coordinates. This Remark illustrates the result that follows. $\square$

PROPOSITION 40.10. *Let L, R be frames and A, B be presheaves over L and R, respectively. Let $\mathfrak{f} = \langle\, f, \lambda \,\rangle : A \longrightarrow B$ be a morphism in $\boldsymbol{pSh}$, with λ a frame-morphism and $J \subseteq I$ be sets. Let $\mathfrak{f}^I = \langle\, f^I, \lambda \,\rangle$.*

a) $T \in \mathfrak{K}_I(B, J) \quad \Rightarrow \quad \mathfrak{f}^{I*}T \in \mathfrak{K}_I(A, J)$.

b) With notation as in the diagram below, the left adjoint of $\mathfrak{f}^*{}_{|\mathfrak{K}_I(B,J)}$ *is* ${}^B\pi_J^* \circ \mathfrak{f}_*^J \circ {}^A\pi_{J*} \;=\; {}^B\pi_J^* \circ {}^B\pi_{J*} \circ \mathfrak{f}_*^I$.

$$\begin{tikzcd}[column sep=large, row sep=large] \bigotimes^I A \arrow[r, "\mathfrak{f}^I"] \arrow[d, "\langle\, {}^A\pi_J, Id \,\rangle"'] & \bigotimes^I B \arrow[d, "\langle\, {}^B\pi_J, Id \,\rangle"] \\ \bigotimes^J A \arrow[r, "\mathfrak{f}^J"'] & \bigotimes^J B \end{tikzcd}$$

PROOF. a) With notation as in 39.1, let ρ be the right adjoint of λ and assume that $\overline{a}$, $\overline{c} \in |A|^I$ verify $J \subseteq \mathfrak{c}(\overline{a}, \overline{c})$. Then, $J \subseteq \mathfrak{c}(f\overline{a}, f\overline{c})$ and so for $T \in \mathfrak{K}_I(B, J)$ we have

$$(1) \qquad Ef\overline{a} \wedge [\![\, T(f\overline{c}) \,]\!] \;=\; Ef\overline{c} \wedge [\![\, T(f\overline{a}) \,]\!].$$

Hence, since $\rho \circ \lambda \geq Id_L$ (7.8.(a)), (1) entails

$$\begin{aligned} E\overline{a} \wedge \mathfrak{f}^{I*}T(\overline{c}) &= E\overline{a} \wedge E\overline{c} \wedge \rho([\![\, T(f\overline{c}) \,]\!]) \\ &\leq E\overline{c} \wedge E\overline{a} \wedge \rho(\lambda(E\overline{a})) \wedge \rho([\![\, T(f\overline{c}) \,]\!]) \\ &= E\overline{c} \wedge E\overline{a} \wedge \rho\Big(Ef\overline{a} \wedge [\![\, T(f\overline{c}) \,]\!]\Big) \\ &= E\overline{c} \wedge E\overline{a} \wedge \rho\Big(Ef\overline{c} \wedge [\![\, T(f\overline{a}) \,]\!]\Big) \\ &= E\overline{c} \wedge E\overline{a} \wedge \rho(\lambda(E\overline{c})) \wedge \rho([\![\, T(f\overline{a}) \,]\!]) \\ &= E\overline{c} \wedge E\overline{a} \wedge \rho([\![\, T(f\overline{a}) \,]\!]) \;=\; E\overline{c} \wedge \mathfrak{f}^{I*}T(\overline{a}). \end{aligned}$$

Since the argument is symmetrical in $\overline{a}$ and $\overline{c}$, we conclude that $\mathfrak{f}^{I*}T \in \mathfrak{K}_I(A, J)$, as desired.

b) The equality of the terms in the statement follows from the commutativity of the displayed diagram and 39.1.(h) [6]. To verify adjointness, let $S \in \mathfrak{K}_I(A, J)$ and $T \in \mathfrak{K}_I(B, J)$. By 40.8, there is $T' \in \mathfrak{K}_J B$, such that ${}^B\pi_J^* T' = T$. Hence, the adjointness of the pairs $\langle \mathfrak{f}_*, \mathfrak{f}^* \rangle$ and $\langle \pi_{J*}, \pi_J^* \rangle$ yield

$$\begin{array}{lll} {}^B\pi_J^* \circ \mathfrak{f}_*^J \circ {}^A\pi_{J*}(S) \ \leq \ T & \text{iff} & {}^B\pi_J^* \circ \mathfrak{f}_*^J \circ {}^A\pi_{J*}(S) \ \leq \ {}^B\pi_J^*(T') \\ & \text{iff} & \mathfrak{f}_*^J \circ {}^A\pi_{J*}(S) \ \leq \ T' \\ & \text{iff} & {}^A\pi_{J*}(S) \ \leq \ \mathfrak{f}^{J*}(T') \\ & \text{iff} & S \ \leq \ {}^A\pi_J^* \circ \mathfrak{f}^{J*}(T') \ = \ \mathfrak{f}^{I*} \circ {}^B\pi_J^*(T') \\ & & \qquad\qquad\qquad\quad\ = \ \mathfrak{f}^{I*}(T), \end{array}$$

completing the proof. □

Remark 40.11. a) By 40.10.(b), the left adjoint of the restriction of $\mathfrak{f}^*$ to $\mathfrak{K}_I(B, J)$ is obtained as follows : given $S \in \mathfrak{K}_I(A, J)$, we take the *saturation* of $\mathfrak{f}_*S$ along π_J, that is, the inverse image of its projection in $\mathfrak{K}_J B$. If λ is the left adjoint of the said restriction, then for $S \in \mathfrak{K}_I(A, J)$ and $\bar{b} \in |B|^I$,

$$\lambda S(\bar{b}) \ = \ E\bar{b} \wedge \bigvee_{\overline{u} \in |B|^{I-J}} \mathfrak{f}_* S(\pi_J(\bar{b});\ \overline{u}).$$

b) The reader can check that statement and proof of 40.10 are valid for Ω-sets. □

Proposition 40.10, together with Theorems 40.8, 39.3 and 39.5 yield

Corollary 40.12. *Let A be a Ω-presheaf and D be the filter of dense elements in Ω. Let*

$$\mathfrak{h} = \langle \varepsilon_D, \pi_D \rangle : A \longrightarrow A/D \quad \textit{and} \quad r : A \longrightarrow rA$$

be the localization of A at D (34.1) *and the regularization associated to double negation* (35.7). *If I is a set, then for all* finite *$J \subseteq I$,*

(1) $\mathfrak{h}^*$ *takes $\mathfrak{K}_I(A/D,\ J)$ isomorphically onto $Reg(\mathfrak{K}_I(A, J))$;*

(2) r^* *takes $Reg(\mathfrak{K}_I(rA, J))$ isomorphically onto $Reg(\mathfrak{K}_I(A, J))$.* □

The analog of Theorem 40.8 for morphisms is left to the reader. Recall that $[A, B]$ is the set of Ω-set morphisms from A to B.

Theorem 40.13. *For $J \subseteq I$, let $\pi_J : \prod_{i \in I} A_i \longrightarrow \prod_{j \in J} A_j$ be the* morphism *that forgets the coordinates outside J, where A_i, $i \in I$, are Ω-sets. If B is a Ω-set, the map*

$$f \in \left[\prod_{j \in J} A_j,\ B\right] \ \longmapsto f \circ \pi_J \in \left[\prod_{i \in I} A_i,\ B\right]$$

is a natural bijection between $\left[\prod_{j \in J} A_j,\ B\right]$ and $\left[\prod_{i \in I} A_i,\ B\right]_J$ [7]. □

The last result of this Chapter shows that $\mathfrak{K}_I(A, J)$ can be naturally identified with $\mathfrak{K}_I(A, K)$, whenever J and K are of the same cardinality.

Proposition 40.14. *Let A be a Ω-set and $J,\ K \subseteq I$ be sets. If $card(J) = card(K)$, there is an automorphism τ of $\mathfrak{K}_I A$ that restricts to an isomorphism between $\mathfrak{K}_I(A, K)$ and $\mathfrak{K}_I(A, J)$.*

[6] A similar observation, together with 40.8, yields another proof of item (a).

[7] $[\prod_{i \in I} A_i,\ B]_J$ is the set of morphisms from that depend on J, as in 40.2.

PROOF. We shall use the Fact that follows, whose proof is left to the reader.

Fact. *For J, K as above, one of the following alternatives hold*

(1) $card(J - K) = card(K - J)$;

(2) *If* $card(J-K) < card(K-J)$, *there is* $T \subseteq J \cap K$ *such that* $card[(J-K) \cup T] = card(K - J)$;

(3) *If* $card(K-J) < card(J-K)$, *there is* $S \subseteq J \cap K$ *such that* $card[(K-J) \cup S] = card(J - K)$.

We now define a permutation, $\sigma : I \longrightarrow I$, such that $\sigma(J) = K$. To start off, set σ equal to the identity in $I - (J \cup K)$. In $J \cup K$,
$*$ If $card(J - K) = card(K - J)$, let $f : (J - K) \longrightarrow (K - J)$ be a bijection and define, for $i \in (J \cup K)$

$$\sigma(i) \quad = \quad \begin{cases} i & \text{if } i \in J \cap K; \\ f(i) & \text{if } i \in J - K; \\ f^{-1}(i) & \text{if } i \in K - J. \end{cases}$$

It is clear that σ is a permutation of I, taking J to K.
$*$ If $card(J-K) < card(K-J)$, by the Fact, there is $T \subseteq (J \cap K)$ and a bijection $f : (J - K) \cup T \longrightarrow (K - J)$. For $i \in I$, define

$$\sigma(i) \quad = \quad \begin{cases} i & \text{if } i \in (J \cap K) - T; \\ f(i) & \text{if } i \in (J - K) \cup T; \\ f^{-1}(i) & \text{if } i \in (K - J). \end{cases}$$

Again, it is straightforward that σ is a permutation of I, taking J to K. The situation of case (3) can be handled similarly. By 37.4.(f), the map τ given by

$$S \in \mathfrak{K}_I A \longmapsto S^\sigma \in \mathfrak{K}_I A$$

is an automorphism of $\mathfrak{K}_I A$. Since $[\![S^\sigma(\overline{a})]\!] \;=\; [\![S(\overline{a}^\sigma)]\!]$, where $\overline{a}_i^\sigma = \overline{a}_{\sigma(i)}$, it is readily verified that the characteristic maps that depend on J correspond bijectively, via τ, to the characteristic maps that depend on K. $\square$

REMARK 40.15. If J, K are *finite* subsets of I of the same cardinality, only case (1) in the statement of the Fact in the proof of 40.14 can occur. Hence, it is possible to give an automorphism of $\mathfrak{K}_I A$ that moves only coordinates in the symmetric difference $J \bigtriangleup K$. If I is well-ordered, such an isomorphism can be constructed from an *increasing* bijection between $(J - K)$ and $(K - J)$. $\square$

Exercises

40.16. Let A be a Ω-presheaf and I be a set. If S is closed in A^I and $J \subseteq I$, then S depends on J iff for all $\overline{a}$, $\overline{c} \in |A|^I$,

$$E\overline{a} = E\overline{c}, \;\; \pi_J(\overline{a}) = \pi_J(\overline{c}) \;\text{ and }\; \overline{a} \in S \quad \Rightarrow \quad \overline{c} \in S. \qquad \square$$

40.17. If A is a Ω-presheaf and I is a set, there are natural isomorphisms

$$\mathfrak{K}_I(A, \emptyset) \;\approx\; (EA)^{\leftarrow} \;\approx\; \{(A^I)_{|p} : p \leq EA\}. \qquad \square$$

CHAPTER 41

Composition and Substitution

The result that follows describes how relations are composed in the context of characteristic maps. As usual, Ω denotes a frame.

LEMMA 41.1. *Let A be a Ω-set and I, J, K be sets. If $R \in \mathfrak{K}_{I \times J}A$ and $S \in \mathfrak{K}_{J \times K}A$, define, for $\overline{a} \in |A|^{I \times K}$* [1]

$$[\![S \circ R(\overline{a})]\!] \;=\; \bigvee_{\overline{c} \in |A|^J} [\![R(\pi_I(\overline{a}); \overline{c})]\!] \wedge [\![S(\overline{c}; \pi_K(\overline{a}))]\!].$$

Then, $S \circ R \in \mathfrak{K}_{I \times K}A$.

PROOF. If $\overline{a} \in |A|^{I \times K}$ and $\overline{c} \in |A|^J$

$$[\![R(\pi_I(\overline{a}); \overline{c})]\!] \;\leq\; E\pi_I(\overline{a}) \quad \text{and} \quad [\![S(\overline{c}; \pi_K(\overline{a}))]\!] \;\leq\; E\pi_K(\overline{a})$$

and so $[\![S \circ R(\overline{a})]\!] \;\leq\; E\overline{a}$. Next, if $\overline{u} \in |A|^{I \times K}$ and $\overline{c} \in |A|^J$, $[=_J]$ and $[ch_J]$ in 38.9 yield

$$\begin{aligned}
&[\![\overline{a} = \overline{u}]\!] \wedge [\![R(\pi_I(\overline{a}); \overline{c})]\!] \wedge [\![S(\overline{c}; \pi_K(\overline{a}))]\!] = \\
&\quad = [\![\pi_I\overline{a} = \pi_I\overline{u}]\!] \wedge [\![\pi_K\overline{a} = \pi_K\overline{u}]\!] \wedge [\![R(\pi_I(\overline{a}); \overline{c})]\!] \wedge [\![S(\overline{c}; \pi_K(\overline{a}))]\!] \\
&\quad \leq [\![R(\pi_I(\overline{u}); \overline{c})]\!] \wedge [\![S(\overline{c}; \pi_K(\overline{u}))]\!],
\end{aligned}$$

and so, taking joins over $\overline{c} \in |A|^J$ we get $[\![S \circ R(\overline{a})]\!] \wedge [\![\overline{a} = \overline{u}]\!] \;\leq\; [\![S \circ R(\overline{u})]\!]$, completing the verification that $S \circ R \in \mathfrak{K}_{I \times K}A$. $\square$

DEFINITION 41.2. *With notation as in* 41.1*, the characteristic map $S \circ R$ is the* **composition of R and S**.

An important operation in the syntax of formal languages and in their interpretation is that of *substitution of terms for variables in a relation or in another term*. This is an example of a situation in which matters are considerably simplified by working with presheaves. Before stating the pertinent results we generalize the notation in 37.10.(b).

41.3. Let A be a Ω-set and $I \neq \emptyset$ be a set. If $J \subseteq I$, $\overline{c} \in |A|^J$ and $\overline{a} \in |A|^I$, define the **substitution of $\overline{c}$ for the coordinates of $\overline{a}$ in J** as the I-sequence given by

$$\overline{a}\,\ulcorner \overline{c} \mid J \urcorner (i) \;=\; \begin{cases} a_i & \text{if } i \notin J \\ c_i & \text{if } i \in J. \end{cases}$$

[1]Notation as in 37.24.

THEOREM 41.4. *Let A be a Ω-set and $J \subseteq I \neq \emptyset$ be sets. Let $A^I \xrightarrow{f_j} A$, $j \in J$, be morphisms and let $f : A^I \longrightarrow A^J$ be their product. If $R \in \mathfrak{K}_I A$ and $\overline{a} \in |A|^I$, set*

$$[\![R\ulcorner f \mid J \urcorner(\overline{a})]\!] \;=\; \bigvee_{\overline{c}\in|A^I|} [\![R(\overline{a}\ulcorner f\overline{c} \mid J\urcorner)]\!] \wedge [\![\overline{a} = \overline{c}]\!].$$

Then, $R\ulcorner f \mid J\urcorner \in \mathfrak{K}_I A$. Moreover,

a) For all $\overline{a} \in |A^I|$, $[\![R\ulcorner f \mid J\urcorner(\overline{a})]\!] = [\![R(\overline{a}\ulcorner f\overline{a} \mid J\urcorner)]\!]$.

b) If A is a Ω-presheaf, then

(1) *For all $\overline{a} \in |A|^I$, $[\![R\ulcorner f \mid J\urcorner(\overline{a})]\!] = [\![R(\overline{a}\ulcorner f(\overline{a}_{|E\overline{a}}) \mid J\urcorner)]\!]$.*

(2) *If R depends on K and f_j depends on K_j, $j \in J$, then $R\ulcorner f \mid J\urcorner$ depends on $(K \cap J^c) \cup \bigcup_{j\in K} K_j$.*

PROOF. It is clear that $[\![R\ulcorner f \mid J\urcorner(\overline{a})]\!] \leq E\overline{a}$. Let $\overline{a}$, $\overline{u} \in |A|^I$; for $\overline{c} \in |A^I|$, since f is a morphism, we get

$$\text{(A)} \qquad \begin{aligned} [\![\overline{a}\ulcorner f\overline{c} \mid J\urcorner = \overline{u}\ulcorner f\overline{c} \mid J\urcorner]\!] &= [\![\pi_{J^c}\overline{a} = \pi_{J^c}\overline{u}]\!] \wedge Ef\overline{c} \\ &= [\![\pi_{J^c}\overline{a} = \pi_{J^c}\overline{u}]\!] \wedge E\overline{c}. \end{aligned}$$

Hence, (A) yields

$$\begin{aligned} &[\![R\overline{a}\ulcorner f\overline{c} \mid J\urcorner]\!] \wedge [\![\overline{a} = \overline{c}]\!] \wedge [\![\overline{a} = \overline{u}]\!] = \\ &\quad= [\![R\overline{a}\ulcorner f\overline{c} \mid J\urcorner]\!] \wedge E\overline{c} \wedge [\![\pi_{J^c}\overline{a} = \pi_{J^c}\overline{u}]\!] \wedge [\![\overline{a} = \overline{c}]\!] \wedge [\![\overline{a} = \overline{u}]\!] \\ &\quad= [\![R\overline{a}\ulcorner f\overline{c} \mid J\urcorner]\!] \wedge [\![\overline{a}\ulcorner f\overline{c} \mid J\urcorner = \overline{u}\ulcorner f\overline{c} \mid J\urcorner]\!] \wedge [\![\overline{a} = \overline{c}]\!] \wedge [\![\overline{a} = \overline{u}]\!] \\ &\quad\leq [\![R\overline{u}\ulcorner f\overline{c} \mid J\urcorner]\!] \wedge [\![\overline{u} = \overline{c}]\!], \end{aligned}$$

and so, taking joins over $\overline{c} \in |A^I|$, we obtain $[\![R\ulcorner f \mid J\urcorner(\overline{a})]\!] \wedge [\![\overline{a} = \overline{u}]\!] \leq [\![R\ulcorner f \mid J\urcorner(\overline{u})]\!]$, and $R\ulcorner f \mid J\urcorner \in \mathfrak{K}_I A$, as asserted.

a) Clearly, for all $\overline{a} \in |A^I|$, we have $[\![R\ulcorner f \mid J\urcorner(\overline{a})]\!] \geq [\![R(\overline{a}\ulcorner f\overline{a} \mid J\urcorner)]\!]$. For the reverse inequality, let $\overline{c} \in |A^I|$. Then,

$$\text{(B)} \qquad [\![\overline{a}\ulcorner f\overline{c} \mid J\urcorner = \overline{a}\ulcorner f\overline{a} \mid J\urcorner]\!] = E\pi_{J^c}\overline{a} \wedge [\![f\overline{a} = f\overline{c}]\!].$$

Since $E\pi_{J^c}\overline{a} \wedge [\![f\overline{a} = f\overline{c}]\!] \geq [\![\overline{a} = \overline{c}]\!]$, (B) entails

$$\begin{aligned} [\![R(\overline{a}\ulcorner f\overline{c} \mid J\urcorner)]\!] \wedge [\![\overline{a} = \overline{c}]\!] &\leq [\![R(\overline{a}\ulcorner f\overline{c} \mid J\urcorner)]\!] \wedge E\pi_{J^c}\overline{a} \wedge [\![f\overline{a} = f\overline{c}]\!] \\ &= [\![R(\overline{a}\ulcorner f\overline{c} \mid J\urcorner)]\!] \wedge [\![\overline{a}\ulcorner f\overline{c} \mid J\urcorner = \overline{a}\ulcorner f\overline{a} \mid J\urcorner]\!] \\ &\leq [\![R(\overline{a}\ulcorner f\overline{a} \mid J\urcorner)]\!]. \end{aligned}$$

Taking joins over $\overline{c} \in |A^I|$, yields $[\![R(\overline{a}\ulcorner f\overline{a} \mid J\urcorner)]\!] \geq [\![R\ulcorner f \mid J\urcorner(\overline{a})]\!]$, establishing the equality of these terms.

b) (1) If A is a Ω-presheaf, 37.4.(a) entails, for $\overline{a} \in |A|^I$,

$$[\![R\ulcorner f \mid J\urcorner(\overline{a})]\!] = [\![R\ulcorner f \mid J\urcorner(\overline{a}_{|E\overline{a}})]\!],$$

and since $\overline{a}_{|E\overline{a}} \in |A^I|$, we get $[\![R\ulcorner f \mid J\urcorner(\overline{a})]\!] = [\![R(\overline{a}_{|E\overline{a}}\ulcorner f\overline{a}_{|E\overline{a}} \mid J\urcorner)]\!]$.

Since $[\![\overline{a}\ulcorner f\overline{a}_{|E\overline{a}} \mid J\urcorner = \overline{a}_{|E\overline{a}}\ulcorner f\overline{a}_{|E\overline{a}} \mid J\urcorner]\!] = E\overline{a}$, [ch 2′] in 37.2 yields

$$[\![R(\overline{a}_{|E\overline{a}}\ulcorner f\overline{a}_{|E\overline{a}} \mid J\urcorner)]\!] = [\![R(\overline{a}\ulcorner f\overline{a}_{|E\overline{a}} \mid J\urcorner)]\!].$$

(2) Write $W = (K \cap J^c) \cup \bigcup_{j \in K} K_j$ and suppose $\overline{a}, \overline{u} \in |A|^I$ satisfy $W \subseteq \mathfrak{c}(\overline{a}, \overline{u})$. Since

$$[\![a_{i_{|E\overline{a}}} = u_{i_{|E\overline{u}}}]\!] \;=\; E\overline{a} \wedge E\overline{u} \wedge [\![a_i = u_i]\!],$$

we have $\mathfrak{c}(\overline{a}, \overline{u}) \subseteq \mathfrak{c}(\overline{a}_{|E\overline{a}}, \overline{u}_{|E\overline{u}})$. Hence, by (b).(1), we may suppose that $\overline{a}$, $\overline{u}$ are in $|A^I|$, otherwise just reason with $\overline{a}_{|E\overline{a}}$ and $\overline{u}_{|E\overline{u}}$, in place of $\overline{a}$, $\overline{u}$, respectively. If $j \in J \cap K$, then $[\![f_j(\overline{a}) = f_j(\overline{u})]\!] \;=\; E\overline{a} \wedge E\overline{u} \;=\; Ef\overline{a} \wedge Ef\overline{u}$, because f_j depends on $K_j \subseteq W$. Hence, $K \;\subseteq\; \mathfrak{c}(\overline{a}\ulcorner f\overline{a} \mid J \urcorner,\ \overline{u}\ulcorner f\overline{u} \mid J \urcorner)$. Since

$$E\overline{a}\ulcorner f\overline{a} \mid J \urcorner \;=\; E\overline{a} \quad \text{and} \quad E\overline{u}\ulcorner f\overline{u} \mid J \urcorner \;=\; E\overline{u}$$

and R depends on K, item (a) entails

$$\begin{aligned} E\overline{a} \wedge [\![R\ulcorner f \mid J \urcorner(\overline{u})]\!] &= E\overline{a}\ulcorner f\overline{a} \mid J \urcorner \wedge [\![R(\overline{u}\ulcorner f\overline{u} \mid J \urcorner)]\!] \\ &= E\overline{u}\ulcorner f\overline{u} \mid J \urcorner \wedge [\![R(\overline{a}\ulcorner f\overline{a} \mid J \urcorner)]\!] \\ &= E\overline{u} \wedge [\![R\ulcorner f \mid J \urcorner(\overline{a})]\!], \end{aligned}$$

ending the proof. $\square$

DEFINITION 41.5. *Notation as in* 41.4*, the characteristic map $R\ulcorner f \mid J \urcorner$ is the* **substitution of f in R at the coordinates in J**. *If $J = \{j\}$, write $R\ulcorner f \mid j \urcorner$ for $R\ulcorner f \mid \{j\} \urcorner$.*

For elements of $\mathfrak{K}_1(A^I)$, A a Ω-set, and elements of $\mathfrak{K}_I A$, A a presheaf, substitution of morphisms in distinct coordinates is commutative :

COROLLARY 41.6. *Let A be a Ω-set and $I \neq \emptyset$ be a set. Let j, k be distinct elements of I and f, $g : A^I \longrightarrow A$ be Ω-set morphisms. If $R \in \mathfrak{K}_1(A^I)$ or if $R \in \mathfrak{K}_I A$ and A is a presheaf, then*

$$R\ulcorner f \mid j \urcorner \ulcorner g \mid k \urcorner \;=\; R\ulcorner g \mid k \urcorner \ulcorner f \mid j \urcorner \;=\; R\ulcorner \langle f, g \rangle \mid \langle j, k \rangle \urcorner.$$

There is an analog of Theorem 41.4 that describes the process of substitution of morphisms for variables in a morphism. We give the pertinent statement, omitting the proof.

PROPOSITION 41.7. *Let A be a Ω-presheaf and $J \subseteq I \neq \emptyset$ be sets. Let g, $f_j : A^I \longrightarrow A$ be morphisms, $j \in J$, and $f : A^I \longrightarrow A^J$ be the product of the f_j. Define*

$$g\ulcorner f \mid J \urcorner : A^I \longrightarrow A, \text{ by } \; g\ulcorner f \mid J \urcorner(\overline{a}) \;=\; g(\overline{a}\ulcorner f\overline{a} \mid J \urcorner),$$

called **the substitution of f in g at the coordinates in J**. *Then,*

a) $g\ulcorner f \mid J \urcorner$ is a Ω-set morphism from A^I to A.

b) If g depends on K and f_j depends on K_j, $j \in J$, then $g\ulcorner f \mid J \urcorner$ depends on $(K \cap J^c) \cup \bigcup_{j \in K} K_j$. $\square$

Exercises

41.8. Determine the dependence on coordinates of a composition of relations as a function of the dependence of its factors. $\square$

41.9. This exercise consists in verifying directly that Theorem 41.4 holds for I-ary relations on a Ω-set. Let A be a Ω-set and $J \subseteq I \neq \emptyset$ be sets. Let $f_j : A^I \longrightarrow A$ be morphisms, $j \in J$, and let f be their product. For $R \in \mathfrak{K}_1(A^I)$ and $\overline{a} \in |A^I|$, define

$$[\![\, R \ulcorner f \mid J \urcorner (\overline{a}) \,]\!] \;=\; [\![\, R(\overline{a} \ulcorner f\overline{a} \mid J \urcorner) \,]\!].$$

Then, $R \ulcorner f \mid J \urcorner \in \mathfrak{K}_1(A^I)$, and is called **the substitution of f in R at the coordinates in J**. Moreover, if R depends on K and f_j depends on K_j, then $R \ulcorner f \mid J \urcorner$ depends on $(J^c \cap K) \cup \bigcup_{j \in K} K_j$. □

CHAPTER 42

Quantifiers

In this Chapter we discuss the operation of *quantification*. The first section is devoted to a general form of this notion, associated to a morphism of Ω-sets. In the second section we specialize to the case of coordinate forgetting projections, that give rise to the classical existential and universal quantifiers.

1. Quantification along a Morphism

Let $f : A \longrightarrow B$ be a morphism of Ω-sets, which will remain fixed throughout this section. By Corollary 38.15, for each set I, f induces an adjoint pair

$$f_* : \mathfrak{K}_I(A) \longrightarrow \mathfrak{K}_I(B) \quad \text{and} \quad f^* : \mathfrak{K}_I(B) \longrightarrow \mathfrak{K}_I(A).$$

From a logical point of view, we can consider f_* as **existential quantification along $\boldsymbol{f}$**. Therefore, some authors write $\exists_f$ for f_*. Hence, 38.15 yields, for $\overline{b} \in |B|^I$, $R \in \mathfrak{K}_I A$ and $S \in \mathfrak{K}_I B$

$$(\exists_f) \qquad \begin{cases} [\![\, \exists_f R(\overline{b}) \,]\!] & = \quad \bigvee_{\overline{a} \in |A|^I} [\![f\overline{a} = \overline{b}]\!] \wedge [\![\, R(\overline{a}) \,]\!]; \\ [\![\, \exists_f R \,]\!] \leq [\![\, S \,]\!] & \text{iff} \quad [\![\, R \,]\!] \leq [\![\, f^* R \,]\!]. \end{cases}$$

Since $\exists_f$ is a left adjoint, it preserves joins, that is, for R, $S \in \mathfrak{K}_I A$

$$[\![\, \exists_f R \,]\!] \vee [\![\, \exists_f S \,]\!] \;=\; [\![\, \exists_f (R \vee S) \,]\!],$$

a familiar law of Logic, that holds even for arbitrary families in $\mathfrak{K}_I A$.

By Corollary 38.15, f^* is an *open* morphism and so Theorem 7.8 implies that it has a *right adjoint*,

$$\mathfrak{u} : \mathfrak{K}_I A \longrightarrow \mathfrak{K}_I B,$$

satisfying for $R \in \mathfrak{K}_I A$ and $T \in \mathfrak{K}_I B$

$$f^* T \;\leq\; R \quad \text{iff} \quad T \;\leq\; \mathfrak{u} R.$$

We have (see proof of 7.8)

$$\mathfrak{u} R \;=\; \bigvee \{T \in \mathfrak{K}_I B : f^* T \leq R\},$$

that is, the *largest* element of $\mathfrak{K}_I B$ whose inverse image is contained in R. Because of this property, the right adjoint of f^* is called **universal quantification along $\boldsymbol{f}$**, written $\forall_f$. Hence, we have constructed a $\bigwedge$-morphism

$$\forall_f : \mathfrak{K}_I A \longrightarrow \mathfrak{K}_I B,$$

given, for $R \in \mathfrak{K}_I A$ by

$$\forall_f R \;=\; \bigvee \{T \in \mathfrak{K}_I B : \forall\, \overline{a} \in |A|^I, \;\; [\![\, T(f\overline{a}) \,]\!] \leq [\![\, R(\overline{a}) \,]\!]\}.$$

Since $\forall_f$ is a $\bigwedge$-morphism, if $\{R_\lambda : \lambda \in \Lambda\} \subseteq \mathfrak{K}_I A$, then

$$\forall_f(\bigwedge_{\lambda\in\Lambda} R_\lambda) \;=\; \bigwedge_{\lambda\in\Lambda} \forall_f R_\lambda,$$

a law that, in its finite version, is well-known in Logic. The next result gives a more convenient description of $\forall_f$.

THEOREM 42.1. *Let $f : A \longrightarrow B$ be a morphism of Ω-sets and I be a set. If $R \in \mathfrak{K}_I A$ and $\bar{b} \in |B|^I$*

$$(\forall_f) \qquad [\![\, \forall_f\, R(\bar{b})\,]\!] \;=\; E\bar{b} \wedge \bigwedge_{\bar{a}\in|A|^I} [\![f\bar{a} = \bar{b}]\!] \to [\![\, R(\bar{a})\,]\!],$$

where $\to$ in the right side of $(\forall_f)$ is implication in Ω.

PROOF. Fix $R \in \mathfrak{K}_I A$ and let $\mathfrak{u} : |B|^I \longrightarrow \Omega$ be given by

$$\mathfrak{u}(\bar{b}) \;=\; E\bar{b} \wedge \bigwedge_{\bar{a}\in|A|^I} [\![f\bar{a} = \bar{b}]\!] \to [\![\, R(\bar{a})\,]\!].$$

It is straightforward that $\mathfrak{u} \in \mathfrak{K}_I B$. We shall verify that for $T \in \mathfrak{K}_I B$

$$f^*T \le R \quad \text{iff} \quad T \le \mathfrak{u},$$

and so $\mathfrak{u} \;=\; \forall_f R$. Assume that for all $\bar{a} \in |A|^I$,

$$[\![\, f^*T(\bar{a})\,]\!] \;=\; [\![\, T(f\bar{a})\,]\!] \;\le\; [\![\, R(\bar{a})\,]\!]. \tag{1}$$

For $\bar{b} \in |B|^I$ and $\bar{a} \in |A|^I$, we have

$$[\![\, T(\bar{b})\,]\!] \wedge [\![f\bar{a} = \bar{b}]\!] \;=\; [\![\, T(f\bar{a})\,]\!] \wedge [\![f\bar{a} = \bar{b}]\!], \tag{2}$$

and (1) entails $[\![\, T(\bar{b})\,]\!] \wedge [\![f\bar{a} = \bar{b}]\!] \;\le\; [\![\, R(\bar{a})\,]\!]$, wherefrom it follows that for arbitrary $\bar{a} \in |A|^I$

$$[\![\, T(\bar{b})\,]\!] \;\le\; [\![f\bar{a} = \bar{b}]\!] \to [\![\, R(\bar{a})\,]\!].$$

Hence,

$$[\![\, T(\bar{b})\,]\!] \;\le\; \bigwedge_{\bar{a}\in|A|^I} [\![f\bar{a} = \bar{b}]\!] \to [\![\, R(\bar{a})\,]\!]. \tag{3}$$

Since $[\![\, T(\bar{b})\,]\!] \le E\bar{b}$, from (3) we conclude that $T \le \mathfrak{u}$, as desired. To establish that $T \le \mathfrak{u}$ implies $f^*T \le R$, it is enough to check, since inverse image is increasing, that $f^*\mathfrak{u} \le R$. For $\bar{c} \in |A|^I$, we have

$$\begin{aligned} f^*\mathfrak{u}(\bar{c}) \;&=\; \mathfrak{u}(f\bar{c}) \;=\; Ef\bar{c} \wedge \bigwedge_{\bar{a}\in|A|^I} [\![f\bar{a} = f\bar{c}]\!] \to [\![\, R(\bar{a})\,]\!] \\ &\le\; E\bar{c} \wedge ([\![f\bar{c} = f\bar{c}]\!] \to [\![\, R(\bar{c})\,]\!]) \;=\; E\bar{c} \wedge (E\bar{c} \to [\![\, R(\bar{c})\,]\!]) \\ &\le\; [\![\, R(\bar{c})\,]\!], \end{aligned}$$

ending the proof. □

In the remainder of this section we describe the laws that connect quantification and negation. We begin with a general result.

PROPOSITION 42.2. *Let $\gamma : L \longrightarrow R$ be an* **open** *frame morphism and let λ, $\rho : R \longrightarrow L$ be the left and right adjoints of γ, respectively. Then, for all p, $q \in R$,*

a) $\rho(p \to q) \;\le\; \rho p \to \rho q \;=\; \rho(\gamma(\rho p) \to q)$.

b) $\rho(\neg\, p) = \neg\, \lambda(\neg\neg\, p)$. *In particular, $\rho(\neg\, p)$ is regular in L.*

c) $\neg\, \rho(\neg\, p) = \neg\neg\, \lambda(\neg\neg\, p)$ *and* $\neg\, \lambda(\neg\, p) = \neg\neg\, \rho(\neg\neg\, p) = \rho(\neg\neg\, p)$.

PROOF. Recall the adjointness relations in 7.8 and 7.9 :

$$\forall\ \langle\, a, p\,\rangle \in L \times R, \qquad \begin{cases} (i)\ \gamma a \le p \quad \text{iff} \quad a \le \rho p; \\ (ii)\ \lambda p \le a \quad \text{iff} \quad p \le \gamma a; \\ (iii)\ \gamma(\rho(p)) \le p \quad \text{and} \quad \gamma(\lambda(p)) \ge p; \\ (iv)\ \rho \circ \gamma \circ \rho = \rho; \quad \lambda \circ \gamma \circ \lambda = \lambda. \end{cases} \qquad \text{(ad)}$$

a) Since ρ preserves meets (it is a right adjoint), we have

$$\rho(p \to q) \le \rho p \to \rho q,$$

and (iv) in (ad) yields

$$\rho(\gamma(\rho p) \to q) \quad \le \quad \rho(\gamma(\rho p)) \to \rho q \quad = \quad \rho p \to \rho q.$$

Let $a \in L$ be such that $a \wedge \rho p \le \rho q$; by relation (i) in (ad), $\gamma(a \wedge \rho p) \le q$. Since γ preserves meets, $\gamma(a \wedge \rho p) = \gamma a \wedge \gamma(\rho p)$ and so $\gamma a \wedge \gamma(\rho p) \le q$. Thus, $\gamma a \le \gamma(\rho p) \to q$ and another application of (i) entails $a \le \rho(\gamma(\rho p) \to q)$, proving (a).

b) To simplify understanding, we first prove

Fact. *For* $\langle\, a, p\,\rangle \in L \times R$, $\ \gamma a \wedge p = \bot$ *iff* $a \wedge \lambda p = \bot$.

Proof. If $a \wedge \lambda p = \bot$, then since $\gamma(\bot) = \bot$, γ preserves meets and $\gamma(\lambda p) \ge p$, it is clear that $\gamma a \wedge p = \bot$. For the converse, $\gamma a \wedge p = \bot$ implies $p \le \neg\,\gamma a = \gamma(\neg\, a)$; by the adjointness relation (ii), $\lambda p \le \neg\, a$ and so $\lambda p \wedge a = \bot$, as desired.

Item (b) is equivalent to

$$\neg\,\lambda(\neg\neg\, p) \le \rho(\neg\, p) \quad \text{and} \quad \rho(\neg\, p) \le \neg\,\lambda(\neg\neg\, p),$$

and so (i) in (ad) and the fact that $\neg * = * \to \bot$ imply that the above are in turn equivalent to

$$(1) \quad \gamma(\neg\,\lambda(\neg\neg\, p)) \wedge p = \bot \quad \text{and} \quad (2) \quad \rho(\neg\, p) \wedge \lambda(\neg\neg\, p) = \bot.$$

Proof of (1) : By the Fact, (1) is equivalent to $\neg\,\lambda(\neg\neg\, p) \wedge \lambda(p) = \bot$. Since λ is increasing (it is a $\bigvee$-morphism) and $p \le \neg\neg\, p$, we get

$$\neg\,\lambda(\neg\neg\, p) \wedge \lambda(p) \quad \le \quad \neg\,\lambda(\neg\neg\, p) \wedge \lambda(\neg\neg\, p) \quad = \quad \bot,$$

as needed.

Proof of (2) : By the Fact, (2) is equivalent to $\gamma(\rho(\neg\, p)) \wedge \neg\neg\, p = \bot$. Thus, the first inequality in (iii) of (ad) yields $\ \gamma(\rho(\neg\,(p)) \wedge \neg\neg\, p \ \le\ \neg\, p \wedge \neg\neg\, p \ =\ \bot$, establishing (b). Item (c) is a straightforward consequence of (b). ☐

Proposition 42.2 applied to f^* yields

COROLLARY 42.3. *If* $A \xrightarrow{f} B$ *is a morphism of* Ω*-sets and* I *is a set, then for all* $R, S \in \mathfrak{K}_I A$,

a) $\forall_f(R \to S) \quad \le \quad \forall_f R \to \forall_f S.$

b) $\forall_f \neg\, R \quad = \quad \neg\, \exists_f \neg\neg\, R$ *and* $\forall_f \neg\neg\, R \quad = \quad \neg\, \exists_f \neg\, R.$

c) $\neg \forall_f \neg\, R \quad = \quad \neg\neg\ \exists_f\ \neg\neg\, R.$

Note that if $\mathfrak{K}_I A$ is a Boolean algebra, (c) and (d) in 42.3 give back the usual relations between the quantifiers and negation.

2. Classical Quantifiers

As mentioned at the beginning of this Chapter, the classical quantifiers arise when we consider projections that forget coordinates. If A is a Ω-set, I is a set and $J \subseteq I$, the adjoints to π_J^* are associated to the existential and universal quantification *with respect to the variables in* $(I - J)$. In 24.30 there is a geometrical discussion of the fundamental ideas concerning this topic.

By 40.8, the following diagram is commutative, where *can.* is the canonical open embedding of $\mathfrak{K}_I(A, J)$ into $\mathfrak{K}_I A$ (40.5.(d)) and $\pi_J^* : \mathfrak{K}_J A \longrightarrow \mathfrak{K}_I(A, J)$ is an *isomorphism* :

$$\begin{tikzcd}
\mathfrak{K}_J A \arrow[rr, "\pi_J^*"] \arrow[dr, "\pi_J^*"'] & & \mathfrak{K}_I(A, J) \arrow[dl, "can."] \\
& \mathfrak{K}_I A &
\end{tikzcd}$$

Hence, we may identify *inverse image by* π_J with the canonical open inclusion of $\mathfrak{K}_I(A, J)$ into $\mathfrak{K}_I A$. To make clear our general framework, we prove

PROPOSITION 42.4. *Let* $L \xrightarrow{h} R$ *be a regular embedding of complete lattices. Let* $P = Im\, h$ *and* $\gamma : L \longrightarrow P$ *be the induced isomorphism. Let* $\iota_P : P \longrightarrow R$ *be the canonical inclusion.*

a) ι_P *is a regular embedding* [1].

b) If λ, ρ *are the left and right adjoints of* ι_P, *then* $\gamma^{-1} \circ \lambda$ *and* $\gamma^{-1} \circ \rho$ *are the left and right adjoints of* h, *respectively.*

PROOF. Item (a) is immediate from the fact that h is a regular embedding. For (b), let λ, $\rho : R \longrightarrow P$ satisfy

$$(*) \qquad \text{For all } \langle p, r \rangle \in P \times R, \quad \begin{cases} (i) & \lambda r \leq p \quad \text{iff} \quad r \leq p; \\ (ii) & p \leq r \quad \text{iff} \quad p \leq \rho r. \end{cases}$$

One should keep in mind that for $p \in P$ and $x \in L$

$$(**) \qquad h(\gamma^{-1}(p)) = p \quad \text{and} \quad \gamma^{-1}(h(x)) = x.$$

For $\langle x, r \rangle \in L \times R$, (i) in (*), the first relation in (**) and the fact that h is an embedding yield

$$\gamma^{-1}(\lambda(r)) \leq x \quad \text{iff} \quad h(\gamma^{-1}(\lambda(r))) \leq hx \quad \text{iff} \quad \lambda r \leq hx \quad \text{iff} \quad r \leq hx,$$

showing that $\gamma^{-1} \circ \lambda$ is the left adjoint of h. On the other hand, the second relation in (**) and (ii) in (*) entail

$$hx \leq r \quad \text{iff} \quad hx \leq \rho x \quad \text{iff} \quad \gamma^{-1}(h(x)) \leq \gamma^{-1}(\rho(x)) \quad \text{iff} \quad x \leq \gamma^{-1}(h(x)),$$

establishing that $\gamma^{-1} \circ \rho$ is the right adjoint of h. $\square$

[1] I.e., P is a regular sublattice of R.

Proposition 42.4 shows that if the regular embedding h is identified, via the isomorphism γ, to the canonical embedding of its image into its codomain, then the left and right adjoints of this embedding are isomorphic to the left and right adjoints of h, respectively.

Applying 42.4 to the open embedding $\pi_J^* : \mathfrak{K}_J A \longrightarrow \mathfrak{K}_I A$, we obtain that the left and right adjoints of the canonical inclusion of $\mathfrak{K}_I(A, J)$ into $\mathfrak{K}_I A$ will represent, up to isomorphism, the left and right adjoints of π_J^*. This is the way we shall treat the classical quantifiers.

42.5. **Notation.** If A is a Ω-set, $K \subseteq I$ are sets and $\overline{a} \in |A|^I$, define

$$\mathfrak{p}(\overline{a}, K) \;=\; \{\overline{c} \in |A|^I : \pi_K(\overline{a}) = \pi_K(\overline{c})\}. \qquad \square$$

THEOREM 42.6. *Let A be a Ω-set, and $J \subseteq I$ be sets. For S in $\mathfrak{K}_I A$ and $\overline{a}$ in $|A|^I$, define*

$$(\exists\pi_J) \qquad [\![\, \exists\pi_J S(\overline{a}) \,]\!] \;=\; E\overline{a} \wedge \textstyle\bigvee_{\overline{c} \in \mathfrak{p}(\overline{a}, J)} [\![\, S(\overline{c}) \,]\!];$$

$$(\forall\pi_J) \qquad [\![\, \forall\pi_J S(\overline{a}) \,]\!] \;=\; E\overline{a} \wedge \textstyle\bigwedge_{\overline{c} \in \mathfrak{p}(\overline{a}, J)} E\overline{c} \rightarrow [\![\, S(\overline{c}) \,]\!],$$

where $\rightarrow$ in the right-hand side of $(\forall\pi_J)$ is implication in Ω. Then,

a) $S \in \mathfrak{K}_I(A, K) \quad \Rightarrow \quad \exists\pi_J S,\ \forall\pi_J S \in \mathfrak{K}_I(A,\, K \cap J)$.

b) The map $\exists\pi_J,\ \ \forall\pi_J : \mathfrak{K}_I A \longrightarrow \mathfrak{K}_I A$ are the left and right adjoints of the canonical inclusion of $\mathfrak{K}_I(A, J)$ into $\mathfrak{K}_I A$, respectively.

PROOF. Initially, it must be shown that if $S \in \mathfrak{K}_I A$, then $\forall\pi_J S, \exists\pi_J S \in \mathfrak{K}_I A$. The case of $\exists\pi_J S$ was treated in the proof of 40.8. In the case of $\forall\pi_J S$, [ch 1] in 37.1 is clear. For [ch 2], if $\overline{x}$, $\overline{y} \in |A|^I$, it must be verified that

$$[\![\overline{x} \,=\, \overline{y}]\!] \wedge [\![\, \forall\pi_J S(\overline{x}) \,]\!] \;\leq\; E\overline{y} \wedge \textstyle\bigwedge_{\overline{v} \in \mathfrak{p}(\overline{y}, J)} E\overline{v} \longrightarrow [\![\, S(\overline{v}) \,]\!],$$

that amounts to proving that for each $\overline{v} \in \mathfrak{p}(\overline{y},\, J)$

$$[\![\overline{x} \,=\, \overline{y}]\!] \wedge [\![\, \forall\pi_J S(\overline{x}) \,]\!] \;\leq\; E\overline{v} \longrightarrow [\![\, S(\overline{v}) \,]\!],$$

that, in view of the adjunction $[\rightarrow]$ in 6.1, reduces to

$$(1) \qquad [\![\overline{x} \,=\, \overline{y}]\!] \wedge [\![\, \forall\pi_J S(\overline{x}) \,]\!] \wedge E\overline{v} \;\leq\; [\![\, S(\overline{v}) \,]\!].$$

Notation as in 37.24, set $\overline{c} = \langle\, \pi_J(\overline{x}); \pi_{J-I}(\overline{v}) \,\rangle$; then, $\overline{c} \in \mathfrak{p}(\overline{x},\, J)$ and, since $\pi_J(\overline{v}) = \pi_J(\overline{y})$, $[=_J]$ in 38.9 yields

$$(2) \qquad \begin{aligned} [\![\overline{c} \,=\, \overline{v}]\!] \;&=\; [\![\pi_J(\overline{x}) \,=\, \pi_J(\overline{v})]\!] \wedge [\![\pi_{I-J}(\overline{v}) \,=\, \pi_{I-J}(\overline{v})]\!] \\ &=\; [\![\pi_J(\overline{x}) \,=\, \pi_J(\overline{y})]\!] \wedge E\pi_{I-J}(\overline{v}). \end{aligned}$$

From (2), we obtain, recalling $[E_J]$ and $[=_J]$ in 38.9,

$$\begin{aligned} &[\![\overline{x} \,=\, \overline{y}]\!] \wedge E\overline{v} \;= \\ &\quad=\; [\![\pi_J(\overline{x}) \,=\, \pi_J(\overline{y})]\!] \wedge [\![\pi_{J-I}(\overline{x}) \,=\, \pi_{I-J}(\overline{y})]\!] \wedge E\pi_J(\overline{v}) \wedge E\pi_{I-J}(\overline{v}) \\ &\quad=\; [\![\overline{c} \,=\, \overline{v}]\!] \wedge [\![\pi_{J-I}(\overline{x}) \,=\, \pi_{I-J}(\overline{y})]\!] \wedge E\pi_J(\overline{v}) \\ &\quad=\; [\![\overline{c} \,=\, \overline{v}]\!] \wedge [\![\pi_{I-J}(\overline{x}) \,=\, \pi_{I-J}(\overline{y})]\!] \;\leq\; [\![\overline{c} \,=\, \overline{v}]\!]. \end{aligned}$$

Going back to (1), the preceding and *Modus Ponens* in 6.4.(b) entail

$$\begin{array}{rcl} [\![\overline{x} = \overline{y}]\!] \wedge [\![\,\forall\pi_J S(\overline{x})\,]\!] \wedge E\overline{v} & \leq & [\![\overline{c} = \overline{v}]\!] \wedge E\overline{x} \wedge \bigwedge_{\overline{u}\in\mathfrak{p}(\overline{x},J)} E\overline{u} \to [\![\,S(\overline{u})\,]\!] \\ & \leq & [\![\overline{c} = \overline{v}]\!] \wedge (E\overline{c} \to [\![\,S(\overline{c})\,]\!]) \\ & = & E\overline{c} \wedge [\![\overline{c} = \overline{v}]\!] \wedge (E\overline{c} \to [\![\,S(\overline{c})\,]\!]) \\ & \leq & [\![\overline{c} = \overline{v}]\!] \wedge [\![\,S(\overline{c})\,]\!] \;\;\leq\;\; [\![\,S(\overline{v})\,]\!] \end{array}$$

completing the proof that $\forall\pi_J S \in \mathfrak{K}_I A$.

a) Let $S \in \mathfrak{K}_I(A, K)$ and $\overline{x}$, $\overline{y} \in |A|^I$ verify $J \subseteq \mathfrak{c}(\overline{x}, \overline{y})$. It must be shown (40.2.(a)) that

(3) $$E\overline{x} \wedge [\![\,\forall\pi_J S(\overline{y})\,]\!] \;=\; E\overline{y} \wedge [\![\,\forall\pi_J S(\overline{x})\,]\!].$$

Fact 1. *For $\overline{x}$, $\overline{y}$ as above, if $\overline{u} \in \mathfrak{p}(\overline{x}, J)$, then*

$$E\overline{u} \wedge [\![\,\forall\pi_J S(\overline{y})\,]\!] \;\leq\; [\![\,S(\overline{u})\,]\!].$$

Proof. Set $\overline{c} = \langle\, \pi_J(\overline{y});\, \pi_{J^c}(\overline{u})\,\rangle$; note that $\overline{c} \in \mathfrak{p}(\overline{y},\, J)$ and

(i) $E\overline{c} \;=\; E\pi_J(\overline{y}) \wedge E\pi_{J^c}(\overline{u}) \;\geq\; E\overline{y} \wedge E\overline{u}$;

(ii) $\mathfrak{c}(\overline{c}, \overline{u}) = I$; indeed, outside J, $\overline{c}$ and $\overline{u}$ coincide, while in J we have $\pi_J\overline{c} = \pi_J\overline{y}$ and $\pi_J\overline{u} = \pi_J\overline{x}$, and so $J \subseteq \mathfrak{c}(\overline{c}, \overline{u})$.

(iii) By 40.3.(a), (ii) implies $E\overline{u} \wedge [\![\,S(\overline{c})\,]\!] = E\overline{c} \wedge [\![\,S(\overline{u})\,]\!]$.

From (i) and (iii) we get

$$\begin{array}{rcl} E\overline{u} \wedge [\![\,\forall\pi_J S(\overline{y})\,]\!] & = & E\overline{u} \wedge E\overline{y} \wedge \bigwedge_{\overline{v}\in\mathfrak{p}(\overline{y},J)} E\overline{v} \to [\![\,S(\overline{v})\,]\!] \\ & \leq & E\overline{c} \wedge E\overline{u} \wedge (E\overline{c} \to [\![\,S(\overline{c})\,]\!]) \\ & \leq & E\overline{u} \wedge [\![\,S(\overline{c})\,]\!] \;\;\leq\;\; [\![\,S(\overline{u})\,]\!], \end{array}$$

completing the proof of Fact 1.

By Fact 1, for all $\overline{u} \in \mathfrak{p}(\overline{x}$, J) we have $[\![\,\forall\pi_J S(\overline{y})\,]\!] \;\leq\; E\overline{u} \to [\![\,S(\overline{u})\,]\!]$, and so, taking meets with respect to $\overline{u}$ on the right-hand side and meets with $E\overline{x}$ on both sides, we arrive at $E\overline{x} \wedge [\![\,\forall\pi_J S(\overline{y})\,]\!] \;\leq\; [\![\,\forall\pi_J S(\overline{x})\,]\!]$, which in turn readily implies $E\overline{x} \wedge [\![\,\forall\pi_J S(\overline{y})\,]\!] \;\leq\; E\overline{y} \wedge [\![\,\forall\pi_J S)(\overline{x})\,]\!]$. Since the argument is symmetrical in $\overline{x}$ and $\overline{y}$, we conclude the validity of (3). The argument for $\exists\pi_J$ is similar (in fact, simpler).

b) It is immediate from (a) that $\forall\pi_J S$, $\exists\pi_J S \in \mathfrak{K}_I(A, J)$, for all S in $\mathfrak{K}_I A$. Let $\iota_J : \mathfrak{K}_I(A, J) \longrightarrow \mathfrak{K}_I A$ be the natural embedding.

$\forall\pi_J$ is right adjoint to ι_J : For $R \in \mathfrak{K}_I(A, J)$ and $S \in \mathfrak{K}_I A$, we must verify that

(4) $$R \;\leq\; S \quad\Leftrightarrow\quad R \;\leq\; \forall\pi_J S.$$

Assume that $[\![\,R(\overline{x})\,]\!] \leq [\![\,S(\overline{x})\,]\!]$, for all $\overline{x} \in |A|^I$. If $\overline{u} \in \mathfrak{p}(\overline{x},\, J)$, the fact that $R \in \mathfrak{K}_I(A, J)$ entails $E\overline{u} \wedge [\![\,R(\overline{x})\,]\!] \;=\; E\overline{x} \wedge [\![\,R(\overline{u})\,]\!] \;\leq\; [\![\,S(\overline{u})\,]\!]$, whence $[\![\,R(\overline{x})\,]\!] \;\leq\; E\overline{u} \to [\![\,S(\overline{u})\,]\!]$. Since u is arbitrary in $\mathfrak{p}(\overline{x},\, J)$ and $[\![\,R(\overline{x})\,]\!] \leq E\overline{x}$, we get

$$[\![\,R(\overline{x})\,]\!] \;\leq\; E\overline{x} \wedge \bigwedge_{\overline{u}\in\mathfrak{p}(\overline{x},J)} E\overline{u} \to [\![\,S(\overline{u})\,]\!] \;=\; [\![\,\forall\pi_J S(\overline{x})\,]\!],$$

verifying $(\Rightarrow)$ in (4). Conversely, if the right-hand side of (4) holds, then

$$[\![\,R(\overline{x})\,]\!] \;\leq\; E\overline{x} \wedge (E\overline{x} \to [\![\,S(\overline{x})\,]\!]) \;\leq\; [\![\,S(\overline{x})\,]\!],$$

as needed.

$\exists\pi_J$ is left adjoint to ι_J : This amounts to proving, for $R \in \mathfrak{K}_I(A, J)$ and $S \in \mathfrak{K}_I A$

$$\exists\pi_J S \;\leq\; R \quad\Leftrightarrow\quad S \;\leq\; R.$$

($\Rightarrow$) is clear; for the converse, fix $\overline{x} \in |A|^I$. If $\overline{u} \in \mathfrak{p}(\overline{x},\, J)$, then

$$E\overline{x} \wedge [\![\, S(\overline{u}) \,]\!] \;\leq\; E\overline{x} \wedge [\![\, R(\overline{u}) \,]\!] \;=\; E\overline{u} \wedge [\![\, R(\overline{x}) \,]\!] \;\leq\; [\![\, R(\overline{x}) \,]\!].$$

Taking joins on both sides with respect to $\overline{u} \in \mathfrak{p}(\overline{x},\, J)$, we get

$$[\![\, \exists\pi_J S(\overline{x}) \,]\!] \;\leq\; [\![\, R(\overline{x}) \,]\!],$$

ending the proof. □

Corollary 42.7. *Let A be a Ω-set and $J \subseteq K \subseteq I$ be sets.*

a) If $R \in \mathfrak{K}_I(A, J)$, then $R \;=\; \forall\pi_K R \;=\; \exists\pi_K R$.

b) If $S \in \mathfrak{K}_I A$, then $Q\pi_K\,(Q'\pi_J\, S) \;=\; Q'\pi_J\, S$, where $Q\pi_K$ and $Q'\pi_J$ stand for existential or universal quantification.

Proof. a) To be precise, let $\iota_J : \mathfrak{K}_I(A, J) \longrightarrow \mathfrak{K}_I A$ be the canonical embedding. We have a commutative diagram of open embeddings

$$\begin{tikzcd}
\mathfrak{K}_I(A,J) \arrow[rr, "\iota_{JK}"] \arrow[dr, "\iota_J"'] & & \mathfrak{K}_I(A,K) \arrow[dl, "\iota_K"] \\
& \mathfrak{K}_I A &
\end{tikzcd}$$

and so $\iota_K \circ \iota_{JK} = \iota_J$. To simplify notation, let λ_K, ρ_K be the left and right adjoints of ι_K. Then, Corollary 7.9 yields

$$(*) \qquad \lambda_K \circ \iota_K \;=\; Id_{\mathfrak{K}_I(A,J)} \;=\; \rho_K \circ \iota_K.$$

Indeed, since $\langle\, \lambda_J,\, \iota_J \,\rangle$ is an adjoint pair and ι_J is injective, 7.9.(a) entails $\lambda_K \circ \iota_J = Id_{\mathfrak{K}_I(A,J)}$. Similarly, 7.9.(b) implies the other relation in (*). From (*), we obtain

$$\lambda_K \circ \iota_J \;=\; \lambda_K \circ (\iota_K \circ \iota_{JK}) \;=\; Id_{\mathfrak{K}_I(A,J)} \circ \iota_{JK} \;=\; \iota_{JK}.$$

The same argument shows that $\rho_K \circ \iota_J = \iota_{JK}$, establishing (a). Item (b) follows readily from (a) and 42.6.(a). □

Corollary 42.7 expresses a familiar law of Logic : quantifying over variables already quantified yields nothing new. Another of these well-known laws is that the *order* which variables are existentially or universally quantified is immaterial. This can be proven by direct computation, but we prefer to use a different approach, that exhibits the structural properties of the concepts in discussion.

Proposition 42.8. *a) A commutative diagram of regular embeddings of complete lattices*

$$(D) \qquad \begin{tikzcd} L \arrow[rr, "f"] \arrow[dr, "h"'] & & P \arrow[dl, "g"] \\ & Q & \end{tikzcd}$$

gives rise to two commutative diagrams

$$\begin{tikzcd} Q \arrow[rr, "\lambda_g"] \arrow[dr, "\lambda_h"'] & & P \arrow[dl, "\lambda_f"] \\ & L & \end{tikzcd} \qquad \begin{tikzcd} Q \arrow[rr, "\rho_g"] \arrow[dr, "\rho_h"'] & & P \arrow[dl, "\rho_f"] \\ & L & \end{tikzcd}$$

where λ_*, ρ_* are the left and right adjoints of the arrows in (D), respectively.
b) Consider a commutative diagram of regular embeddings of complete lattices :

$$(E) \qquad \begin{tikzcd} P' \arrow[dr, "g'"'] & L \arrow[l, "f'"'] \arrow[r, "f"] \arrow[d, "h"] & P \arrow[dl, "g"] \\ & Q & \end{tikzcd}$$

Assume that for all $\langle p, p' \rangle \in P \times P'$ [2]

$(*)$ $\qquad$ *(i)* $\lambda_g g'(p')$, $\rho_g g'(p') \in Im\, f$; $\qquad$ *(ii)* $\lambda_{g'} g(p)$, $\rho_{g'} g(p) \in Im\, f'$.

Then, $g\lambda_g g'\lambda_{g'} = g'\lambda_{g'} g\lambda_g$ *and* $g\rho_g g'\rho_{g'} = g'\rho_{g'} g\rho_g$, *where λ_* and ρ_* are the left and right adjoints of the arrows in (E).*

Proof. As already noted, composition will be written as superposition.
a) To show that the diagram of adjoints is commutative, it is sufficient to check that

$$(1) \qquad \langle \lambda_f \lambda_g, h \rangle \quad \text{and} \quad \langle h, \rho_f \rho_g \rangle$$

are adjoint pairs, because adjoints are uniquely determined by this relation (7.8). For $\langle q, \ell \rangle \in Q \times L$,

$$\lambda_f \lambda_g(q) \le \ell \quad \text{iff} \quad \lambda_g(q) \le f(\ell) \quad \text{iff} \quad q \le g(f(\ell)) = h(\ell),$$

verifying that the first adjunction in (1); the second can be obtained similarly.

[2] Composition is written as superposition.

b) Since all arrows in (E) are injective, it follows from items (a) and (b) in 7.9 that

(**) $$\lambda_* \circ * \;=\; Id_{dom\,*} \;=\; \rho_* \circ * \,.$$

We start with the following

Fact. *1)* $\lambda_g g' = f\lambda_{f'}$ *and* $\rho_g g' = f\rho_{f'}$. *2)* $\lambda_{g'} g = f'\lambda_f$ *and* $\rho_{g'} g = f'\rho_f$.

Proof. 1) We shall show that hypothesis (i) in (*) implies (1). For the first equation, let p' be an element in P' and set $\lambda_g g'(p') = f(\ell)$; then, (**) yields

$$\ell \;=\; \lambda_f f(\ell) \;=\; \lambda_f \lambda_g g'(p').$$

Hence, (a) and another application of (**) entail

$$\lambda_g g'(p') \;=\; f\lambda_f \lambda_g g'(p') \;=\; f\lambda_h g'(p') \;=\; f\lambda_{f'}\lambda_{g'} g'(p') \;=\; f\lambda_{f'}(p'),$$

as desired. The second equation in (1), as well as, the implication $(*).(ii) \Rightarrow (2)$ can be obtained similarly.

The first equation in (1) of the Fact and item(a) yield

$$\lambda_g g' = f\lambda_{f'} \;\;\Rightarrow\;\; g\lambda_g g'\lambda_{g'} = gf\lambda_{f'}\lambda_{g'} = h\lambda_h.$$

Similarly, the first equation in (2) of the Fact entails

$$g'\lambda_{g'} g\lambda_g = h\lambda_h,$$

wherefrom we conclude that $g\lambda_g g'\lambda_{g'} = g'\lambda_{g'} g\lambda_g$. The same argument applied to the second equations in (1) and (2) of the Fact shows that

$$g\rho_g g'\rho_{g'} = h\rho_h = g'\rho_{g'} g\rho_g,$$

ending the proof. $\square$

Corollary 42.9. *If A is a Ω-set and J, $K \subseteq I$ are sets, then*

$$Q\pi_J Q\pi_K \;=\; Q\pi_K Q\pi_J,$$

where Q is either $\exists$ or $\forall$.

Proof. By 42.6.(a), the following commutative diagram of open embeddings verifies the hypothesis (i) and (ii) in $(*)$ of 42.8 :

$$\begin{tikzcd}
\mathfrak{K}_I(A,K) \arrow[dr, "\iota_K"'] & \mathfrak{K}_I(A, J\cap K) \arrow[l, "\iota_{J\cap K,K}"'] \arrow[r, "\iota_{J\cap K,J}"] \arrow[d, "\iota_{J\cap K}"] & \mathfrak{K}_I(A,J) \arrow[dl, "\iota_J"] \\
 & \mathfrak{K}_I A &
\end{tikzcd}$$

Indeed, if $R \in \mathfrak{K}_I(A,K)$, then $Q\pi_J R$ is in $\mathfrak{K}_I(A, J\cap K)$; of course, the same applies, with the roles of J and K reversed and the conclusion follows from 42.8. $\square$

Proposition 42.2.(c) applied to the open embedding $\iota_J : \mathfrak{K}_I(A,J) \longrightarrow \mathfrak{K}_I A$ yields (analogously to 42.3)

Corollary 42.10. *If A is a Ω-set, $J \subseteq I$ are sets and $R,\ S \in \mathfrak{K}_I A$,*

a) $\forall \pi_J(R \to S) \ \leq\ \forall \pi_J R \to \forall \pi_J S$.

b) $\forall \pi_J \neg R \ =\ \neg\exists \pi_J \neg\neg R$ *and* $\forall \pi_J \neg\neg R \ =\ \neg\exists \pi_J \neg R$.

c) $\neg\forall \pi_J \neg R \ =\ \neg\neg\exists \pi_J \neg\neg S$.

d) $R \in \mathfrak{K}_I(A, K) \ \Rightarrow\ \forall v_i \neg R \in Reg(\mathfrak{K}_I(A, (K - J)))$. $\Box$

All the intuitionistic laws involving negation and quantifiers follow from 42.10.

The classical quantifiers along the i^{th}-coordinate, $\exists v_i$, $\forall v_i$, are special cases of the above, with $J = I - \{i\}$. Here, instead of the $\mathfrak{p}$ notation in 42.5, it is better to use the *substitution* of an element of $|A|$ into the i^{th}-coordinate of a I-sequence, introduced in 37.10.(b). This is because for $\overline{x} \in |A|^I$,

$$\mathfrak{p}(\overline{x},\ I - \{i\}) \ =\ \{\overline{x}\ulcorner a \mid i\urcorner \in |A|^I : a \in |A|\}.$$

The formulas for $\exists v_i$ and $\forall v_i$ are so important that we state them explicitly.

Corollary 42.11. *Let A be a Ω-set and I be a set.*

a) For $i \in I$ and $S \in \mathfrak{K}_I A$,

$$(\exists v_i) \qquad [\![\exists v_i S(\overline{x})]\!] \ =\ E\overline{x} \wedge \bigvee_{a \in |A|} [\![S(\overline{x}\ulcorner a \mid i\urcorner)]\!].$$

$$\begin{aligned}(\forall v_i) \qquad [\![\forall v_i S(\overline{x})]\!] \ &=\ E\overline{x} \wedge \bigwedge_{a \in |A|} Ea \to [\![S(\overline{x}\ulcorner a \mid i\urcorner)]\!] \\ &=\ E\overline{x} \wedge \bigwedge_{a \in |A|} Ea \to \Big(Ea \wedge [\![S(\overline{x}\ulcorner a \mid i\urcorner)]\!]\Big).\end{aligned}$$

b) $S \in \mathfrak{K}_I(A, K) \ \Rightarrow\ Qv_i S \in \mathfrak{K}_I(A, K - \{i\})$.

Proof. Immediate from the preceding results. For (b), just note that since Qv_i corresponds to $Q\pi_{I-\{i\}}$, 42.6.(a) guarantees that for $S \in \mathfrak{K}_I(A, K)$, $Qv_i S$ is in $\mathfrak{K}_I(A, K \cap I - \{i\}) \ =\ \mathfrak{K}_I(A,\ K - \{i\})$. $\Box$

Proposition 42.12. *Let L, R be frames and A, B be presheaves over L and R, respectively. Let $\mathfrak{f} = \langle f, \lambda \rangle : A \longrightarrow B$ be a morphism in* ***pSh****, with λ a frame morphism. Let I be a set and $R \in \mathfrak{K}_I B$. With notation as in* 39.1,

a) For all $i \in I$, $\mathfrak{f}^*(\exists v_i\, R) \ \geq\ \exists v_i\, \mathfrak{f}^* R$ *and* $\mathfrak{f}^*(\forall v_i\, R) \ \leq\ \forall v_i\, \mathfrak{f}^* R$.

b) Assume that f verifies $Eb = \bigvee_{a \in |A|} [\![fa = b]\!]$ and $\rho \circ \lambda = Id_L$, where ρ is the right adjoint of λ.

(1) *If ρ is a $\bigvee$-morphism, then* $\mathfrak{f}^*(\exists v_i R) \ =\ \exists v_i\ \mathfrak{f}^* R$.

(2) *If ρ is an open morphism, then* $\mathfrak{f}^*(\forall v_i R) \ =\ \forall v_i\ \mathfrak{f}^* R$.

Proof. a) Let $\rho : R \longrightarrow L$ be the right adjoint of λ. We shall use the following observations, all straightforward :

$*$ If $g : L \longrightarrow R$ is increasing and $S \subseteq L$, then $g(\bigvee S) \geq \bigvee_{s \in S} g(s)$;

$*$ $g : L \longrightarrow R$ is a semilattice morphism, then $g(p \to q) \leq g(p) \to g(q)$.

One should also keep in mind that $\rho \circ \lambda \geq Id_L$ (7.8.(a)).

We first treat the case of the universal quantifier.

FACT 42.13. *For $\overline{x} \in |A|^I$, $i \in I$, $a \in |A|$ and $R \in \mathfrak{K}_I B$*

$$E\overline{x} \wedge \rho\Big(Efa \to [\![\, R(f\overline{x}\,\ulcorner a \mid i\urcorner)\,]\!]\Big) \;\;\le\;\; Ea \to \mathfrak{f}^* R(\overline{x}\,\ulcorner a|i\urcorner)$$

Proof. Since $Efa = \lambda(Ea)$, by the second observation in the beginning of the proof, it is enough to check that

$$E\overline{x} \wedge \Big(\rho(\lambda(Ea)) \to \rho\Big([\![\, R(f\overline{x}\,\ulcorner a \mid i\urcorner)\,]\!]\Big)\Big) \;\;\le\;\; Ea \to \mathfrak{f}^* R(\overline{x}\,\ulcorner a \mid i\urcorner).$$

By the adjunction $[\to]$ in 6.1, this is equivalent to

$$Ea \wedge E\overline{x} \wedge \Big(\rho(\lambda(Ea)) \to \rho\Big([\![\, R(f\overline{x}\,\ulcorner a \mid i\urcorner)\,]\!]\Big)\Big) \;\;\le\;\; \mathfrak{f}^* R(\overline{x}\,\ulcorner a \mid i\urcorner).$$

Since $Ea \le \rho(\lambda(Ea))$, *Modus Ponens* and 37.11.(a).(3) yield

$$\begin{aligned}
&Ea \wedge E\overline{x} \wedge \Big(\rho(\lambda(Ea)) \to \rho\Big([\![\, R(f\overline{x}\,\ulcorner a \mid i\urcorner)\,]\!]\Big)\Big) \;\;= \\
&\quad= \;\; Ea \wedge E\overline{x} \wedge \rho(\lambda(Ea)) \wedge \Big(\rho(\lambda(Ea)) \to \rho\Big([\![\, R(f\overline{x}\,\ulcorner a \mid i\urcorner)\,]\!]\Big)\Big) \\
&\quad\le \;\; Ea \wedge E\overline{x} \wedge \rho\Big(([\![\, R(f\overline{x}\,\ulcorner a \mid i\urcorner)\,]\!]\Big) \\
&\quad= \;\; E\overline{x} \wedge E\overline{x}\,\ulcorner a \mid i\urcorner \wedge \rho\Big(([\![\, R(f\overline{x}\,\ulcorner a \mid i\urcorner)\,]\!]\Big) \\
&\quad\le \;\; E\overline{x}\,\ulcorner a \mid i\urcorner \wedge \rho\Big(([\![\, R(f\overline{x}\,\ulcorner a \mid i\urcorner)\,]\!]\Big) \\
&\quad= \;\; \mathfrak{f}^* R(\overline{x}\,\ulcorner a \mid i\urcorner),
\end{aligned}$$

establishing the Fact.

Since ρ is a $\bigwedge$-morphism we then obtain

$$\begin{aligned}
\mathfrak{f}^*(\forall v_i\, R)(\overline{x}) \;&=\; E\overline{x} \wedge \rho\Big([\![\, \forall v_i\, R(f\overline{x})\,]\!]\Big) \\
&=\; E\overline{x} \wedge \rho\Big(Ef\overline{x} \wedge \textstyle\bigwedge_{y\in|B|} Ey \to [\![\, R([f\overline{x}]\,\ulcorner y \mid i\urcorner)\,]\!]\Big) \\
&=\; E\overline{x} \wedge \textstyle\bigwedge_{y\in|B|} \rho\Big(Ey \to [\![\, R([f\overline{x}]\,\ulcorner y \mid i\urcorner)\,]\!]\Big) \\
&\le\; E\overline{x} \wedge \textstyle\bigwedge_{a\in|A|} \rho\Big(Efa \to [\![\, R([f\overline{x}]\,\ulcorner fa \mid i\urcorner)\,]\!]\Big) \\
&\le\; E\overline{x} \wedge \textstyle\bigwedge_{a\in|A|} Ea \to \mathfrak{f}^* R(\overline{x}\,\ulcorner a \mid i\urcorner) \;\;=\;\; \forall v_i\, \mathfrak{f}^* R(\overline{x}),
\end{aligned}$$

as needed. For the existential quantifier, 42.11 and 37.11.(a).(3) yield

$$\begin{aligned}
\mathfrak{f}^*(\exists v_i\, R)(\overline{x}) \;&=\; E\overline{x} \wedge \rho([\![\, \exists v_i\, R(f\overline{a})\,]\!]) \\
&=\; E\overline{x} \wedge \rho\Big(\textstyle\bigvee_{y\in|B|} Ey \wedge [\![\, R([f\overline{x}]\,\ulcorner y \mid i\urcorner)\,]\!]\Big) \\
&\ge\; E\overline{x} \wedge \rho\Big(\textstyle\bigvee_{a\in|A|} Efa \wedge [\![\, R([f\overline{x}]\,\ulcorner fa \mid i\urcorner)\,]\!]\Big) \\
&\ge\; E\overline{x} \wedge \rho\Big(\textstyle\bigvee_{a\in|A|} \lambda(Ea) \wedge [\![\, R(f\overline{x}\,\ulcorner a \mid i\urcorner)\,]\!]\Big) \\
&\ge\; E\overline{x} \wedge \textstyle\bigvee_{a\in|A|} \rho(\lambda(Ea)) \wedge \rho\Big([\![\, R(f\overline{x}\,\ulcorner a \mid i\urcorner)\,]\!]\Big) \\
&\ge\; E\overline{x} \wedge \textstyle\bigvee_{a\in|A|} E\overline{x} \wedge Ea \wedge \rho\Big([\![\, R(f\overline{x}\,\ulcorner a \mid i\urcorner)\,]\!]\Big) \\
&=\; E\overline{x} \wedge \textstyle\bigvee_{a\in|A|} E\overline{x}\,\ulcorner a \mid i\urcorner \wedge \rho\Big([\![\, R(f\overline{x}\,\ulcorner a \mid i\urcorner)\,]\!]\Big) \\
&=\; E\overline{x} \wedge \textstyle\bigvee_{a\in|A|} \mathfrak{f}^* R(\overline{x}\,\ulcorner a \mid i\urcorner) \;\;=\;\; \exists v_i\, \mathfrak{f}^* R(\overline{x}).
\end{aligned}$$

b) (1) First note that for $\overline{a} \in |A|^I$ and $b \in |B|$,

$$\begin{aligned}
[\![R(f\overline{a}\ulcorner b \mid i\urcorner)]\!] &= [\![R(f\overline{a}\ulcorner b \mid i\urcorner)]\!] \wedge Eb \\
&= [\![R(f\overline{a}\ulcorner b \mid i\urcorner)]\!] \wedge \bigvee_{c\in|A|} [\![fc = b]\!] \\
&= \bigvee_{c\in|A|} [\![R(f\overline{a}\ulcorner b \mid i\urcorner)]\!] \wedge [\![fc = b]\!] \\
&= \bigvee_{c\in|A|} [\![R(f(\overline{a}\ulcorner c \mid i\urcorner))]\!] \wedge [\![fc = b]\!].
\end{aligned}$$

Hence, since $\rho(Efc) = \rho(\lambda(Ec)) = Ec$,

$$\begin{aligned}
\mathfrak{f}^*(\exists v_i R)(\overline{a}) &= E\overline{a} \wedge \rho([\![\exists v_i R(\overline{a})]\!]) \\
&= E\overline{a} \wedge \rho\left(\bigvee_{b\in|B|} [\![R(f\overline{a}\ulcorner b \mid i\urcorner)]\!]\right) \\
&= E\overline{a} \wedge \bigvee_{b\in|B|} \rho([\![Rf(\overline{a}\ulcorner b \mid i\urcorner)]\!]) \\
&= E\overline{a} \wedge \bigvee_{b\in|B|} \bigvee_{c\in|A|} \rho([\![R(f(\overline{a}\ulcorner c \mid i\urcorner))]\!] \wedge [\![fc = b]\!]) \\
&= E\overline{a} \wedge \bigvee_{c\in|A|} \rho\left([\![R(f(\overline{a}\ulcorner c \mid i\urcorner))]\!]) \wedge \bigvee_{b\in|B|} [\![fc = b]\!]\right) \\
&= E\overline{a} \wedge \bigvee_{c\in|A|} \rho([\![R(f(\overline{a}\ulcorner c \mid i\urcorner))]\!]) \wedge \rho(Efc) \\
&= E\overline{a} \wedge \bigvee_{c\in|A|} \rho([\![R(f(\overline{a}\ulcorner c \mid i\urcorner))]\!]) \wedge Ec \\
&= E\overline{a} \wedge \bigvee_{c\in|A|} E\overline{a}\ulcorner c \mid i\urcorner \wedge \rho([\![R(f(\overline{a}\ulcorner c \mid i\urcorner))]\!] \;=\; \exists v_i \mathfrak{f}^* R(\overline{a}).
\end{aligned}$$

(2) For the universal quantifier, recall (8.16.(b)) that

$$(*) \qquad \left(\bigvee_{\lambda\in\Lambda} p_\lambda\right) \;\to\; q \;=\; \bigwedge_{\lambda\in\Lambda} (p_\lambda \to q).$$

Since $Eb = \bigvee_{c\in|A|} [\![fc = b]\!]$, (*) yields

$$\begin{aligned}
Eb \to [\![R(f\overline{a}\ulcorner b \mid i\urcorner)]\!] &= (\bigvee_{c\in|A|} [\![fc = b]\!]) \to [\![R(f\overline{a}\ulcorner b \mid i\urcorner)]\!] \\
&= \bigwedge_{c\in|A|} [\![fc = b]\!] \to [\![R(f\overline{a}\ulcorner b \mid i\urcorner)]\!] \\
&= \bigwedge_{c\in|A|} [\![fc = b]\!] \;\to\; ([\![fc = b]\!] \wedge [\![R(f\overline{a}\ulcorner b \mid i\urcorner)]\!]) \\
&= \bigwedge_{c\in|A|} [\![fc = b]\!] \;\to\; ([\![fc = b]\!] \wedge [\![R(f\overline{a}\ulcorner fc \mid i\urcorner)]\!]) \\
&= \bigwedge_{c\in|A|} [\![fc = b]\!] \to [\![R(f(\overline{a}\ulcorner c \mid i\urcorner))]\!].
\end{aligned}$$

Thus, another application of (*) entails

$$\begin{aligned}
\bigwedge_{b\in|B|} Eb \to [\![R(f\overline{a}\ulcorner b \mid i\urcorner)]\!] &= \\
&= \bigwedge_{b\in|B|} \bigwedge_{c\in|A|} ([\![fc = b]\!] \to [\![R(f(\overline{a}\ulcorner c \mid i\urcorner))]\!]) \\
&= \bigwedge_{c\in|A|} \bigwedge_{b\in|B|} ([\![fc = b]\!] \to [\![R(f(\overline{a}\ulcorner c \mid i\urcorner))]\!]) \\
&= \bigwedge_{c\in|A|} \left(\bigvee_{b\in|B|} [\![fc = b]\!]\right) \to [\![R(f(\overline{a}\ulcorner c \mid i\urcorner))]\!] \\
&= \bigwedge_{c\in|A|} Efc \to [\![R(f(\overline{a}\ulcorner c \mid i\urcorner))]\!].
\end{aligned}$$

Since ρ is open and $\rho(Efc) = Ec$, we then obtain

$$\bigwedge_{b\in|B|} \rho(Eb \to [\![R(f\overline{a}\ulcorner b \mid i\urcorner)]\!]) \;=\; \bigwedge_{c\in|A|} Ec \to \rho([\![R(f(\overline{a}\ulcorner c \mid i\urcorner))]\!]),$$

and the conclusion in (2) is immediately forthcoming. □

For localization at the filter of dense elements and regularization by double negation we have, complementing 39.3 and 39.5 :

Theorem 42.14. *Let A be a Ω-presheaf and $I \neq \emptyset$ be a finite set. Let*

$$\mathfrak{h} = \langle \varepsilon_D, \pi_D \rangle : A \longrightarrow A/D \qquad \textit{and} \qquad A \xrightarrow{r} rA$$

be the localization of A at the filter D of dense elements in Ω (34.6, 39.3) *and the regularization associated to double negation* (35.7, 39.5).

a) For $T \in \mathfrak{K}_I(rA)$, $\begin{cases} (1) \quad r^*\forall v_i \neg T \;=\; \forall v_i r^* \neg T; \\ (2) \quad r^* \neg \exists v_i T \;=\; \neg \exists v_i r^* T. \end{cases}$

b) If $T \in Reg(\mathfrak{K}_I(rA))$, *then* $r^*(\forall v_i T) \;=\; \forall v_i r^* T$.

c) For $T \in \mathfrak{K}_I(A/D)$, $\begin{cases} (1) \quad \mathfrak{h}^*\forall v_i \neg T \;=\; \forall v_i \neg \mathfrak{h}^* T; \\ (2) \quad \mathfrak{h}^* \neg \exists v_i T \;=\; \neg \exists v_i \mathfrak{h}^* T. \end{cases}$

d) If $T \in Reg(\mathfrak{K}_I(A/D))$, *then* $\mathfrak{h}^*\forall v_i T \;=\; \forall v_i \mathfrak{h}^* T$.

PROOF. We prove (a) and (b), leaving (c), (d) to the reader. Recall the following facts concerning implication and negation in a frame :

$*$ $p \wedge (q \to r) \;=\; p \wedge ((p \wedge q) \to (p \wedge r))$; (6.4.(i))
$*$ $p \wedge q \;=\; p \to (p \wedge q)$;
$*$ $\neg\neg$ distributes over $\wedge$ and $\to$; (6.8.(g), 6.8.(j))
$*$ $q \in Reg(\Omega) \;\Rightarrow\; p \to q = \neg\neg p \to q \in Reg(\Omega)$; (6.20.(b))
$*$ $\neg\neg \bigwedge_{s \in S} \neg\neg s \;=\; \bigwedge_{s \in S} \neg\neg s$; (8.16.(g))
$*$ $\neg(\bigvee S) \;=\; \neg \bigvee_{s \in S} \neg\neg s$; (8.16.(h))

These relations will be used forthwith, without comment.

FACT 42.15. *For* $\langle x, p \rangle \in |rA|$, $\overline{a} \in |A|^I$ *and* $T \in \mathfrak{K}_I(rA)$

$$\neg [\![T([r\overline{a}] \ulcorner \langle x, p \rangle \mid i \urcorner)]\!] \;=\; \neg [\![T(r(\overline{a} \ulcorner x \mid i \urcorner))]\!].$$

Proof. From Fact 39.6.(a) comes

$$[\![T(r(\overline{a} \ulcorner x \mid i \urcorner))]\!] \;=\; \textstyle\bigwedge_{j \neq i} Ea_j \wedge Ex \wedge [\![T([r\overline{a}] \ulcorner \langle x, p \rangle \mid i \urcorner)]\!].$$

Taking double negation on both sides, recalling that $Ex \leq p \leq \neg\neg Ex$, we get

$$\begin{aligned} \neg\neg [\![T(r(\overline{a} \ulcorner x \mid i \urcorner))]\!] \;&=\; \textstyle\bigwedge_{i \neq j} \neg\neg Ea_j \wedge \neg\neg Ex \wedge \neg\neg [\![T([r\overline{a}] \ulcorner \langle x, p \rangle \mid i \urcorner)]\!] \\ &=\; \neg\neg E[r\overline{a}] \ulcorner \langle x, p \rangle \mid i \urcorner \wedge \neg\neg [\![T([r\overline{a}] \ulcorner \langle x, p \rangle \mid i \urcorner)]\!] \\ &=\; \neg\neg [\![T([r\overline{a}] \ulcorner \langle x, p \rangle \mid i \urcorner)]\!], \end{aligned}$$

which is equivalent to the desired result.

Proof of (1) : From 42.15 we get, since $E\overline{a} = Er\overline{a}$ and $E\langle x, p \rangle = p$ (35.1) :

$$\begin{aligned} r^*(\forall v_i \neg T)(\overline{a}) \;&=\; [\![\forall v_i \neg T(r\overline{a})]\!] \\ &=\; E\overline{a} \wedge \textstyle\bigwedge_{\langle x,p \rangle \in |rA|} E\langle x, p \rangle \to [\![\neg T([r\overline{a}] \ulcorner \langle x, p \rangle \mid i \urcorner)]\!] \\ &=\; E\overline{a} \wedge \textstyle\bigwedge_{\langle x,p \rangle \in |rA|} p \to E[r\overline{a}] \ulcorner \langle x, p \rangle \mid i \urcorner \wedge \neg [\![T([r\overline{a}] \ulcorner \langle x, p \rangle \mid i \urcorner)]\!] \\ &=\; E\overline{a} \wedge \textstyle\bigwedge_{\langle x,p \rangle \in |rA|} (E\overline{a} \wedge p) \;\to \\ &\qquad \to\; E\overline{a} \wedge E[r\overline{a}] \ulcorner \langle x, p \rangle \mid i \urcorner \wedge \neg [\![T(r(\overline{a} \ulcorner \langle x, p \rangle \mid i \urcorner)]\!]. \end{aligned}$$

By Lemma 37.11.(a).(3), we have

$$E\overline{a} \wedge E[r\overline{a}] \ulcorner \langle x, p \rangle \mid i \urcorner \;=\; Er\overline{a} \wedge E[r\overline{a}] \ulcorner \langle x, p \rangle \mid i \urcorner = E\overline{a} \wedge E\langle x, p \rangle = E\overline{a} \wedge p,$$

which substituted in the expression above yields

$$\text{(I)} \quad \begin{aligned} r^*(\forall v_i \neg T)(\overline{a}) &= E\overline{a} \wedge \bigwedge_{\langle x,p\rangle \in |rA|} (E\overline{a} \wedge p) \to \\ &\qquad\qquad \to E\overline{a} \wedge p \wedge \neg [\![T(r(\overline{a} \ulcorner x \mid i \urcorner))]\!] \\ &= E\overline{a} \wedge \bigwedge_{\langle x,p\rangle \in |rA|} p \to \neg [\![T(r(\overline{a} \ulcorner x \mid i \urcorner))]\!]. \end{aligned}$$

Since $\neg [\![T(r(\overline{a} \ulcorner x \mid i \urcorner))]\!]$ is regular in Ω, for $\langle x, p \rangle \in |rA|$, we have

$$\begin{aligned} p \to \neg [\![T(r(\overline{a} \ulcorner x \mid i \urcorner))]\!] &= \neg\neg\, p \to \neg [\![T(r(\overline{a} \ulcorner x \mid i \urcorner))]\!] \\ &= \neg\neg\, Ex \to \neg [\![T(r(\overline{a} \ulcorner x \mid i \urcorner))]\!] \\ &= Ex \to \neg [\![T(r(\overline{a} \ulcorner x \mid i \urcorner))]\!] \end{aligned}$$

and so (I) leads to $\begin{aligned} r^*(\forall v_i \neg T)(\overline{a}) &= E\overline{a} \wedge \bigwedge_{x \in |A|} Ex \to \neg [\![T(r(\overline{a} \ulcorner x \mid i \urcorner))]\!] \\ &= \forall v_i r^* \neg T(\overline{a}), \end{aligned}$

establishing (1). Item (b) is immediate from this and Theorem 39.5.

Proof of (2) : For $\overline{a} \in |A|^I$, 42.15 entails

$$\begin{aligned} r^*(\neg \exists v_i T)(\overline{a}) &= [\![\neg \exists v_i T(r\overline{a})]\!] = E\overline{a} \wedge \neg [\![\exists v_i T(r\overline{a})]\!] \\ &= E\overline{a} \wedge \neg \left(\bigvee_{\langle x,p\rangle \in |rA|} [\![T([r\overline{a}] \ulcorner \langle x,p \rangle \mid i \urcorner)]\!] \right) \\ &= E\overline{a} \wedge \neg \left(\bigvee_{\langle x,p\rangle \in |rA|} \neg\neg [\![T([r\overline{a}] \ulcorner \langle x,p \rangle \mid i \urcorner)]\!] \right) \\ &= E\overline{a} \wedge \neg \left(\bigvee_{\langle x,p\rangle \in |rA|} \neg\neg [\![T(r(\overline{a} \ulcorner x \mid i \urcorner))]\!] \right) \\ &= E\overline{a} \wedge \neg \left(\bigvee_{x \in |A|} [\![T(r(\overline{a} \ulcorner x \mid i \urcorner))]\!] \right) = \neg \exists v_i r^* T(\overline{a}), \end{aligned}$$

ending the proof. □

Exercises

42.16. Let A be a Ω-presheaf and $E : A \longrightarrow \mathbf{1}$ be the unique presheaf morphism from A to the final object $\mathbf{1}$ (25.5, 26.10). Let S be a closed subpresheaf of A and $p \in \Omega$.

a) The following are equivalent :

(1) $\exists_E S = p$; (2) $ES = p$ [3];

(3) $\exists \{p_i : i \in I\} \subseteq p$ with $p = \bigvee_{i \in I} p_i$ and $S(p_i) \neq \emptyset$, $\forall\, i \in I$.

b) The following are equivalent :

(1) $\neg [\![S = \emptyset]\!] = p$ [4].

(2) $p = \bigvee \left\{ q \in \Omega : \begin{array}{c} \forall\, r \leq q, \text{ if } r \neq \bot, \text{ there is } \bot \neq r' \leq r, \\ \text{such that } S(r') \neq \emptyset. \end{array} \right\}$

(3) $p \in Reg(\Omega)$, $ES \leq p$ and there is a $q \leq p$, such that $\neg\neg\, q = p$ and $S(q) \neq \emptyset$.

c) What can be said about the relation between the following elements of Ω : $\exists_E S$ and $[\![S \neq \emptyset]\!] =_{def} \neg [\![S = \emptyset]\!]$? □

[3] ES is the support of S, as in 26.1.(c).

[4] $\emptyset$ is the empty subpresheaf of A; this equality is defined in 36.6.

CHAPTER 43

Relations. Equivalence Relations and Quotients

As in section 24.3, a **relation** on presheaves $A_1, \ldots, A_n$ is a subpresheaf of $\prod A_i$. Since we are mainly interested in the subobjects describable via characteristic maps, we **define** a relation on $A_1, \ldots, A_n$ as a **closed** subpresheaf of $\prod A_i$. Any set of sections in this product generates a relation on $A_1, \ldots, A_n$: its closure in $\prod A_i$. Notice that this automatically prescribes that a relation on a finite family of sheaves is a **subsheaf** of their product.

DEFINITION 43.1. *Let $n \geq 1$ be an integer. A* ***n-ary relation*** *on a Ω-presheaf A is a characteristic map in $\mathfrak{K}_n A$. If R is a binary relation on A, that is $R \in \mathfrak{K}_2 A$, we frequently use the so called* infix notation *for R, that is,*

$$[\![\, x\ R\ y \,]\!] \text{ stands for } [\![\, R(x,y) \,]\!].$$

All subsets $S \subseteq |A^n|$ generate a n-ary relation on A, namely $[\![\, S(*) \,]\!]$.

LEMMA 43.2. *Let A be a L-set. For $R \in \mathfrak{K}_n A$, $n \geq 2$, the following are equivalent*[1] *:*

(1) *For all $\overline{x} \in |A|^n$, $[\![\, \Delta_n(\overline{x}) \,]\!] \ \leq\ [\![\, R(\overline{x}) \,]\!]$;*

(2) *For all $a \in |A|$, $[\![\, R(\widehat{a}) \,]\!] \ =\ Ea$.*

PROOF. By 37.19, $[\![\, \Delta_n(\overline{x}) \,]\!] \ = \ \bigwedge_{i=1}^{n-1} [\![x_i \ = \ x_{i+1}]\!]$, and (1) $\Rightarrow$ (2) follows immediately. For the converse, given $\overline{x} \in |A|^n$, formula ($\sharp$) in the proof of 37.19 (page 408), yields, with $a = x_n$

$$\begin{aligned} \bigwedge_{i=1}^{n-1} [\![x_i \ = \ x_{i+1}]\!] \ &= \ Ex_n \wedge \bigwedge_{i=1}^{n-1} [\![x_i \ = \ x_{i+1}]\!] \\ &= \ Ex_n \wedge \bigwedge_{i=1}^{n} [\![x_n \ = \ x_i]\!] \\ &= \ [\![\, R(\widehat{x}_n) \,]\!] \wedge \bigwedge_{i=1}^{n} [\![x_n \ = \ x_i]\!] \ \leq \ [\![\, R(\overline{x}) \,]\!], \end{aligned}$$

as needed. $\square$

DEFINITION 43.3. *If R is a binary relation on A, define the* **inverse of R** *by the rule*

$$\textit{For all } \langle\, x, y \,\rangle \in |A|^2, \quad [\![\, y\ R^{-1}\ x \,]\!] \ = \ [\![\, x\ R\ y \,]\!].$$

Classically, an *equivalence relation* on a set A is a subset $\theta \subseteq A \times A$, such that $\Delta \subseteq \theta$, $\theta = \theta^{-1}$ and $\theta \circ \theta \subseteq \theta$. By analogy, we set down

[1] Notation as in 37.19; moreover, $\widehat{a} = \langle\, a, a, \ldots, a \,\rangle \in A^n$.

DEFINITION 43.4. *An* **equivalence relation** *on a L-set A is a binary relation θ on A, such that for all x, y, $z \in |A|$*

[ER 1] : $[\![x = y]\!] \ \leq \ [\![\, x\ \theta\ y \,]\!]$;

[ER 2] : $[\![\, x\ \theta\ y \,]\!] \ = \ [\![\, y\ \theta\ x \,]\!]$;

[ER 3] : $[\![\, x\ \theta\ z \,]\!] \wedge [\![\, z\ \theta\ y \,]\!] \ \leq \ [\![\, x\ \theta\ y \,]\!]$.

Write $Eq(A)$ for the poset of equivalence relations on A, with the order induced by $\mathfrak{K}_2A$.

REMARK 43.5. Let A be an L-set and $\theta \in Eq(A)$.

a) By 43.2, condition [ER 1] in 43.4 is equivalent to

[ER 1′] : $\forall\, x \in |A|, \ \ [\![\, x\,\theta\,x \,]\!] = Ex$.

Note the similarity of [ER 1′] to the usual reflexive law "$x\ \theta\ x$".

b) For x, $y \in |A|$, we have $[\![x = y]\!] \ \leq \ [\![\, x\ \theta\ y \,]\!] \ \leq \ Ex \wedge Ey$. It is clear that

$$[\![\, x \top_2 y \,]\!] = Ex \wedge Ey \quad \text{and} \quad [\![x = y]\!],$$

are both equivalence relations on A, in fact, the top and bottom, respectively, of the poset $Eq(A)$. Exercise 43.9 describes some of the lattice-theoretic properties of $Eq(A)$. $\square$

EXAMPLE 43.6. Let A be a Ω-set and $p \in \Omega$. For x, $y \in |A|$, define

$$[\![\, x\ \theta_p\ y \,]\!] \ = \ Ex \wedge Ey \wedge [p \to (p \wedge [\![x = y]\!])].$$

It is easily established that $\theta_p \in \mathfrak{K}_2A$ and that it satisfies [ER 1′] in 43.5.(a) and [ER 2] in 43.4. Hence, to see that θ_p is an equivalence relation on A, it remains to verify transitivity; for x, y, $z \in |A|$, this comes down to

$$p \wedge Ex \wedge Ey \wedge Ez \wedge [\![\, x\ \theta_p\ y \,]\!] \wedge [\![\, y\ \theta_p\ z \,]\!] \ \leq \ p \wedge [\![x = z]\!].$$

But the rule of *Modus Ponens* in 6.4 yields

$$\begin{aligned} p \wedge Ex \wedge Ey \wedge Ez \wedge [p \to (p \wedge [\![x = y]\!])] \wedge [p \to (p \to [\![y = z]\!])] \ \leq& \\ \leq \ p \wedge ([\![x = y]\!] \wedge [\![y = z]\!]) \ \leq \ p \wedge [\![x = z]\!],& \end{aligned}$$

as needed. In the notation of 37.16, we have

$$S_{\theta_p} \ = \ \{\langle\, s, t\,\rangle \in |A^2| : p \wedge Es \wedge Et \ = \ p \wedge [\![s = t]\!]\},$$

which, in case A is a Ω-presheaf, may be written

$$S_{\theta_p} \ = \ \{\langle\, s, t\,\rangle \in |A^2| : s_{|p} = t_{|p}\}.$$

Hence, the equivalence relation θ_p corresponds to *compatibility over* p in A, as defined in 25.26. $\square$

43.7. **Quotients.** If θ is an equivalence relation on an L-set A, let $\widehat{\theta}$ be the following binary relation on $|A|$:

$$x\,\widehat{\theta}\,y \quad \text{iff} \quad Ex \ = \ Ey \ = \ [\![\, x\ \theta\ y \,]\!].$$

It is clear $\widehat{\theta}$ is an equivalence relation on $|A|$. Write x/θ for the class of $x \in |A|$ with respect to $\widehat{\theta}$. Define a L-set A/θ by the following prescriptions :

$(i)\ |A/\theta| \ = \ \{x/\theta : x \in |A|\} \ = \ |A|/\widehat{\theta}; \quad (ii)\ [\![x/\theta = y/\theta]\!] \ = \ [\![\, x\ \theta\ y \,]\!].$

Then A/θ is an extensional L-set and the natural map

$$\pi_\theta : |A| \longrightarrow |A/\theta|, \quad x \longmapsto x/\theta$$

is a morphism of L-sets. The diagram $A \xrightarrow{\pi_\theta} A/\theta$ is the **quotient of $\boldsymbol{A}$ by the equivalence relation $\boldsymbol{\theta}$**. The universal property of the diagram $A \xrightarrow{\pi_\theta} A/\theta$ appears in Exercise 43.10.

Now suppose A is a L-presheaf. For x, $y \in |A|$ and $p \in L$,

(*) $$x\ \widehat{\theta}\ y \quad \Rightarrow \quad x_{|p}\ \widehat{\theta}\ y_{|p}.$$

To see this, note that $[\![\, x_{|p}\ \theta\ x \,]\!] \geq [\![x_{|p} = x]\!] = p \wedge Ex$, with a similar relation holding with y in place of x. Thus, [ER 3] in 43.4 and the hypothesis that $x\ \widehat{\theta}\ y$ entail

$$[\![\, x_{|p}\ \theta\ y_{|p} \,]\!] \ \geq \ [\![\, x_{|p}\ \theta\ x \,]\!] \wedge [\![\, x\ \theta\ y \,]\!] \wedge [\![\, y\ \theta\ y_{|p} \,]\!] \ = \ p \wedge Ex \wedge Ey.$$

Since $Ex_{|p} = p \wedge Ex = p \wedge Ey = Ey_{|p}$ and θ is a characteristic function, the above inequality implies $[\![\, x_{|p}\ \theta\ y_{|p} \,]\!] \ = \ Ex_{|p} \ = \ Ey_{|p}$, establishing (*). Since θ is a congruence with respect to restriction, there is a natural way to make the quotient A/θ into a L-presheaf and π_θ into a presheaf morphism : for $x \in |A|$ and $p \in L$, define

(**) $$(x/\theta)_{|p} \ = \ x_{|p}/\theta.$$

The diagram $A \xrightarrow{\pi_\theta} A/\theta$ is the **quotient of $\boldsymbol{A}$ by $\boldsymbol{\theta}$** in the category $\boldsymbol{pSh(L)}$. In general. A/θ *will not* be a sheaf even if A is a sheaf. **In this case, we define the quotient of $\boldsymbol{A}$ by $\boldsymbol{\theta}$ to be the completion of the presheaf constructed above**, still indicating it by A/θ. □

As an application of equivalence relations, we prove a general result concerning the gluing of families of presheaves. For topological spaces, this appears in section 3.3 of Chapter 0 in [**24**] (and Exercise II.1.22 in [**25**]). This is the process by which a general scheme is constructed out of affine schemes in Algebraic Geometry.

If $p \in \Omega$, the ideal $p^{\leftarrow} = \{q \in \Omega : q \leq p\}$ is a frame, with p in the role of $\top$. The concept of restriction of a L-set A to an element p of L, $A_{|p}$, was defined in 25.3. With these preliminaries, we state

Theorem 43.8. *Let Ω be a frame and B be a subset of Ω. Suppose we are given the following data :*

(1) *For each $b \in B$, a presheaf A_b over $b^{\leftarrow}$;*

(2) *For each $\langle\, a, b\,\rangle \in B \times B$, an isomorphism $f_{ab} : A_{a_{|a \wedge b}} \longrightarrow A_{b_{|a \wedge b}}$, such that for all $a, b, c \in B$ we have, with $p = a \wedge b \wedge c$,*

a) $f_{bb} = Id_{A_b}$.

b) If $x \in |A_a|$ is such that $Ex \leq p$, then $f_{ac}(x) \ = \ [f_{bc} \circ f_{ab}](x)$.

$$\begin{tikzcd} A_{a|p} \arrow[r, "f_{ab}"] \arrow[d, "f_{ac}"'] & A_{b|p} \arrow[dl, "f_{bc}"] \\ A_{c|p} & \end{tikzcd}$$

Then, there is a unique presheaf A over Ω, together with isomorphisms, $g_b : A_b \longrightarrow A_{|b}$, $b \in B$, such that for all $a, b \in B$ and $x \in |A_a|$,

$$Ex \leq a \wedge b \quad \Rightarrow \quad [g_b \circ f_{ab}](x) \;=\; g_a(x).$$

Moreover, if each A_b is a sheaf over $b^{\leftarrow}$, then A is a sheaf over Ω.

PROOF. For each $b \in B$, let $\mathfrak{e}A_b$ be the base extension from $b^{\leftarrow}$ to Ω, as in 31.3. By 32.11.**(III)** and 32.12, we have, for $u \in \Omega$

$$\mathfrak{e}A_b(u) \;=\; \begin{cases} A_p(u) & \text{if } u \leq p \\ \emptyset & \text{otherwise,} \end{cases}$$

and $\mathfrak{e}A_p$ is a Ω-sheaf, whenever A_p is a $p^{\leftarrow}$-sheaf.

Let $\langle C, \{k_b : b \in B\} \rangle$ be the coproduct of the $\mathfrak{e}A_b$ in $\boldsymbol{pSh(\Omega)}$, where the $k_b : \mathfrak{e}A_b \longrightarrow C$ are the associated monics, as in 25.19 and 26.20. Recall that $|C| \;=\; \coprod_{b \in B} |\mathfrak{e}A_b|$, identifying the unique sections over $\perp$ in each $\mathfrak{e}A_b$. Since $\mathfrak{e}A_b(u) = \emptyset$ if $u \notin b^{\leftarrow}$, we may write the elements of $|C|$ as pairs $\langle x, a \rangle$ with $x \in |A_a|$; in particular,

$$E\langle x, a \rangle \;=\; Ex \;\leq\; a.$$

Equality and restriction in C are given by

$$(*) \qquad \begin{cases} [\![\langle x, a \rangle = \langle y, b \rangle]\!] \;=\; \begin{cases} [\![x = y]\!] & \text{if } a = b \\ \perp & \text{otherwise.} \end{cases} \\[2ex] \langle x, a \rangle_{|q} \;=\; \langle x_{|a \wedge q}, \, a \wedge q \rangle. \end{cases}$$

By 26.20 and 32.12, C is a sheaf whenever each A_b is a sheaf over Ω.

Define $\theta : |C| \times |C| \longrightarrow \Omega$ by

$$[\![\, \langle x, a \rangle \;\theta\; \langle y, b \rangle \,]\!] \;=\; [\![f_{ab}(x_{|b}) = y]\!].$$

The definition of θ makes sense because $x_{|b} \in A_{a|b}$. Hence,

$$\begin{aligned} [\![\, \langle x, a \rangle \;\theta\; \langle y, b \rangle \,]\!] \;&\leq\; Ef_{ab}(x_{|b}) \wedge Ey \;=\; b \wedge Ex \wedge Ey \\ &=\; Ex \wedge Ey \;=\; E\langle x, a \rangle \wedge E\langle y, b \rangle. \end{aligned}$$

To verify that $\theta \in \mathfrak{K}_2C$, it must be shown that

$$\begin{aligned} [\![\, \langle x, a \rangle \;\theta\; \langle y, b \rangle \,]\!] \wedge [\![\langle x, a \rangle = \langle x', a' \rangle]\!] \wedge [\![\langle y, b \rangle = \langle y', b' \rangle]\!] \;\leq& \\ \leq\; [\![\, \langle x, a' \rangle \;\theta\; \langle y', b' \rangle \,]\!]&. \end{aligned}$$

By (*), we may assume $a = a'$ and $b = b'$. Thus,

$$\begin{aligned}
[\![f_{ab}(x_{|b}) = y]\!] \wedge [\![\langle x, a \rangle = \langle x', a \rangle]\!] \wedge [\![\langle y, b \rangle = \langle y', b \rangle]\!] \\
&= [\![f_{ab}(x_{|b}) = y]\!] \wedge [\![x = x']\!] \wedge [\![y = y']\!] \\
&\leq [\![f_{ab}(x_{|b}) = y]\!] \wedge b \wedge [\![x = x']\!] \wedge [\![y = y']\!] \\
&= [\![f_{ab}(x_{|b}) = y]\!] \wedge [\![x_{|b} = x'_{|b}]\!] \wedge [\![y = y']\!] \\
&\leq [\![f_{ab}(x_{|b}) = y]\!] \wedge [\![f_{ab}(x_{|b}) = f_{ab}(x'_{|b})]\!] \wedge [\![y = y']\!] \\
&\leq [\![f_{ab}(x'_{|b}) = y]\!] \wedge [\![y = y']\!] \;\leq\; [\![f_{ab}(x'_{|b}) = y']\!]
\end{aligned}$$

and property [ch 2] in 37.1 follows immediately. It remains to check that θ is an equivalence relation on C. We have

$*$ Property (a) in the statement implies $[\![\langle x, a \rangle \; \theta \; \langle x, a \rangle]\!] = Ex$; by 43.2, θ verifies [ER 1] in 43.4;

$*$ By (2).(b), f_{ba} restricted to $A_{a_{|a \wedge b}}$ has f_{ab} as its inverse. Thus, definition θ is symmetrical, i.e., $[\![\langle x, a \rangle \; \theta \; \langle y, b \rangle]\!] = [\![\langle y, b \rangle \; \theta \; \langle x, a \rangle]\!]$.

$*$ To check transitivity, let $\langle x, a \rangle$, $\langle y, b \rangle$ and $\langle z, c \rangle$ be sections in $|C|$. Then, if $p = a \wedge b \wedge c$, (2).(b) and 26.8.(b) yield

$$\begin{aligned}
[\![\langle x, a \rangle \; \theta \; \langle y, b \rangle]\!] \wedge [\![\langle b, y \rangle \; \theta \; \langle z, c \rangle]\!] = \\
&= [\![f_{ab}(x_{|b}) = y]\!] \wedge [\![f_{bc}(y_{|c}) = z]\!] \\
&= p \wedge [\![f_{ab}(x_{|b}) = y]\!] \wedge [\![f_{bc}(y_{|c}) = z]\!] \\
&= [\![f_{ab}(x_{|p}) = y_{|p}]\!] \wedge [\![f_{bc}(y_{|p}) = z_{|p}]\!] \\
&\leq [\![f_{bc}(f_{ab}(x_{|p})) = f_{bc}(y_{|p})]\!] \wedge [\![f_{bc}(y_{|p}) = z_{|p}]\!] \\
&= [\![f_{ac}(x_{|p}) = f_{bc}(y_{|p})]\!] \wedge [\![f_{bc}(y_{|p}) = z_{|p}]\!] \\
&\leq [\![f_{ac}(x_{|p}) = z_{|p}]\!] \;=\; p \wedge [\![f_{ac}(x_{|c}) = z]\!] \\
&\leq [\![f_{ac}(x_{|c}) = z]\!],
\end{aligned}$$

and so θ is transitive. Let A be the quotient presheaf C/θ and $\pi_\theta : C \longrightarrow C/\theta$ be the quotient map. Recall (43.7) that the domain of C/θ is $|A|/\widehat{\theta}$, where $\widehat{\theta}$ is the equivalence relation

$$\langle t, a \rangle \; \widehat{\theta} \; \langle z, b \rangle \quad \text{iff} \quad Et = Ez = [\![f_{ab}(t_{|b}) = z]\!].$$

Since A_b is extensional and $Ez \leq a \wedge b$, this equivalence can be written [2]

$$(+) \qquad \langle t, a \rangle \; \widehat{\theta} \; \langle z, b \rangle \quad \text{iff} \quad f_{ab}(t) = z.$$

For $b \in B$, consider the presheaf morphisms $g_b = \pi_\theta \circ k_b : A_b \longrightarrow A$; we shall prove that they satisfy the requirements in the conclusion of the Theorem.

If $\langle z, c \rangle/\theta \in |A_{|b}|$, then $E\langle z, c \rangle/\theta = Ez \leq b$, and so $Ez \leq b \wedge c$. Hence, (+) entails $\langle z, c \rangle \; \widehat{\theta} \; \langle f_{cb}(z), b \rangle$; therefore

$$\pi_\theta \circ k_b(f_{cb}(z)) \;=\; \pi_\theta \left(\langle f_{cb}(z), b \rangle \right) \;=\; \langle z, c \rangle/\theta,$$

[2]Because $t_{|b} = t$.

and $g_b : A_p \longrightarrow A_{|p}$ is surjective. For injectivity if $x, y \in |A_b|$, then [3],

$$\begin{array}{rcl} [\![g_b(x) = g_b(y)]\!] & = & [\![\langle x, b\rangle/\theta = \langle y, b\rangle/\theta]\!] \;=\; [\![\langle x, b\rangle\ \theta\ \langle y, b\rangle]\!] \\ & = & [\![f_{bb}(x) = y]\!] \;=\; [\![x = y]\!] \end{array}$$

and the conclusion follows from 26.17.(b). To finish the proof in the presheaf case, let $a, b \in B$ and $x \in |A_a|$ satisfy $Ex \leq a \wedge b$. Then, since $\langle x, a\rangle\ \widehat{\theta}\ \langle f_{ab}(x), b\rangle)$ (by (+) above), we have

$$\begin{array}{rcl} g_b(f_{ab}(x))) & = & \pi_\theta(k_b(f_{ab}(x))) \;=\; \pi_\theta(\langle f_{ab}(x), b\rangle) \;=\; \langle f_{ab}(x), b\rangle)/\theta \\ & = & \langle x, a\rangle/\theta \;=\; g_a(x), \end{array}$$

as desired. The case in which A_b is a sheaf, $b \in B$, let cA be the completion of the presheaf A constructed above. If we substitute g_* by $c \circ \pi_\theta \circ k_*$, we obtain monics from A_* into cA, with the properties in the statement. Since $A_{|b} \approx A_b$, $A_{|b}$ is complete over all $q \in p^{\leftarrow}$. We now invoke Theorem 27.20 to conclude that $cA_{|b} = A_{|b}$ and the newly defined monics are in fact isomorphisms, ending the proof. $\square$

The equivalence relation generated by a set of sections in $|A \times A|$ is instrumental in proving that $\boldsymbol{\Omega}$ ***set***, $\boldsymbol{pSh(\Omega)}$ and $\boldsymbol{Sh(\Omega)}$ have coequalizers, as was the case for sheaves and presheaves over topological spaces in section 24.3. The details are left as Exercise 43.11.

Exercises

43.9. Let A be a Ω-presheaf.

a) $Eq(A)$ is a complete lattice with the po induced by $\mathfrak{K}_2A$ [4]. Is it a complete **sublattice** of $\mathfrak{K}_2A$?

b) If $S \subseteq |A^2|$, define the **equivalence relation generated by S** to be

$$[\![x\ \theta_S\ y]\!] = \bigwedge \left\{ [\![x\ R\ y]\!] \ : \begin{array}{c} R \in Eq(A) \text{ and } \forall \langle s_1, s_2\rangle \in S \\ [\![s_1\ R\ s_2]\!] = Es_1 = Es_2 \end{array} \right\}$$

If $S = \{\langle x, y\rangle\}$, write θ_{xy} for θ_S. Then,

(1) For all $\langle x, y\rangle \in |A^2|$, θ_{xy} is compact in $Eq(A)$.

(2) For all $R \in Eq(A)$, $R = \bigvee_{\langle x,y\rangle \in R} \theta_{xy}$.

(3) $Eq(A)$ is an algebraic frame. $\square$

43.10. a) Let A be an L-set and $\theta \in Eq(A)$. Let $f : A \longrightarrow$ B be a morphism of L-sets, such that

$$\forall\, x, y \in |A|, \ \ [\![x\ \theta\ y]\!] \ \leq \ [\![fx = fy]\!].$$

Then, there is a **unique** L-set morphism, $g : A/\theta \longrightarrow B$ satisfying $g \circ \pi_\theta = f$.

[3]Recall (see (ii) in 43.7) that $[\![*/\theta = */\theta]\!] = [\![*\ \theta\ *]\!]$.

[4]Hence, meets and joins are computed pointwise.

$$\begin{tikzcd} A \arrow[rr, "\pi_\theta"] \arrow[dr, "f"'] & & A/\theta \arrow[dl, "g"] \\ & B & \end{tikzcd}$$

b) The universal property in (a) holds in $\boldsymbol{pSh(L)}$, in the category of finitely complete L-sets and in $\boldsymbol{Sh(\Omega)}$.

c) π_θ is an epic. Is it always surjective as a set map from $|A|$ to $|A/\theta|$? $\square$

43.11. a) The categories $\boldsymbol{\Omega\, set}$, $\boldsymbol{pSh(\Omega)}$ and $\boldsymbol{Sh(\Omega)}$ have coequalizers.

b) The categories $\boldsymbol{\Omega\, set}$, $\boldsymbol{pSh(\Omega)}$ and $\boldsymbol{Sh(\Omega)}$ are cocomplete. $\square$

43.12. Let A be a L-set. A **partial order on $\boldsymbol{A}$** is a characteristic map, $P \in \mathfrak{K}_2 A$ satisfying, for all x, y, $z \in |A|$

[ppo 1] : $\llbracket x\ Px \rrbracket = Ex$;

[ppo 2] : $\llbracket x\ P\ y \rrbracket \wedge \llbracket y\ P\ x \rrbracket \ \leq\ \llbracket x\ =\ y \rrbracket$;

[ppo 3] : $\llbracket x\ P\ y \rrbracket \wedge \llbracket y\ P\ z \rrbracket \ \leq\ \llbracket x\ P\ z \rrbracket$.

a) If A is a presheaf and $P \in \mathfrak{K}_2 A$ is a po on A, then for all $u \in \Omega$,

$$P(u) \ = \ \{\langle x, y \rangle \in A(u)^2 : \llbracket x\ P\ y \rrbracket = u\}$$

is a (classical) partial order on $A(u)$.

b) Let P be a closed subpresheaf of A^2 such that for each $u \in \Omega$, $P(u)$ is a partial order on $A(u)$. Then, the characteristic map of P verifies conditions [ppo i], $i =$ 1, 2, 3. $\square$

43.13. Let X be a topological space and $A = \mathbb{C}(X)$ be the sheaf of continuous real valued maps on X. For f, $g \in |\mathbb{C}(X)|$, set [5]

$$\llbracket f \leq g \rrbracket \ = \ \text{int}\ \{x \in Ef \cap Eg : fx \leq gx\}.$$

Then, $\llbracket * \leq * \rrbracket$ is a partial order on A, i.e., it verifies [ppo 1] $-$ [ppo 3] in 43.12. $\square$

[5] int $(\cdot)$ is the *interior* operation in a topological space.

CHAPTER 44

Finitary Relations and Operations on a Presheaf

This Chapter collects results on finitary relations and operations on Ω-presheaves, although many of them hold true for arbitrary Ω-sets.

DEFINITION 44.1. *If A is a Ω-presheaf, let* [1]

$$\mathfrak{K}_f A \;=\; \bigcup_{J \subseteq_f \mathbb{N}} \mathfrak{K}_{\mathbb{N}}(A, J); \qquad op(A) \;=\; \bigcup_{J \subseteq_f \mathbb{N}} [A^I,\ A]_J,$$

be the set of **finitary relations and operations**, *respectively, on A. For $K \subseteq_f \mathbb{N}$, write $op(A, K)$ for the operations on A that depend on K and $op(A, n)$ for those that depend on $\underline{n} = \{1, 2, \ldots, n\}$.*

From Propositions 37.4 and 40.5 we get

COROLLARY 44.2. *With the operations defined in 37.4, $\mathfrak{K}_f A$ is a Heyting algebra, whose bottom is $\bot_{\mathbb{N}}$ and whose top is $\top_{\mathbb{N}}$. Moreover, if Ω is a Boolean algebra, the same is true of $\mathfrak{K}_f A$.*

In general, $\mathfrak{K}_f A$ is not a complete lattice for joins and meets of an arbitrary family of finitary relations might not be finitary. It will become apparent that $\mathfrak{K}_f A$ and $op(A)$ have just enough structure to interpret first-order Logic. We now introduce

DEFINITION 44.3. *For $R \in \mathfrak{K}_f A$ and $\omega \in op(A)$, define*

$$\begin{aligned} \mathrm{fv}(R) &= \bigcap \{J \subseteq \mathbb{N} : R \text{ depends on } J\} \\ \mathrm{fv}(\omega) &= \bigcap \{J \subseteq \mathbb{N} : \omega \text{ depends on } J\}, \end{aligned}$$

called the sets of **free variables** *in R and ω, respectively.*

COROLLARY 44.4. *If $R \in \mathfrak{K}_f A$ and $\omega \in op(A)$, A a Ω-presheaf, then R depends on $\mathrm{fv}(R)$ and ω depends on $\mathrm{fv}(\omega)$. Moreover, $\mathrm{fv}(*)$ is the smallest subset of $\mathbb{N}$ on which $*$ depends.*

PROOF. Since R and ω depend on a finite subset of $\mathbb{N}$, the intersections defining $\mathrm{fv}(*)$ in 44.3 are finite. The result is then a consequence of 40.5.(e). $\square$

The behavior of $\mathrm{fv}(*)$ with respect to substitution and quantification is the usual one :

[1]Notation as in 40.2; recall that $J \subseteq_f \mathbb{N}$ means that J is a finite subset of $\mathbb{N}$.

COROLLARY 44.5. *Let A be a presheaf over Ω, $R \in \mathfrak{K}_f A$ and $\omega \in op(A)$. For $J \subseteq \mathbb{N}$, let τ_j, $j \in J$, be finitary operations in A and let τ be their product.*

a) $\forall\, i \in \mathbb{N}$, $\exists v_i R, \forall v_i R \in \mathfrak{K}_f A$ *and* $\mathrm{fv}(\exists v_i R) = \mathrm{fv}(\forall v_i R) = \mathrm{fv}(R) - \{i\}$.

b) If $J \subseteq \mathrm{fv}(R)$, *then* $R\ulcorner \tau \mid J \urcorner \in \mathfrak{K}_f A$ *and*

$$\mathrm{fv}(R\ulcorner \tau \mid J \urcorner) = (\mathrm{fv}(R) - J) \cup \bigcup_{j\in J} \mathrm{fv}(\tau_j).$$

c) If $J \subseteq \mathrm{fv}(\omega)$, *then* $\omega\ulcorner \tau \mid J \urcorner \in op(A)$ *and*

$$\mathrm{fv}(\omega\ulcorner \tau \mid J \urcorner) = (\mathrm{fv}(\omega) - J) \cup \bigcup_{j\in J} \mathrm{fv}(\tau_j).$$

PROOF. Item (a) follows from Theorem 42.6.(a) (or 42.11.(b)). Items (b) and (c) are a consequence of Theorem 41.4.(b) and Proposition 41.7. $\square$

Let L, R be frames and A, B be presheaves over L, R, respectively. Let $\mathfrak{f} = \langle f, \lambda \rangle : A \longrightarrow B$ be a morphism in $\boldsymbol{pSh}$, with λ a frame morphism. By Remark 40.9, image by $\mathfrak{f}$ does not produce a morphism from $\mathfrak{K}_f A$ to $\mathfrak{K}_f B$. However, for inverse image, we have

COROLLARY 44.6. *With notation as above, let ρ be the right adjoint of λ.*

a) $\mathfrak{f}^* : \mathfrak{K}_f B \longrightarrow \mathfrak{K}_f A$ *is a semilattice morphism. If ρ preserves joins or implication, the same is true of* $\mathfrak{f}^*$.

b) For $T \in \mathfrak{K}_f B$, $\mathfrak{f}^*(\exists v_i R) \geq \exists v_i\, \mathfrak{f}^* R$ *and* $\mathfrak{f}^*(\forall v_i R) \leq \forall v_i\, \mathfrak{f}^* R$.

c) If for all $b \in |B|$, $Eb = \bigvee_{c\in|A|} [\![fc = b]\!]$ *and* $\rho \circ \lambda = Id_L$, *then*

(1) ρ *is a* $\bigvee$*-morphism* $\Rightarrow$ $\mathfrak{f}^*(\exists v_i R) = \exists v_i\, \mathfrak{f}^* R$.

(2) ρ *is open* $\Rightarrow$ $\mathfrak{f}^*(\forall v_i R) = \forall v_i\, \mathfrak{f}^* R$.

PROOF. a) By Proposition 40.10.(a), $\mathfrak{f}^*$ takes $\mathfrak{K}_f B$ to $\mathfrak{K}_f A$. The preservation of meets, joins and implication follow from the corresponding assertions in 39.1. Items (b) and (c) are straightforward consequences of Proposition 42.12. $\square$

The next result is basic in connecting $\mathfrak{K}_f A$ and $op(A)$ with the Intuitionistic Predicate Calculus $\mathcal{H}$, presented in 17.5.

PROPOSITION 44.7. *Let A be a Ω-presheaf and J be a finite subset of $\mathbb{N}$. Let ω, $\tau_j \in op(A)$, $j \in J$, and let $\tau : A^I \longrightarrow A^J$ be the product of the τ_j. Let R, $S \in \mathfrak{K}_f A$.*

a) If $\diamond$ is one of the operations $\wedge$, $\vee$ or $\rightarrow$, then

$$(R \diamond S)\ulcorner \tau \mid J \urcorner = R\ulcorner \tau \mid J \urcorner \diamond S\ulcorner \tau \mid J \urcorner.$$

Moreover, $(\neg R)\ulcorner \tau \mid J \urcorner = \neg (R\ulcorner \tau \mid J \urcorner)$.

b) If $k \notin \mathrm{fv}(\tau) \cup J$, *then* $\begin{cases} (\exists v_k R)\ulcorner \tau \mid J \urcorner = \exists v_k (R\ulcorner \tau \mid J \urcorner); \\ (\forall v_k R)\ulcorner \tau \mid J \urcorner = \forall v_k (R\ulcorner \tau \mid J \urcorner). \end{cases}$

c) For all $k \in \mathbb{N}$, $\forall v_k R \leq R\ulcorner \omega \mid k \urcorner \leq \exists v_k R$.

d) If $k \notin \mathrm{fv}(R)$, *then for all* $\overline{x} \in A^{\mathbb{N}}$ *and* $c \in A$

(1) $[\![R(\overline{x})]\!] \leq [\![S(\overline{x})]\!] \;\Rightarrow\; Ec \wedge [\![R(\overline{x})]\!] \leq [\![S(\overline{x}\ulcorner c \mid k \urcorner)]\!]$.

(2) $[\![\, S(\overline{x}) \,]\!] \;\leq\; [\![\, R(\overline{x}) \,]\!] \quad\Rightarrow\quad E\overline{x} \wedge [\![\, S(\overline{x}\ulcorner c \mid k\urcorner) \,]\!] \;\leq\; [\![\, R(\overline{x}) \,]\!]$.

e) If $k \notin \mathrm{fv}(R)$, then

(1) $R \;\leq\; S \;\Rightarrow\; R \;\leq\; \forall v_k S$; (2) $S \;\leq\; R \;\Rightarrow\; \exists v_k S \;\leq\; R$.

PROOF. a) All relations follow from 41.4.(b).(1). As an example, for implication we have, with $\overline{a} \in |A|^I$, $\overline{c} = \tau(\overline{a}_{|E\overline{a}})$ and recalling that

$$E\overline{a} \;=\; E\overline{a}\ulcorner\tau\overline{a}_{|E\overline{a}} \mid J\urcorner \;=\; E\overline{a}\ulcorner\overline{c} \mid J\urcorner,$$

$$\begin{aligned}
[\![\,(R \to S)\ulcorner\tau \mid J\urcorner(\overline{a})\,]\!] \;&=\; [\![\,(R \to S)(\overline{a}\ulcorner\overline{c} \mid J\urcorner)\,]\!] \;=\\
&=\; E\overline{a}\ulcorner\overline{c} \mid J\urcorner \wedge \Big([\![\, R(\overline{a}\ulcorner\overline{c} \mid J\urcorner)\,]\!] \to [\![\, S(\overline{a}\ulcorner\overline{c} \mid J\urcorner)\,]\!]\Big)\\
&=\; E\overline{a} \wedge [\![\, R\ulcorner\tau \mid J\urcorner(\overline{a})\,]\!] \to [\![\, S\ulcorner\tau \mid J\urcorner(\overline{a})\,]\!]\\
&=\; [\![\,(R\ulcorner\tau \mid J\urcorner \to S\ulcorner\tau \mid J\urcorner)(\overline{a})\,]\!].
\end{aligned}$$

b) We treat the universal quantifier; the analogous argument for the existential quantifier is left to the reader. For $\overline{a} \in |A|^I$ and $u \in |A|$, set

$$\overline{x} \;=\; \overline{a}_{|E\overline{a}} \quad\text{and}\quad \overline{y}_u \;=\; (\overline{a}\ulcorner u \mid k\urcorner)_{|E\overline{a}\ulcorner u|k\urcorner}.$$

Then, $\overline{x}$, $\overline{y}_u \in A^I$, with

$$E\overline{x} = E\overline{a} \quad\text{and}\quad E\overline{y}_u \;=\; E\overline{a}\ulcorner u \mid k\urcorner = \textstyle\bigwedge_{i\neq k} Ea_i \wedge Eu.$$

Hence, $E\overline{x} \wedge E\overline{y}_u \;=\; E\overline{a} \wedge Eu$. Since $k \notin \mathrm{fv}(\tau) = \bigcup_{j\in J} \tau_j$ (44.8.(d)), $\overline{a}$ and $\overline{a}\ulcorner u \mid k\urcorner$ coincide in all coordinates except k and

$$E\overline{a}_{|E\overline{a}\wedge Eu} \;=\; E(\overline{a}\ulcorner u \mid k\urcorner)_{|E\overline{a}\wedge Eu} \;=\; E\overline{a} \wedge Eu,$$

we conclude that

(*) $$\tau(\overline{x})_{|E\overline{a}\wedge Eu} \;=\; \tau(\overline{x}_{|E\overline{a}\wedge Eu}) \;=\; \tau(\overline{y}_{u|E\overline{a}\wedge Eu}) \;=\; \tau(\overline{y}_u)_{|E\overline{a}\wedge Eu}.$$

Hence, since $k \notin J$, 41.4.(b).(1) and (*) yield,

$$\begin{aligned}
&[\![\,(\forall v_k)R\ulcorner\tau \mid J\urcorner(\overline{a})\,]\!] \;=\; E\overline{a} \wedge \textstyle\bigwedge_{u\in|A|} Eu \to [\![\, R\ulcorner\tau \mid J\urcorner(\overline{a}\ulcorner u \mid k\urcorner)\,]\!]\\
&\quad=\; E\overline{a} \wedge \textstyle\bigwedge_{u\in|A|} Eu \to [\![\, R(\overline{a}\ulcorner u \mid k\urcorner\ulcorner\tau(\overline{y}_u) \mid J\urcorner)\,]\!]\\
&\quad=\; E\overline{a} \wedge \textstyle\bigwedge_{u\in|A|} E\overline{a} \wedge Eu \;\to\; E\overline{a} \wedge Eu \wedge [\![\, R(\overline{a}\ulcorner u \mid k\urcorner\ulcorner\tau(\overline{y}_u) \mid J\urcorner)\,]\!]\\
&\quad=\; E\overline{a} \wedge \textstyle\bigwedge_{u\in|A|} E\overline{a} \wedge Eu \;\to\; [\![\, R(\overline{a}\ulcorner u \mid k\urcorner\ulcorner\tau(\overline{y}_u)_{|E\overline{a}\wedge Eu} \mid J\urcorner)\,]\!]\\
&\quad=\; E\overline{a} \wedge \textstyle\bigwedge_{u\in|A|} E\overline{a} \wedge Eu \;\to\; [\![\, R(\overline{a}\ulcorner u \mid k\urcorner\ulcorner\tau(\overline{x})_{|E\overline{a}\wedge Eu} \mid J\urcorner)\,]\!]\\
&\quad=\; E\overline{a} \wedge \textstyle\bigwedge_{u\in|A|} E\overline{a} \wedge Eu \;\to\; E\overline{a} \wedge Eu \wedge [\![\, R(\overline{a}\ulcorner u \mid k\urcorner\ulcorner\tau(\overline{x}) \mid J\urcorner)\,]\!]\\
&\quad=\; E\overline{a} \wedge \textstyle\bigwedge_{u\in|A|} E\overline{a} \wedge Eu \;\to\; E\overline{a} \wedge Eu \wedge [\![\, R(\overline{a}\ulcorner\tau(\overline{x}) \mid J\urcorner\ulcorner u \mid k\urcorner)\,]\!]\\
&\quad=\; E\overline{a} \wedge \textstyle\bigwedge_{u\in|A|} Eu \;\to\; [\![\, R(\overline{a}\ulcorner\tau(\overline{x}) \mid J\urcorner\ulcorner u \mid k\urcorner)\,]\!]\\
&\quad=\; E\overline{a}\ulcorner\tau(\overline{x}) \mid J\urcorner \;\wedge\; \textstyle\bigwedge_{u\in|A|} Eu \;\to\; [\![\, R(\overline{a}\ulcorner\tau(\overline{x}) \mid J\urcorner\ulcorner u \mid k\urcorner)\,]\!]\\
&\quad=\; [\![\,(\forall v_k R)\ulcorner\tau \mid J\urcorner(\overline{a})\,]\!],
\end{aligned}$$

as desired.

c) For $\overline{x} \in |A|^I$, let $c = \omega(\overline{x}_{|E\overline{x}})$. Then, $Ec = E\overline{x} = E\overline{x}\ulcorner c \mid k\urcorner$. Hence, Theorem 41.4.(b).(1) yields

$$\begin{aligned}[\![\forall v_k S(\overline{x})]\!] &\leq E\overline{x} \wedge (Ec \longrightarrow [\![S(\overline{x}\ulcorner c \mid k \urcorner)]\!]) \leq [\![S(\overline{x}\ulcorner c \mid k\urcorner)]\!] \\ &= [\![S\ulcorner \omega \mid k\urcorner(\overline{x})]\!],\end{aligned}$$

establishing the first inequality in (d). The second is similar.

d) For (1), since $k \notin \mathrm{fv}(R)$ it follows that

$$\begin{aligned}Ec \wedge [\![R(\overline{x})]\!] &= Ec \wedge \textstyle\bigwedge_{i \neq k} Ex_i \wedge [\![R(\overline{x})]\!] = E\overline{x}\ulcorner c \mid k\urcorner \wedge [\![R(\overline{x})]\!] \\ &= E\overline{x} \wedge [\![R(\overline{x}\ulcorner c \mid k\urcorner)]\!] \leq E\overline{x} \wedge [\![S(\overline{x}\ulcorner c \mid k\urcorner]\!] \\ &\leq [\![S(\overline{x}\ulcorner c \mid k\urcorner)]\!].\end{aligned}$$

Analogously, one obtains (2); item (e) is a straightforward consequence of (d) and the pertinent definitions. □

Exercises

44.8. Let A be a Ω-presheaf and R, $S \in \mathfrak{K}_f A$. Let $J \subseteq_f \mathbb{N}$, $\tau_j \in op(A)$ and let τ be the product of the τ_j.

a) $\mathrm{fv}(R \to S) = \mathrm{fv}(R \wedge S) = \mathrm{fv}(R \vee S) = \mathrm{fv}(R) \cup \mathrm{fv}(S)$.

b) $\mathrm{fv}(\neg R) = \mathrm{fv}(R)$.

c) $\mathrm{fv}(\bot_{\mathbb{N}}) = \mathrm{fv}(\top_{\mathbb{N}}) = \emptyset$.

d) Set $\mathrm{fv}(\tau) = \bigcap \{J \subseteq \mathbb{N} : \tau \text{ depends on } J\}$. Then, $\mathrm{fv}(\tau) = \bigcup_{j \in J} \mathrm{fv}(\tau_j)$. □

44.9. Let A be a Ω-presheaf. All the axioms and rules of the Intuitionistic Propositional Calculus in 17.5 hold in $\mathfrak{K}_f A$, with the following conventions :

∗ In the axioms and rules of $\mathcal{H}$, the symbols ϕ, ψ and χ stand for elements of $\mathfrak{K}_f A$;

∗ An axiom is *satisfied* if its value is $\top_{\mathbb{N}}$ for all ϕ, ψ, $\chi \in \mathfrak{K}_f A$;

∗ A rule is *valid* whenever the conjunction of its antecedents is satisfied, so is its conclusion. For instance, to check that the rule of *Modus Ponens* is valid it must be shown that

$$\phi \wedge (\phi \to \psi) = \top_{\mathbb{N}} \quad \Rightarrow \quad \psi = \top_{\mathbb{N}},$$

for arbitrary ϕ, ψ in $\mathfrak{K}_f A$. □

CHAPTER 45

Graded Frames

Let A be a Ω-presheaf. Recall (37.1) that if $n \geq 1$ is an integer, $\mathfrak{K}_n A$ is the frame of n-characteristic maps on A. By Proposition 40.14, if $J \subseteq_f \mathbb{N}$ has cardinal $n \geq 1$, there is an automorphism of $\mathfrak{K}_{\mathbb{N}} A$ that takes $\mathfrak{K}_{\mathbb{N}}(A, J)$ isomorphically onto $\mathfrak{K}_{\mathbb{N}}(A, \underline{n})$ [1], which may be identified with $\mathfrak{K}_n A$ (40.8). Hence, associated to A there is a sequence of frames

$$\mathfrak{K}_*(A) \;\; = \;\; \langle\, \mathfrak{K}_0(A),\, \mathfrak{K}_1(A),\, \ldots,\, \mathfrak{K}_n(A), \ldots \rangle$$

where $\mathfrak{K}_0(A) = (EA)^{\leftarrow}$ (see (!) in 37.3). In this Chapter we introduce the structure that is adequate to study this type of object. It is analogous to the graded rings and modules that appear in Homology, Cohomology or in Topological and Algebraic K-theory. A general reference for homological constructs is MacLane's [**45**]; for Algebraic K-theory the reader may consult [**48**], [**4**], [**74**] or [**73**]

1. Introduction

We start with the following

DEFINITION 45.1. *A* **graded frame** *is of a sequence of frames*

$$L_* \;\; = \;\; \langle\, L_0,\, L_1, \ldots, L_n,\, \ldots \rangle$$

together with a binary operation, $\otimes : L_* \times L_* \longrightarrow L_*$*, consisting for each pair of integers* $n,\, m \geq 0$ *of an operation*

$$\otimes : L_n \times L_m \longrightarrow L_{n+m},$$

satisfying the following conditions, where $\perp_n$*,* $\top_n$ *are the top and bottom of* L_n*, respectively :*

[$\otimes$ 1] *:* $\otimes$ *is associative;*

[$\otimes$ 2] *:* $\otimes$ *distributes over finite meets and arbitrary joins;*

[$\otimes$ 3] *:* $\top_n \otimes \top_m = \top_m \otimes \top_n = \top_{n+m}$*;*

[$\otimes$ 4] *: For all* $\eta \in L_n$, $\;\eta \otimes \eta \;=\; (\eta \otimes \top_n) \wedge (\top_n \otimes \eta)$*;*

[$\otimes$ 5] *: For all* $p \in L$ *and* $\eta \in L_n$, $\;p \otimes \eta = \eta \otimes p = \eta \wedge (p \otimes \top_n)$*;*

[$\otimes$ 6] *: For all* $\eta \in L_n$, $\;\top_0 \otimes \eta = \eta \otimes \top_0 = \eta$.

The frame L_n *is the* **component of degree *n* of L_****; a member of* L_n *is called an* **element of degree *n*** *of* L_**. When context allows write* $\top$ *for* $\top_0$

[1] $\underline{n} = \{1, 2, \ldots, n\}$.

(the top of L_0). A **graded cBa** *is a graded frame in which all components are complete Boolean algebras.*

The following comments are included to help make the statement of Definition 45.1 clearer :

$*$ The only commutativity assumptions are [$\otimes$ 5] and [$\otimes$ 6]. Note that [$\otimes$ 5] implies $(p \otimes \top_n) = (\top_n \otimes p)$, for all $p \in L$. In general, the operation $\otimes$ *is not commutative*;

$*$ Associativity means that if $\alpha \in L_n$, $\beta \in L_m$ and $\gamma \in L_k$, then in L_{n+m+k} we have

$$(\alpha \otimes \beta) \otimes \gamma = \alpha \otimes (\beta \otimes \gamma).$$

$*$ Distributivity over finite meets means that if α, $\beta \in L_n$ and $\gamma \in L_m$, then the following relations hold in L_{n+m} :

$$\begin{cases} (\alpha \wedge \beta) \otimes \gamma & = & (\alpha \otimes \gamma) \wedge (\beta \otimes \gamma); \\ \gamma \otimes (\alpha \wedge \beta) & = & (\gamma \otimes \alpha) \wedge (\gamma \otimes \beta). \end{cases}$$

$*$ Distributivity over arbitrary joins means that if $\{\alpha_i : i \in I\} \subseteq L_n$ and $\gamma \in L_m$, then the following relations hold in L_{n+m} :

$$\begin{cases} \left(\bigvee_{i \in I} \alpha_i\right) \otimes \gamma & = & \bigvee_{i \in I} \alpha_i \otimes \gamma; \\ \gamma \otimes \bigvee_{i \in I} \alpha_i & = & \bigvee_{i \in I} \gamma \otimes \alpha_i. \end{cases}$$

$*$ Axiom [$\otimes$ 4] is the form of *idempotency* that is compatible with the graded structure.

$*$ For all $\alpha \in L_m$ and $n \geq 0$, $\alpha \otimes \bot_n = \bot_n \otimes \alpha = \bot_{n+m}$ [2].

Example 45.2. If L is a frame, the constant sequence

$$\widehat{L} \;=\; \langle\, L, L, \ldots, L, \ldots \rangle$$

is a graded frame, where the operation $\otimes$ is the meet in L. Thus, for $p \in L_n$, $q \in L_m$, we set $p \otimes q \;=\; p \wedge q$, computed in $L_{n+m} = L$. $\square$

Example 45.3. Let A be a set. The sequence

$$2^A_* \;=\; \langle\, 2,\, 2^A,\, 2^{A \times A}, \ldots, 2^{A^n}, \ldots \rangle$$

where $2 = \{\bot, \top\}$ is the two element cBa, is a graded cBa, where the operation $\otimes$ is defined, for $S \subseteq A^n$ and $T \subseteq A^m$, by

$$S \otimes T \;=\; \left\{\langle\, a_1, \ldots, a_n, c_1, \ldots, c_m \,\rangle \in A^{n+m} : \begin{array}{c} \langle\, a_1, \ldots, a_n \,\rangle \in S \\ \text{and} \\ \langle\, c_1, \ldots, c_m \,\rangle \in T \end{array}\right\}$$

In this case, meets and joins are simply intersection and union, whether finite or not. Observe that for $S \subseteq A^n$

$$S \otimes \top \;=\; \top \otimes S \;=\; S \quad \text{and} \quad S \otimes \bot \;=\; \bot \otimes S \;=\; \bot_n,$$

where $\bot_n$ is $\emptyset$, understood as a subset of A^n. $\square$

[2] $\bot$ is the join of the empty family.

LEMMA 45.4. *Let L_* be a graded frame and n, m, $l \geq 0$ be integers.*

a) $\otimes$ is increasing in both coordinates : for α, $\beta \in L_n$ and $\gamma \in L_m$, then

$$\alpha \leq \beta \quad \Rightarrow \quad \begin{cases} (\alpha \otimes \gamma) \leq (\beta \otimes \gamma); \\ (\gamma \otimes \alpha) \leq (\gamma \otimes \beta). \end{cases}$$

b) For p, $q \in L_0$, $p \otimes q = p \wedge q$.

c) For all p, $q \in L_0$ and n, $m \geq 0$,

$$(p \otimes \top_n) \otimes (q \otimes \top_m) = (p \wedge q) \otimes \top_{n+m}.$$

d) For $\alpha \in L_n$, the maps $\begin{cases} (\cdot) \otimes \alpha : L_m \longrightarrow L_{n+m}, & \beta \mapsto \beta \otimes \alpha; \\ \alpha \otimes (\cdot) : L_m \longrightarrow L_{n+m}, & \beta \mapsto \alpha \otimes \beta, \end{cases}$

are frame-morphisms, i.e., preserve finite meets and arbitrary joins.

e) Right-tensoring by $\top_n$ provides a family of frame-morphisms from L_l to L_{l+n}, such that the following diagram is commutative [3] :

$$\begin{tikzcd} L_l \arrow[r, "\otimes \top_n"] \arrow[d, "\otimes \top_m"'] & L_{l+n} \arrow[d, "\otimes \top_m"] \\ L_{l+m} \arrow[r, "\otimes \top_n"'] & L_{l+m+n} \end{tikzcd}$$

f) For $n \geq 1$, the frame-morphism $(*) \otimes \top_n$ is the n-fold composition of tensoring on the right with $\top_1$.

PROOF. a) If $\alpha \leq \beta$ iff $\beta = \alpha \vee \beta$; hence, distributivity over joins yields $\beta \otimes \gamma = (\alpha \vee \beta) \otimes \gamma = (\alpha \otimes \gamma) \vee (\beta \otimes \gamma)$, proving that $(\alpha \otimes \gamma) \leq (\beta \otimes \gamma)$. Similarly, one verifies the other relation.

b) From [$\otimes$ 5] and [$\otimes$ 6] we get : $p \otimes q = p \wedge (q \otimes \top) = p \wedge q$, as needed.

c) First note that associativity ([$\otimes$ 1]) and [$\otimes$ 5] yield

$$(p \otimes \top_n) \otimes (q \otimes \top_m) = p \otimes (\top_n \otimes \top_m) \otimes q = (p \otimes \top_{n+m}) \otimes q.$$

Next, since $(p \otimes \top_{n+m})$ is an element of degree $n + m$ in L^*, another application of [$\otimes$ 5] and distributivity of meets over $\otimes$ yields

$$(p \otimes \top_{n+m}) \otimes q = (p \otimes \top_{n+m}) \wedge (q \otimes \top_{n+m}) = (p \wedge q) \otimes \top_{n+m},$$

as needed. Item (d) is an immediate consequence of distributivity of finite meets and arbitrary joins over $\otimes$, while (e) and (f) follow from (d) and [$\otimes$ 3] in 45.1. $\square$

To present the various concepts of morphism of graded frames and the inductive systems associated to them, we introduce a set of "standard" operations in frames, namely

[3] Notation is thoughtfully ambiguous.

($\mathfrak{O}$) $\mathfrak{O}$ = {semilattice, $\bigwedge$, $\bigvee$, frame [4], $[\bigwedge, \bigvee]$, open}

REMARK 45.5. By Lemma 45.4.(e), a graded frame L_* gives rise to an inductive system of frame and frame-morphisms

$$\mathcal{I}(L_*) \;\; = \;\; L_0 \xrightarrow{\otimes\top_1} L_1 \cdots \xrightarrow{\otimes\top_1} L_n \xrightarrow{\otimes\top_1} L_{n+1} \cdots$$

that will be important in what follows. Note that a morphism

$$g = \langle\, g_n \,\rangle_{n\geq 0} : \mathcal{I}(L_*) \longrightarrow \mathcal{I}(K_*)$$

consists of a sequence $g_n : L_n \longrightarrow K_n$, $n \geq 0$, such that the following diagram is commutative :

$$\begin{tikzcd}
L_n \arrow[r, "g_n"] \arrow[d, "(*) \otimes \top_1"'] & K_n \arrow[d, "(*) \otimes \top_1"] \\
L_{n+1} \arrow[r, "g_{n+1}"'] & K_{n+1}
\end{tikzcd}$$

If $\mathfrak{o} \in \mathfrak{O}$ is one of the "standard" operations in ($\mathfrak{O}$), then g is a **$\mathfrak{o}$-morphism** iff for all $n \geq 0$, g_n is a $\mathfrak{o}$-morphism from L_n to K_n. ☐

DEFINITION 45.6. *a) Let $d \geq 1$ be an integer and L_*, K_* be graded frames. A* **map of degree d**, *$f : L_* \longrightarrow K_*$, consists of a sequence $f = \langle\, f_n \,\rangle_{n\geq 0}$ such that*

[grm 1] : $\forall\, n \geq 0, \;\; f_n : L_n \longrightarrow K_{dn}$;

[grm 2] : *For all $\alpha \in L_n$ and $\eta \in L_m$,* $\;f_n(\alpha) \otimes f_m(\eta) \;\; = \;\; f_{n+m}(\alpha \otimes \eta)$.

A **map of graded frames** *is map of degree* 1.

b) A map of graded frames, $f : L_* \longrightarrow K_*$, is **stable** *if it induces a morphism from $\mathcal{I}(L_*)$ to $\mathcal{I}(K_*)$, as in* 45.5.

c) Let $\mathfrak{o} \in \mathfrak{O}$ be one of the standard operations in frames (see ($\mathfrak{O}$), above) and let $f : L_* \longrightarrow K_*$ be a map of degree d. f is a **$\mathfrak{o}$-morphism** *if for all $n \geq 0$, $f_n : L_n \longrightarrow K_{dn}$ is a $\mathfrak{o}$-morphism of frames. The expression* **$\mathfrak{o}$-morphism** *stands for a $\mathfrak{o}$-morphism of degree* 1.

e) A $\mathfrak{o}$-morphism is **stable**, *if its carrier is a stable map.*

EXAMPLE 45.7. Let L_* be a graded frame. There is a natural map (of degree 1), $j : \widehat{L}_0 \longrightarrow L_*$, where $\widehat{L}_0$ is the constant graded frame of 45.2. For $p \in \widehat{L}_n = L_0$, set

$$j_n(p) = p \otimes \top_n.$$

By items (b) and (c) in 45.4, j is a graded frame-morphism (or $[\wedge, \bigvee]$-morphism). More examples will emerge in section 2. ☐

[4]That is, $[\wedge, \bigvee]$-morphism.

45.8. **Categories of Graded Frames.** a) If L_* is a graded frame, its **identity morphism**, Id_{L_*}, is the morphism whose n^{th} map is the identity of the n^{th} component of L_*. Clearly, Id_{L_*} has degree 1.

b) Let $\mathfrak{o} \in \mathfrak{O}$ (see ($\mathfrak{O}$) in page 470). If $f : L_* \longrightarrow R_*$ and $g : R_* \longrightarrow S_*$ are $\mathfrak{o}$-morphisms of graded frames, of degrees d, d', respectively, then $g \circ f : L_* \longrightarrow S_*$, defined, for $n \geq 0$, by

$$(g \circ f)_n \quad = \quad g_{dn} \circ f_n,$$

is a $\mathfrak{o}$-morphism of graded frames, of degree dd'. In particular, the composition of $\mathfrak{o}$-morphisms of degree 1 is a $\mathfrak{o}$-morphism of degree 1. Graded frames with $\mathfrak{o}$-morphisms of arbitrary degree, with $\mathfrak{o}$-morphisms of degree 1 or stable $\mathfrak{o}$-morphisms are categories. $\square$

2. The Graded Frame of Relations on a Presheaf

Let A be a Ω-set and $\mathfrak{K}_*(A)$ be the sequence

$$\mathfrak{K}_*(A) \quad = \quad \langle\, (EA)^{\leftarrow}, \mathfrak{K}_1(A), \mathfrak{K}_2(A), \ldots, \mathfrak{K}_n(A), \ldots \rangle$$

consisting of the frames of n-characteristic maps on A. Recall that $\mathfrak{K}_0(A)$ is identified with $(EA)^{\leftarrow}$ (by (!) in 37.3). As in 37.22.(b) and (c), write $\top_n$, $\bot_n$ for the n-characteristic maps on A given by

$$\overline{x} \longmapsto E\overline{x} \quad \text{ and } \quad \overline{x} \longmapsto \bot,$$

which by 37.4.(a) are the top and bottom of $\mathfrak{K}_n(A)$. If A is a Ω-presheaf, $\mathfrak{K}_*(A)$ is a graded frame, as follows : define a binary operation on $\mathfrak{K}_*(A)$,

$$\otimes : \mathfrak{K}_*(A) \times \mathfrak{K}_*(A) \longrightarrow \mathfrak{K}_*(A)$$

where for $R \in \mathfrak{K}_n(A)$, $S \in \mathfrak{K}_m(A)$ and $\overline{x} \in |A|^{n+m}$

$$[\![\, R \otimes S(\overline{x})\,]\!] \quad = \quad [\![\, R(x_1, \ldots, x_n)\,]\!] \wedge [\![\, S(x_{n+1}, \ldots, x_{n+m})\,]\!].$$

It is straightforward that $R \otimes S \in \mathfrak{K}_{n+m}(A)$. The geometrical significance of this operation is described in 37.27 (or 45.3). Note that

$*$ If $n = 0$, then $\otimes : (EA)^{\leftarrow} \times \mathfrak{K}_m(A) \longrightarrow \mathfrak{K}_m(A)$ takes $\langle\, p, R\,\rangle$ to $p \wedge R \in \mathfrak{K}_m(A)$ (by 37.4.(b)). In particular,

$$\forall\, p \in \Omega,\ \forall\, R \in \mathfrak{K}_m(A), \quad p \otimes R = R \otimes p.$$

In terms of the isomorphism in 37.16, $p \otimes R$ is the characteristic map of $R_{|p}$;

$*$ In general, $\otimes$ **is not commutative**.

Besides being a graded frame, $\mathfrak{K}_*(A)$ has other special properties, described in

PROPOSITION 45.9. *Let A be a Ω-presheaf and n, $m \geq 0$ be integers. Then,*

a) $\mathfrak{K}_*(A)$ is a graded frame.

b) For R, $R' \in \mathfrak{K}_n(A)$ and S, $S' \in \mathfrak{K}_m(A)$,

$$R \otimes (S \to S') \quad = \quad (R \otimes \top_m) \wedge \Big((R \otimes S) \to (R \otimes S')\Big);$$

$$(R \to R') \otimes S \quad = \quad \Big((R \otimes S) \to (R' \otimes S)\Big) \wedge (\top_n \otimes S).$$

c) The map $(\cdot) \otimes \top_1 : \mathfrak{K}_n(A) \longrightarrow \mathfrak{K}_{n+1}(A)$*, given by* $R \mapsto R \otimes \top_1$*, is an open* (8.1.($b$)) *injection.*

d) Right-tensoring by $\top_m$ *provides a family of open injections from* $\mathfrak{K}_n(A)$ *into* $\mathfrak{K}_{n+m}(A)$.

Proof. Item (a) is straightforward and left to the reader.

b) To simplify exposition, if $\overline{x} \in |A|^{n+m}$, we write $\overline{z} = \langle x_1, \ldots, x_n \rangle$ and $\overline{y} = \langle x_{n+1}, \ldots, x_{n+m} \rangle$. Hence, $\overline{x} = \langle \overline{z}, \overline{y} \rangle$ and $E\overline{x} = E\overline{z} \wedge E\overline{y}$.

Let $R \in \mathfrak{K}_n(A)$ and $S, S' \in \mathfrak{K}_m(A)$; 6.4.(i) and [*car* 1] in 37.1 yield

$$\begin{aligned}
[\![R \otimes (S \to S')(\overline{x})]\!] \;&=\; [\![R(\overline{z})]\!] \wedge ([\![(S \to S')(\overline{y})]\!] \;= \\
&= [\![R(\overline{z})]\!] \wedge E\overline{y} \wedge ([\![S(\overline{y})]\!] \to [\![S'(\overline{y})]\!]) \\
&= [\![R(\overline{z})]\!] \wedge E\overline{z} \wedge E\overline{y} \wedge ([\![R(\overline{y})]\!] \to [\![S'(\overline{y})]\!]) \\
&= [\![R(\overline{z})]\!] \wedge E\overline{x} \wedge \Big(([\![R(\overline{z})]\!] \wedge [\![S(\overline{y})]\!]) \to ([\![R(\overline{z})]\!] \wedge [\![S'(\overline{y})]\!])\Big) \\
&= [\![R(\overline{z})]\!] \wedge E\overline{x} \wedge \Big([\![R \otimes S(\overline{x})]\!] \to [\![R \otimes S'(\overline{x})]\!]\Big) \\
&= [\![R(\overline{z})]\!] \wedge [\![(R \otimes S \to R \otimes S')(\overline{x})]\!] \\
&= [\![R(\overline{z})]\!] \wedge E\overline{y} \wedge [\![(R \otimes S \to R \otimes S')(\overline{x})]\!] \\
&= [\![(R \otimes \top_m)(\overline{x})]\!] \wedge [\![(R \otimes S \to R \otimes S')(\overline{x})]\!],
\end{aligned}$$

establishing the first equality; the other can be proven similarly.

c) By 45.4.(d) (or (e)), right-tensoring with $\top_1$ is a frame-morphism. It follows easily from the definition that it preserves arbitrary meets [5]. To see it is open, let $R, R' \in \mathfrak{K}_n(A)$. Item (b) and [$\otimes$ 3] in 45.1 then yield

$$\begin{aligned}
(R \to R') \otimes \top_1 \;&=\; (R \otimes \top_1 \to R' \otimes \top_1) \wedge (\top_n \otimes \top_1) \\
&= (R \otimes \top_1 \to R' \otimes \top_1) \wedge \top_{n+1} \;=\; (R \otimes \top_1 \to R' \otimes \top_1),
\end{aligned}$$

showing that $(*) \otimes \top_1$ preserves implication; it remains to check that it is injective. Note that

$$R \otimes \top_1 = R' \otimes \top_1, \tag{1}$$

implies that for all $\overline{x} \in |A|^n$

$$[\![R(\overline{x})]\!] \wedge Ex_n = [\![R'(\overline{x})]\!] \wedge Ex_n. \tag{2}$$

To see this, just compute the value of each member in (1) at the $n+1$-tuple $\langle x_1, \ldots, x_n, x_n \rangle$ to obtain the equality in (2). Since $E\overline{x} \leq Ex_n$, property [ch 1] and (2) immediately imply that $R = R'$. For (d) observe that

$*$ The composition of open injections yields an open injection;

$*$ By 45.4.(f), right-tensoring with $\top_n$ is the n-fold composition of right-tensoring with $\top_1$. ☐

The relation between $\mathfrak{K}_*(A^d)$ and $\mathfrak{K}_*(A)$ yields an example of a graded morphism of degree d.

[5] In fact, in $\mathfrak{K}_*(A)$, right or left tensoring by any element is a $[\bigwedge, \bigvee]$-morphism.

Proposition 45.10. *Let A be a Ω-presheaf and $d \geq 1$ be an integer. There is a natural graded morphism of degree d over $(EA)^{\leftarrow}$, $g : \mathfrak{K}_*(A^d) \longrightarrow \mathfrak{K}_*(A)$, such that for all $m \geq 0$, g_m is a frame isomorphism.*

Proof. Set $g_0 = Id_{(EA)^{\leftarrow}}$; for $m \geq 1$ and $\overline{x} \in |A|^{md}$, we shall write

$$\overline{x} \;=\; \langle\, \overline{x}_1, \ldots, \overline{x}_m \,\rangle,$$

where $\overline{x}_{j+1} = \langle\, x_{(j-1)d+1}, \ldots, x_{jd} \,\rangle \in |A|^d$, $1 \leq j \leq m$. Hence,

$$\overline{x}_{j_{|E\overline{x}_j}} \in |A^d| \quad \text{and} \quad E\overline{x} = \textstyle\bigwedge_{j=1}^m E\overline{x}_j. \tag{1}$$

Consider the map

$$\gamma_m : |A|^{md} \longrightarrow |A^d|^m, \text{ given by } \gamma_m(\overline{x}) = \langle\, \overline{x}_{1_{|E\overline{x}_1}}, \ldots, \overline{x}_{m_{|E\overline{x}_m}} \,\rangle,$$

and observe that for $\overline{x}$, $\overline{y} \in |A|^{md}$

$$E\gamma_m(\overline{x}) = E\overline{x} \quad \text{and} \quad [\![\overline{x} \,=\, \overline{y}]\!] = [\![\gamma_m(\overline{x}) \,=\, \gamma_m(\overline{y})]\!]. \tag{2}$$

Define $g_m : \mathfrak{K}_m(A^d) \longrightarrow \mathfrak{K}_{md}(A)$ by [6]

$$g_m(h)(\overline{x}) = h(\gamma_m(\overline{x})).$$

It follows easily from (2) that $g_m(h) \in \mathfrak{K}_{md}(A)$. If h, $h' \in \mathfrak{K}_m(A^d)$, then

$$g_m(h \wedge h')(\overline{x}) = h \wedge h'(\gamma_m(\overline{x})) = h(\gamma_m(\overline{x})) \wedge h'(\gamma_m(\overline{x})),$$

and g_m preserves meets; similarly, one verifies that it preserves joins. To show that $g = \langle\, g_n \,\rangle_{n\geq 0}$ is a graded frame morphism, we use the following notational convention : if $\overline{x} \in |A|^{(m+n)d}$, write

$$\overline{z} \;=\; \langle\, x_1, \ldots, x_{md} \,\rangle \quad \text{and} \quad \overline{y} \;=\; \langle\, x_{md+1}, \ldots, x_{md+nd} \,\rangle.$$

Hence, $\overline{z} \in |A|^{md}$, while $\overline{y} \in |A|^{nd}$. Moreover,

$$\gamma_{m+n}(\overline{x}) = \langle\, \gamma_m(\overline{z}), \gamma_n(\overline{y}) \,\rangle. \tag{3}$$

Given $h \in \mathfrak{K}_m(A^d)$ and $k \in \mathfrak{K}_n(A^d)$, (3) yields, for $\overline{x} \in |A|^{(m+n)d}$,

$$\begin{aligned} g_m(h) \otimes g_n(k)(\overline{x}) &= g_m(h)(\overline{z}) \wedge g_n(k)(\overline{y}) \;=\; h(\gamma_m(\overline{z})) \wedge k(\gamma_n(\overline{y})) \\ &= h \otimes k(\gamma_m(\overline{z}), \gamma_n(\overline{y})) \;=\; h \otimes k(\gamma_{m+n}(\overline{x})) \\ &= g_{m+n}(h \otimes k)(\overline{x}), \end{aligned}$$

as needed. It remains to show that g_m is an isomorphism. Note that if $h \in \mathfrak{K}_m(A^d)$ and $\overline{x} \in |A|^{md}$, the first relation in (1) and 37.12.(a) entail

$$\begin{aligned} h(\gamma_m(\overline{x})) &= h(\overline{x}_{1_{|E\overline{x}_1}}, \ldots, \overline{x}_{m_{|E\overline{x}_m}}) \;=\; \textstyle\bigwedge_{j=1}^m E\overline{x}_j \wedge h(\overline{x}) \\ &= E\overline{x} \wedge h(\overline{x}) \;=\; h(\overline{x}), \end{aligned} \tag{4}$$

which immediately implies that g_m is injective. For surjectivity, fix $k \in \mathfrak{K}_{md}(A)$. Note that if $t \in |A^d|$, then

$$t \;=\; \langle\, t_1, \ldots, t_d \,\rangle, \;\; \text{with } Et_k \;=\; Et_1,\, 1 \leq k \leq d.$$

Define $h : |A^d|^m \longrightarrow \Omega$ by

$$h(t_1, \ldots, t_m) \;=\; k(t_{11}, \ldots, t_{1d}, t_{21}, \ldots, t_{2d}, \ldots, t_{m1}, \ldots, t_{md}).$$

It is straightforward that $h \in \mathfrak{K}_m(A^d)$. For $\overline{x} \in |A|^{md}$, we have

[6] To simplify presentation, we momentarily abandon the infix notation.

$$\begin{array}{rcl} g_m(h)(\overline{x}) & = & h(\gamma_m(\overline{x})) \;=\; h(\overline{x}_{1_{|E\overline{x}_1}}, \ldots, \overline{x}_{m_{|E\overline{x}_m}}) \\ & = & k(\overline{x}_{1_{|E\overline{x}_1}}, \ldots, \overline{x}_{m_{|E\overline{x}_m}}) \;=\; E\overline{x} \wedge k(\overline{x}) \;=\; k(\overline{x}), \end{array}$$

establishing that $k = g_m(h)$ and ending the proof. □

Our next task is the description of the functorial properties of $\mathfrak{K}_*$. For a morphism of presheaves over the same frame, 38.11 (or 39.1) yields

Corollary 45.11. *If Ω is a frame, $\mathfrak{K}_*$ is a covariant functor from the category of Ω-presheaves to the category of graded frames, with $\bigvee$-morphisms. A morphism of Ω-presheaves, $f : A \longrightarrow B$, induces a $\bigvee$-morphism*

$$f_* \;=\; \langle\, f_{*n} \,\rangle_{n \geq 0} : \mathfrak{K}_*A \longrightarrow \mathfrak{K}_*B,$$

that in degree 0 is the inclusion of $(EA)^{\leftarrow}$ into $(EB)^{\leftarrow}$, and for $n \geq 1$, $R \in \mathfrak{K}_n(A)$ and $\overline{b} \in |B|^n$,

$$[\![\, f_{*n}R(\overline{b}) \,]\!] \;=\; \textstyle\bigvee_{\overline{a} \in |A|^n} [\![f\overline{a} = \overline{b}]\!] \wedge [\![\, R(\overline{a}) \,]\!].$$

Proof. Recall (25.36) that $EA \leq EB$ and so f_{*0} is well defined. In view of 38.11, it is enough to check that f_* is a morphism of graded frames, that is, that for $n, m \geq 0$, $R \in \mathfrak{K}_nA$ and $S \in \mathfrak{K}_mA$

$$f_{*n}R \otimes f_{*m}S \;=\; f_{*(n+m)}(R \otimes S).$$

If $\overline{b} \in |B|^n$ and $\overline{d} \in |B|^m$, write $\langle \overline{b}, \overline{d} \rangle = \langle b_1, \ldots, b_n, d_1, \ldots, d_m \rangle \in |B|^{n+m}$. Then,

$$\begin{array}{l} [\![\, f_{*n}R \otimes f_{*m}S(\langle \overline{b}, \overline{d} \rangle) \,]\!] \;=\; [\![\, f_{*n}R(\overline{b}) \,]\!] \wedge [\![\, f_{*m}(\overline{d}) \,]\!] \;= \\ \quad = \; \bigvee_{\overline{a} \in |A|^n} \bigvee_{\overline{c} \in |A|^m} [\![f\overline{a} = \overline{b}]\!] \wedge [\![f\overline{c} = \overline{d}]\!] \wedge [\![\, R(\overline{a}) \,]\!] \wedge [\![\, S(\overline{c}) \,]\!] \\ \quad = \; \bigvee_{\langle \overline{a}, \overline{c} \rangle \in |A|^{n+m}} [\![f\langle \overline{a}, \overline{c} \rangle = \langle \overline{b}, \overline{d} \rangle]\!] \wedge [\![\, R \otimes S(\langle \overline{a}, \overline{c} \rangle) \,]\!] \\ \quad = \; [\![\, f_{*(n+m)}(R \otimes S)(\langle \overline{b}, \overline{d} \rangle) \,]\!], \end{array}$$

as needed. □

By Remark 40.9 [7], in general f_* **is not a stable** morphism of graded frames. However, for *inverse image* we have

Corollary 45.12. *If Ω is a frame, $\mathfrak{K}_*$ is a contravariant functor from* $\boldsymbol{pSh(\Omega)}$ *to the category of graded frames with open stable morphism. If $f : A \longrightarrow B$ is a morphism of Ω-presheaves, then*

$$f^* \;=\; \langle\, f_n^* \,\rangle_{n \geq\, 0} : \mathfrak{K}_*B \longrightarrow \mathfrak{K}_*A$$

that in degree 0 is the right adjoint of the inclusion of $(EA)^{\leftarrow}$ into $(EB)^{\leftarrow}$, and for $n \geq 1$, $T \in \mathfrak{K}_nB$ and $\overline{a} \in |A|^n$

$$[\![\, f_n^*T(\overline{a}) \,]\!] \;=\; [\![\, T(f\overline{a}) \,]\!].$$

In particular, $f_n^*(\top_n) = \top_n$ *and* $f_n^*(\bot_n) = \bot_n$.

[7] See also 40.10 and 40.11.

Proof. It is clear that f^*_n is well-defined for all $n \geq 0$. Taking 38.11 into account, it remains to check that f^* is a *stable map* of graded frames. To see that it respects grading, let $R \in \mathfrak{K}_n(B)$ and $S \in \mathfrak{K}_m(B)$. For $\overline{a} \in |A|^{n+m}$, we have

$$\begin{array}{rcl} [\![f^*_n R \otimes f^*_m S(\overline{a})]\!] & = & [\![R(fa_1, \ldots, fa_n)]\!] \wedge [\![S(fa_{n+1}, \ldots, fa_{n+m})]\!] \\ & = & [\![R \otimes S(f\overline{a})]\!] \;=\; [\![f^*_{n+m}(R \otimes S)(\overline{a})]\!], \end{array}$$

as desired. For stability, write $\langle \overline{a}, c \rangle = \langle a_1, \ldots, a_n, c \rangle$ for a typical element of $|A|^{n+1}$. A similar convention applies to elements in $|B|^{n+1}$. The meaning of $f\langle \overline{a}, c \rangle$ is then clear. For $T \in \mathfrak{K}_n B$, it must be shown that

$$f^*_{n+1}(T \otimes \top_1) \;=\; f^*_n T \otimes \top_1. \qquad \begin{tikzcd} \mathfrak{K}_n B \arrow[r, "\otimes \top_1"] \arrow[d, "f^*_n"'] & \mathfrak{K}_{n+1} B \arrow[d, "f^*_{n+1}"] \\ \mathfrak{K}_n A \arrow[r, "\otimes \top_1"'] & \mathfrak{K}_{n+1} A \end{tikzcd}$$

If $\langle \overline{a}, c \rangle \in |A|^{n+1}$,

$$\begin{array}{rcl} [\![f^*_{n+1}(T \otimes \top_1)(\overline{a}, c)]\!] & = & [\![T \otimes \top_1(f\langle \overline{a}, c \rangle)]\!] \;=\; [\![T(f\overline{a})]\!] \wedge Efc \\ & = & [\![f^*_n T(\overline{a})]\!] \wedge Ec \;=\; [\![(f^*_n T \otimes \top_1)(\overline{a}, c)]\!], \end{array}$$

ending the proof. □

If $f : A \longrightarrow B$ is a morphism of presheaves, the graded morphisms f_* and f^* form an adjoint pair, as follows :

Corollary 45.13. *If $f : A \longrightarrow B$ is a morphism of Ω-presheaves, then*

$$f_* \circ f^* \leq Id_{\mathfrak{K}_*(B)} \quad \text{and} \quad f^* \circ f_* \geq Id_{\mathfrak{K}_*(A)}$$

and $\langle f_*, f^* \rangle$ is an adjoint pair [8]*. Moreover,*

a) The following conditions are equivalent :

(1) f is monic; (2) $\forall\, n \geq 0$, f_{*n} is injective; (3) $\forall\, n \geq 0$, f^*_n is surjective.

b) The following are equivalent :

(1) f is epic; (2) $\forall\, n \geq 0$, f_{*n} is surjective; (3) $\forall\, n \geq 0$, f^*_n is injective. □

Since the completion in Theorem 27.9 is monic and epic, we obtain

Corollary 45.14. *If A is a Ω-presheaf, there is a natural graded frame isomorphism, $c_* : \mathfrak{K}_*(A) \longrightarrow \mathfrak{K}_*(cA)$, where cA is the completion of A over Ω.* □

The existential and universal quantifiers yield examples of morphisms of the inductive system $\mathcal{I}(\mathfrak{K}_* A)$ (45.5), that *are not morphisms of graded frames* :

Corollary 45.15. *Let A be a Ω-presheaf and let $k \geq 1$ be an integer. Define*

$$\begin{cases} \exists v_k & : \; \mathcal{I}(\mathfrak{K}_* A) \longrightarrow \mathcal{I}(\mathfrak{K}_* A) \\ \forall v_k & : \; \mathcal{I}(\mathfrak{K}_* A) \longrightarrow \mathcal{I}(\mathfrak{K}_* A) \end{cases}$$

[8] That is, for all $n \geq 0$, $\langle f_{*n}, f^*_n \rangle$ is an adjoint pair as in 7.8.

by the following prescriptions :

$*$ *If* $n < k$, *then* $(Qv_k)_n \;=\; Id_{\mathfrak{K}_n A}$, $\;(Q \;=\; \exists, \forall)$;

$*$ *If* $k \leq n$, $R \in \mathfrak{K}_n A$ *and* $\overline{a} \in |A|^n$, *then*

$$\begin{cases} [\![\, \exists v_k R(\overline{a}) \,]\!] & = \; E\overline{a} \wedge \bigvee_{c \in |A|} [\![\, R(\overline{a} \ulcorner c \mid k \urcorner) \,]\!]; \\ [\![\, \forall v_k R(\overline{a}) \,]\!] & = \; E\overline{a} \wedge \bigwedge_{c \in |A|} Ec \to [\![\, R(\overline{a} \ulcorner c \mid \urcorner) \,]\!]. \end{cases}$$

Then, $\exists v_k$ *is a* $\bigvee$*-morphism and* $\forall v_k$ *is a* $\bigwedge$*-morphism on* $\mathcal{I}(\mathfrak{K}_* A)$.

PROOF. By Theorem 42.6, $\exists v_k$ and $\forall v_k$ are, in each degree, right and left adjoints, and so, $\bigvee$ and $\bigwedge$-morphisms, respectively; it remains to show (see 45.5) that if $n \geq 0$ and $R \in \mathfrak{K}_n A$, then

$$\text{(1)} \qquad \begin{cases} \exists v_k(R \otimes \top_1) & = \; \exists v_k R \otimes \top_1 \\ \forall v_k(R \otimes \top_1) & = \; \forall v_k R \otimes \top_1. \end{cases}$$

If $k > n+1$, it follows immediately from the definitions that the equations in (1) are satisfied. If $k \geq n+1$, we have two possibilities :

$*$ $\underline{k > n+1 :}$ Then, for $Q = \exists, \forall$, $\;Qv_k(R \otimes \top_1) \;=\; R \otimes \top_1 \;=\; Qv_k R \otimes \top_1$, as needed.

$*$ $\underline{k = n+1 :}$ Then, for $\langle \overline{a}, c \rangle = \langle a_1, \ldots, a_n, c \rangle \in |A|^{n+1}$ we have, recalling 8.16.(b), 6.4.(c) and that $p \to q = p \to (p \wedge q)$

$$\begin{aligned}
[\![\, \forall v_k(R \otimes \top_1)(\langle \overline{a}, c \rangle) \,]\!] \; &= \; E\overline{a} \wedge Ec \wedge \textstyle\bigwedge_{u \in |A|} Eu \to [\![\, R \otimes \top_1(\langle \overline{a}, c \rangle \ulcorner u \mid k \urcorner) \,]\!] \\
&= \; E\overline{a} \wedge Ec \wedge \textstyle\bigwedge_{u \in |A|} Eu \to [\![\, R \otimes \top_1(\langle \overline{a}, u \rangle) \,]\!] \\
&= \; E\overline{a} \wedge Ec \wedge \textstyle\bigwedge_{u \in |A|} Eu \to ([\![\, R(\overline{a}) \,]\!] \wedge Eu) \\
&= \; E\overline{a} \wedge Ec \wedge \textstyle\bigwedge_{u \in |A|} Eu \to [\![\, R(\overline{a}) \,]\!] \\
&= \; E\overline{a} \wedge Ec \wedge \Big[(\textstyle\bigvee_{u \in |A|} Eu) \to [\![\, R(\overline{a}) \,]\!] \Big] \\
&= \; E\overline{a} \wedge Ec \wedge (EA \to [\![\, R(\overline{a}) \,]\!]) \\
&= \; E\overline{a} \wedge Ec \wedge (E\overline{a} \to [\![\, R(\overline{a}) \,]\!]) \\
&= \; E\overline{a} \wedge Ec \wedge [\![\, R(\overline{a}) \,]\!] \; = \; [\![\, \forall v_k R(\overline{a}) \,]\!] \wedge Ec \\
&= \; [\![\, (\forall v_k R \otimes \top_1)(\langle \overline{a}, c \rangle) \,]\!],
\end{aligned}$$

as desired. The existential quantifier can be treated similarly. $\square$

We now show how the algebra $\mathfrak{K}_f A$, of finitary relations on A, can be recovered from the graded frame $\mathfrak{K}_* A$. Let A be a Ω-presheaf; for $n \geq 0$ define

$$\iota_n : \mathfrak{K}_n A \longrightarrow \mathfrak{K}_f A, \quad \text{by } [\![\, \iota_n R(\overline{a}) \,]\!] \;=\; E\overline{a} \wedge [\![\, R(a_1, \ldots, a_n) \,]\!],$$

where $\overline{a} \in |A|^{\mathbb{N}}$. Then

COROLLARY 45.16. *With notation as above*

a) For all $n \geq 0$, ι_n *is an open morphism.*

b) The system $\langle \mathfrak{K}_f A; \{\iota_n : n \geq 0\} \rangle$ *is the colimit of* $\mathcal{I}(\mathfrak{K}_* A)$, *that is,*

$$\mathfrak{K}_f A \;=\; \varinjlim \; \mathcal{I}(\mathfrak{K}_* A).$$

PROOF. Item (a) follows from 40.8; indeed, considered as a map from $\mathfrak{K}_n A$ to $\mathfrak{K}_{\mathbb{N}} A$, it is inverse image by the projection that forgets the coordinates outside $\underline{n}$, and so, open and injective. Since its image is $\mathfrak{K}_{\mathbb{N}}(A, \underline{n})$, a complete subalgebra of $\mathfrak{K}_f A$, ι_n is an open morphism (although $\mathfrak{K}_f A$ is only a Heyting algebra). Since for all $n \geq 0$, the following diagram is commutative

$$\begin{tikzcd}
\mathfrak{K}_n A \arrow[rr, "\otimes \top_1"] \arrow[dr, "\iota_n"'] & & \mathfrak{K}_{n+1} A \arrow[dl, "\iota_{n+1}"] \\
& \mathfrak{K}_f A &
\end{tikzcd}$$

the system $\langle\, \mathfrak{K}_f A; \{\iota_n : \geq 0\}\,\rangle$ is a dual cone over $\mathcal{I}(\mathfrak{K}_* A)$; it is straightforward that $\mathfrak{K}_f A = \varinjlim \mathcal{I}(A)$. $\square$

REMARK 45.17. Since $\mathfrak{K}_f A = \varinjlim \mathcal{I}(\mathfrak{K}_* A)$, inverse image by a morphism, as well as quantification, may be recovered from their graded counterparts defined for $\mathfrak{K}_*$. Indeed :

$*$ If $f : A \longrightarrow B$ is a morphism of Ω-presheaves, $f^* : \mathfrak{K}_* B \longrightarrow \mathfrak{K}_* A$ is, by 45.12, a stable open morphism. Hence [9], $\varinjlim f^* : \mathfrak{K}_f B \longrightarrow \mathfrak{K}_f A$ is a Heyting algebra morphism; it is easily established that $\varinjlim f^*$ is precisely the inverse image by f described in 44.6, with $\lambda = Id_\Omega$.

$*$ By 45.15, $\exists v_k$, $\forall v_k$ constitute morphism of the inductive system $\mathcal{I}(\mathfrak{K}_* A)$. Hence, $\varinjlim \exists v_k$ and $\varinjlim \forall v_k$ are maps from $\mathfrak{K}_f A$ to $\mathfrak{K}_f A$; it is straightforward that these colimits maps correspond to existential and universal quantification as in 42.11. $\square$

Exercises

45.18. If $L_* \xrightarrow{f} K_*$ is a map of degree d and $\alpha \in L_m$, then, for all $n \geq 0$, the following diagram is commutative :

$$\begin{tikzcd}
L_n \arrow[r, "f_n"] \arrow[d, "(*) \otimes \alpha"'] & K_{dn} \arrow[d, "(*) \otimes f(\alpha)"] \\
L_{n+m} \arrow[r, "f_{n+m}"'] & K_{(n+m)d}
\end{tikzcd}$$

$\square$

[9] See Proposition 17.19.

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